Generalized numerical range as a versatile tool to study quantum entanglement

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Let X be an operator acting on an N-dimensional complex Hilbert space $\mathcal{H}_N$. Let $W(X)$ denote its *numerical range* [1], i.e. the set of all λ such that there exists a normalized state $|\psi\rangle \in \mathcal{H}_N$, $||\psi|| = 1$, which satisfies $\langle\psi|X|\psi\rangle = \lambda$. We are going to analyze various generalizations of this definition in view of their possible applications in the theory of quantum information.

Take any integer number k such that $1 \leq k \leq N$ and define a subset of the complex plane given by

$$W_k(X) = \{\lambda \in \mathbb{C} : P_k X P_k = \lambda P_k\}, \tag{1}$$

where P_k is an arbitrary k–dimensional projection operator. Note that this definition reduces the to standard numerical range for $k = 1$. For $k > 1$ the sets $W_k(X)$ are called *higher-rank numerical ranges* [2, 3] and they satisfy the following inclusion relation $W_1(X) \supseteq W_2(X) \supseteq \ldots \supseteq W_N(X)$.

It was recently shown that for any normal operator X its higher rank numerical range forms a convex set [4, 5]. This generalization of the standard numerical range is interesting from the mathematical perspective [6] and also in relation to quantum error correction codes [7, 8, 9].

Let us now take an arbitrary composite number, $N = KM$, and consider the Hilbert space $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$ with a tensor product structure. Following [10, 11] we define the *product numerical range* $W_\otimes$ of X, with respect to this tensor product structure,

$$W_\otimes(X) \ := \ \{\langle\psi_A \otimes \psi_B|X|\psi_A \otimes \psi_B\rangle : |\psi_A\rangle \in \mathcal{H}_K, |\psi_B\rangle \in \mathcal{H}_M\}, \tag{2}$$

where the states $|\psi_A\rangle \in \mathcal{H}_K$ and $|\psi_B\rangle \in \mathcal{H}_M$ are normalized.

We analyze operators acting on a tensor product Hilbert space and investigate their product numerical range, product numerical radius and product C–numerical radius. Concrete bounds for the product numerical range for Hermitian operators are derived. Product numerical range of a non-Hermitian operator forms a subset of the standard numerical range. While the latter set is convex, the product range need not be convex nor simply connected [12].

The product numerical range of a tensor product is equal to the Minkowski product of numerical ranges of individual factors. As an exemplary application of these algebraic tools in the theory of quantum information, we study block positive matrices, entanglement witnesses and consider the problem of finding minimal output entropy of a quantum channel. Furthermore, we apply product numerical range to solve the problem of local distinguishability for a family of two unitary gates.

For an arbitrary operator A which acts on an N dimensional complex Hilbert space $\mathcal{H}_N$ we introduce its *numerical shadow* as a probability distribution P_A

defined on the complex plane

$$
(3) \qquad P_A(z) \ := \ \int_{\Omega_N} \mathrm{d}\mu(\psi)\ \delta\Big(z - \langle\psi|A|\psi\rangle\Big) \ ,
$$

where $\mu(\psi)$ denotes the unique unitarily invariant (Fubini-Study) measure on the set Ω_N of N–dimensional pure quantum states.

We show that for any normal operator A acting on $\mathcal{H}_N$, such that $AA^* = A^*A$, its shadow covers its numerical range with the probability corresponding to a projection of a *regular* N–simplex embedded in $\mathbb{R}^{N-1}$ into the plane. As the numerical range of a generic non–normal matrix is not a polygon, the corresponding numerical shape occurs to be a more complicated probability distribution. Numerical shadow of such an exemplary matrix of order three, $A_3 = \begin{pmatrix} 0 & 1 & 1 \\ 0 & i & 1 \\ 0 & 0 & -1 \end{pmatrix}$ with respect to real states shown in Fig. 1a resembles an artist's image of $\mathbb{R}P^3$.

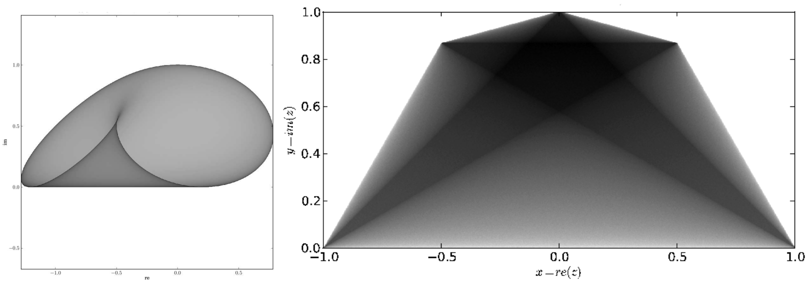

FIGURE 1. Numerical shadow restricted to real states for the operators A_3 and $A_5 = \mathrm{diag}[1, \exp(i\pi/3), i, \exp(i2\pi/3), -1]$. Observe that the inner dark pentagon in the right panel allows one to identify the numerical range of rank $k = 2$ of A_5.

This notion may also be generalized to give a *restricted* shadow of an operator,

$$
(4) \qquad P_A^R(z) \ := \ \int_{\Omega_R} \mathrm{d}\mu_R(\psi)\ \delta\big(z - \langle\psi|A|\psi\rangle\big) \ ,
$$

where $\mu_R(\psi)$ denotes the Fubini-Study measure restricted to the set Ω_R and normalized, $\int_{\Omega_R} \mathrm{d}\mu_R(\psi) = 1$. For instance, one can consider the set of real states and analyze the shadow restricted to real states.

Assume now that the dimension N is composite, so one can define the sets of separable pure states and maximally entangled states. In analogy to the notion of product numerical range one can thus analyze numerical shadow restricted to separable (maximally entangled) states only. In the simplest case of $N = 2 \times 2$ the numerical shadow of a unitary matrix of size 4 is presented in Fig. 2

Investigating numerical shadows of several operators of a given composite dimension with respect to the set of separable (maximally entangled) states one

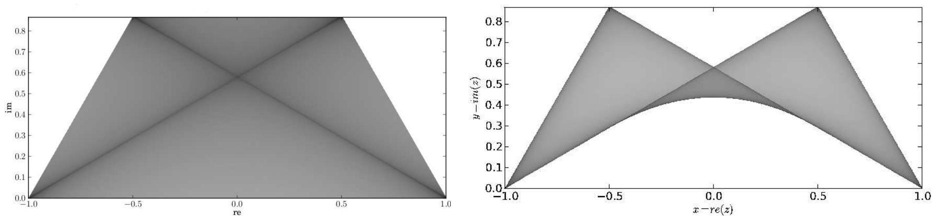

FIGURE 2. Numerical shadow for the operator $A_4 = \mathrm{diag}[1, \exp(i\pi/3), \exp(i2\pi/3), -1]$ restricted to a) real states and b) real separable states.

gains information about the structure of these multi-dimensional sets. On the other hand, knowing the numerical shadow of an unknown observable it is possible to identify this observable. We believe that the advocated approach based on geometrization of the algebraic notions provides a further contribution to our understanding of the geometry of quantum entanglement [13].

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