On the stability of the solar system

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... if we conceive of an intelligence that at a given instant comprehends all the relations of the entities of this universe, it could state the respective position, motions, and general affects of all these entities at any time in the past or future.

Pierre-Simon Laplace

If the above statement of Laplace [1] were true, we could predict the future of any dynamical system by measuring its initial conditions. For instance, knowing the masses, positions and velocities of the sun and the planets of the Solar System one could determine their trajectories for an arbitrary long time. In fact, an apparent regularity of the dynamics of all the planets, based on centuries of observations, might support such a point of view.

The description of motion of planets on the sky was one of the main problems, which stimulated advances of the natural sciences over the centuries. Decisive discoveries explaining the rules governing the motion of planets were made by the founding fathers of contemporary astronomy, mathematics, and physics, including Copernicus, Galileo, Kepler, and Newton. While celestial mechanics of any *two* bodies, interacting by the gravitational force, is well understood, the famous *three body problem* is by far more complex.

Although important partial results were obtained during the last three centuries by Leonard Euler (1707-1783), Louis Lagrange (1736-1813), C.G. Jacobi (1804-1851), George W. Hill (1938-1914), Henri Poincaré (1854-1912), Tullio Levi-Civitá (1873-1941), George D. Birkhoff (1884-1944) and many others, the general problem of solving the dynamics of three interacting bodies cannot be solved analytically, and one needs to rely on numerical methods. Even a simplified version of the model, the so-called restricted three body problem, in which the mass of one body is negligible in comparison with the total mass of the system, may exhibit complicated dynamics. Vaguely speaking, the probe body may rotate around one source of the gravitational field, but at a certain point in time it might start to encircle the other body. If the energy and the angular momentum of the total system are sufficiently high, it is hardly possible to specify the exact time of such a transition, so the dynamics of the system becomes unpredictible. Such a motion of a dynamical system, in which arbitrarily small variation of the initial conditions causes exponentially large changes of the future trajectory is called *chaotic.* In the opposite case the dynamics is called stable or regular.

Can one predict the motion of a single planet for, say, billion of years from now*? Is the solar system stable?*

This very question attracted attention of several generations of researchers. Initial results of Laplace and Lagrange strongly suggested positive answer. Already at the end of XIX century the issue of stability of the Solar System was considered as one of the crucial challenges for natural sciences and the king of Sweden, Oskar II, founded special prize for solving the problem. The prize was awarded in 1887 to the French mathematician Henri Poincaré, who had obtained important, but not entirely decisive results though. He demonstrated that the frequently used perturbation techniques may not lead to the correct solution, since the series taking into account terms of higher and higher orders may not converge.

Mathematical theory of stability of motion has been initiated by Aleksander M. Lapunov (1857-1918), who analysed how fast the distance between two neighbouring trajectories increases in time. If the system in question is chaotic, such a distance grows exponentially with time, and the coefficient in the exponent, called *Lapunov exponent*, is positive. Although such systems were known to mathematicians since the beginnings of the nineteenth century, they were considered rather a mathematical curiosity and researchers were not aware of their implications for physics, astronomy and science in general. The situation has changed during the last thirty years, when the significance of the pioneering work of Edward Lorenz was accepted. In his 1963 paper [2], published in a meteorological journal, Lorenz analysed numerically a certain nonlinear dynamical system and demonstrated that a minor variation of the initial conditions changed dramatically the behaviour of the system. This property, nowadays called the *"Butterfly effect"*: (flap of wings of a butterfly in Australia may cause a tornado in Florida) turned out to be typical for a majority of dynamical systems used to model various phenomena in physics, chemistry or biology.

What may be the origin of chaos in a dynamical system? Just as in the restricted three body problem, the instability may arise if there exist two forces of the same order of magnitude, which act on a given body. Fortunately, the mass of Sun is almost a thousand times larger than the total mass of the planets. Hence, the gravitational interaction of any given planet with other planets is much smaller than the interaction with the Sun, so the degree of chaos, expressed in the Lapunov exponent, cannot be large. On the other hand, taking into account the discovery of Lorenz, it is not so unnatural to expect chaos in the Solar System. To the contrary, it would be surprising if a system of so many coupled degrees of freedom (say, 7 major planets and the Sun, times 3 degrees each, gives 24 degrees of freedom), were regular.

To receive any answer to this issue by means of numerical simulations one needed computer performance which became available only recently, (for instance numerical integrations of the Solar System dynamics for the total times span of 35 Gyr performed by Mitchtchenko and Ferraz-Mello [3] took about 15 weeks of CPU time on a 660 MHz Alpha 21264A workstation). First numerical evidence of the chaotic nature of the dynamics of Pluto was obtained in 1988 by Sussmann and Wisdom [4], who estimated the Lapunov time (the inverse of the Lapunov exponent, which sets the time horizon of a possible prediction of a system trajectory) of order of 10 million years. Further numerical investigations of the full Solar System performed by Laskar and co-workers and included the Newtonian interaction of 8 major planets with relativistic and lunar corrections, allowed to estimate the Lapunov time of the entire System (with Pluto ignored) of 5 million years [5-7]. To put some light onto this number please note that $e^{19} \approx 1.5 \times 10^8$, which is equal to the distance from Earth to the Sun (one astronomical unit), expresses in kilometers. Thus the uncertainty of 1 km in determining the initial position of a planet may increase to one astronomical unit after the time 19 x 5Myr = 95 Myrs. Although the above value of the Lapunov exponent of the Solar System was later confirmed in an investigation of a more realistic model by Sussmann and Widom [8], the source of the chaos has not been convincingly established.

Although from the mathematical point of view, such numerical results do not provide a rigorous proof that dynamics of the Solar System is chaotic, we now clearly see that Laplace was wrong: even measuring initial conditions of the system to an arbitrarily high, but *finite* accuracy, we will not be able to describe the system dynamics "at any time in the past or future". To predict future of a chaotic system for arbitrary long times, one would need to know the initial conditions with *infinite* accuracy, and this is by no means possible. Furthermore, we do not know exactly the *complete Hamiltonian* of the Solar System: apart from Newtonian gravitational forces, there are many effects of various origin (e.g. relativistic corrections, variation of the mass of the sun, consequences of asphericity of the planets and the spin-orbit resonances, thermal radiation, and many others), which cannot be exactly taken into account.

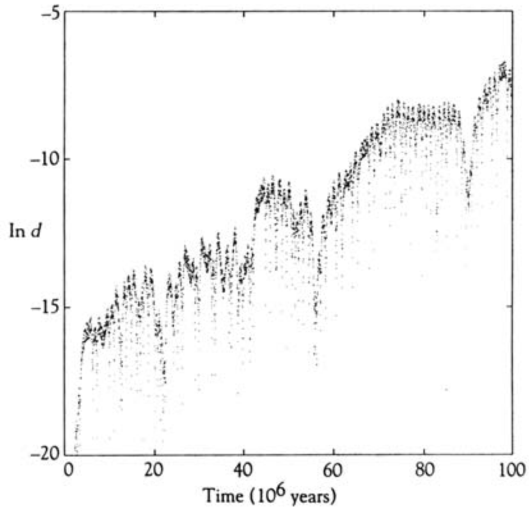

Fig.1. *Divergence d (in astronomical units) between two initially close orbits of Pluto grows exponentially in time (from Susmann and Wisdom [4]). Linear fit of the data, represented in semilog scale, gives an approximate slope 1/12, which corresponds to the Lapunov time of the order of 10 Myrs.*

Physics is an experimental science, and it is not reasonable to expect any progress of the theory without a close relation to experiment. Astronomy, on the other hand, is an observational science: instead of performing experiments, which in principle could be repeated infinitely many times, astronomers have to rely on observations, some of which may hardly be repeated. The situation is different, when one considers 'numerical astronomy', for instance, numerical computations of the trajectories of planets in the Solar System. In such a case it is not difficult to perform 'Gedankenexperiments' by varying arbitrarily the actual parameters of the system, and studying the consequences of the changes made.

The most natural question to be asked, concerns the long term fate of the Earth. Using his frequency map analysis (FMA) method of numerical integration of equations governing the motion of the planets of the Solar System, Laskar was varying the initial conditions of various planets. Changing the present position of the Earth only by 150 meters (10^{-9} variation in eccentricity) after 100 Myr one obtains quite different trajectory [7], what is a confirmation of chaotic dynamics. On the other hand, all these reference trajectories are *similar* to the 'real' one. Hence the Solar System is *structurally stable*: although one can not predict exactly the future trajectory for such a long time span, we have good arguments to expect that the overall structure of the System will not be destroyed. A related phenomenon called *shadowing* is known in the theory of dynamical systems: numerical iteration of any chaotic map for times much longer than the Lapunov time cannot provide the *exact* solution. However, under some technical assumptions one may rigorously prove the existence of an exact trajectory of the system, which originates from a slightly different initial conditions, and is *shadowed* by the computer generated 'approximate orbit', i.e. is arbitrary close to it [9]. This important result indicates that long time computer simulations of dynamics of a chaotic system, e.g. the Solar System, are not entirely meaningless [10].

Numerical results show [4-8] that planets of the Solar system exhibit rather different behaviour. Beside chaotic trajectory of Pluto (see Fig. 1), the dynamics of the outer planets (Jupiter to Neptun) is essentially regular, while the motion of the inner planets (Mercury to Mars) is largely chaotic. This fact is easily seen in Fig.2, which shows how eccentricity and inclination of orbits of the planets vary in time. While the behaviour of the large planets is so regular that all curves appear as straight lines, the corresponding curves for inner planets show erratic fluctuations, which may be interpreted as signatures of chaos.

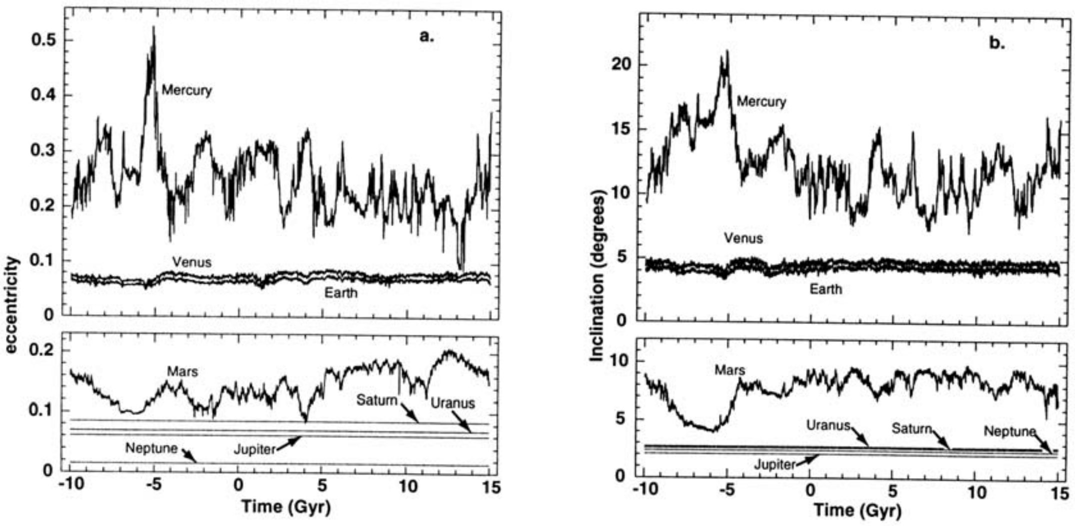

Fig.2. *Maximal eccentricity (a) and inclination (b) of 8 planets of the Solar system as a function of time. Picture obtained by Laskar [7] by means of numerical integration. Note different character of the dynamics of outer and inner planets.*

To investigate the role of the Earth-Moon system in the dynamical stability of the inner Solar System, Innanen, Mikkola and Wiegert [11] performed several numerical experiments integrating the equations of motion without certain inner planets. They have obtained rather unexpected results: Earth plays a crucial role in the long-term stability of the orbits of the inner planets. In the absence of Earth the orbits of Venus and Mercury would be heavily exposed to strong destabilising resonances with giant planets.

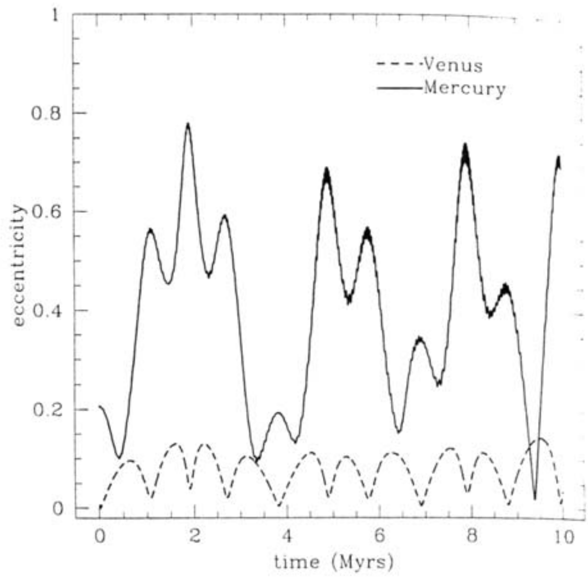

Fig.3. *Eccentricity of orbits of Mercury and Venus as a function of time obtained by Innanen et al [11] by numerically integrating the equations of motion in absence of Earth. Fluctuations of this parameter become much larger than these of the actual system (see Fig. 2), so we may be proud of our Earth: it stabilises the dynamics of the Inner Solar System.*

The role of the resonances may be illustrated by discussing the dynamics of a simple model system - the mathematical pendulum subjected to a constant gravitational field. Depending on its energy, there exist two qualitatively different kinds of the motion: *rotations*, for which the pendulum rotates in one direction, so the sign of the angular momentum is constant, or *librations* around a stable fix point, for which the sign of the angular momentum changes twice during every period of oscillations. Looking at the phase space diagram of the motion, in which any point in the plane (angle θ, angular momentum p) generates a certain trajectory, (see Fig. 4), there exist a curve which separates different kinds of the dynamics. It is called *separatrix* and it consists of two branches connected at the unstable fixed point, (θ= π, p=0). The stable fixed point of the system is located at (θ=0, p=0), in the center of the area bounded by the separatrix. The pendulum pointing down, (θ=0), is stable and perturbed in any direction returns back to its initial position. On the other hand, the unstable fixed point corresponds to the pendulum situated 'upside down', so an arbitrarily small perturbation would drive it out of this position.

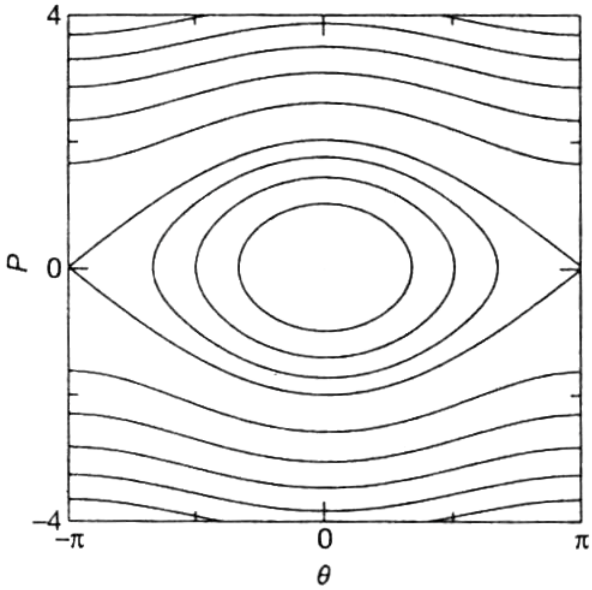

Fig.4. *Phase space diagram for the pendulum resembles a cylinder. The elliptical fixed point (stable) in the center of the plot is surrounded by ellipses representing librations, while the hyperbolic fixed point (unstable) is located at the cut, (θ=π=-π), and belongs to the separatrix. Curves encircling horizontaly the cylinder represent rotations.*

The separatrix with the unstable fixed point is crucial for emergence of chaos. If there exist (at least) two interacting oscillators in the system, a region of chaotic motion emerges in the system. Such an effect may be easily observed in a simplified model of a periodically kicked rotator, defined by the following Hamiltonian

$$H = p^2/2 + K \Sigma_n \ \delta(t-n) \ \cos(\pi\theta) \ . \qquad (1)$$

The system rotates without friction and gravitational field but is subjected to infinitely short periodic perturbations (kicks), which take place at t=1,2,3..., in units of the kicking period. The dynamics depends on the kicking strength *K:* for *K=0* there is no perturbation and the system performs free rotations, corresponding to horizontal lines in the phase space (θ, p). For positive values of the parameter *K* the dynamics becomes more complex: resonances with frequencies commensurate with the kicking period emerge around stable periodic orbits – see Fig.5. There exist also unstable periodic orbits. In contrast with the case of the pendulum shown in Fig. 4, the infinitely thin separatrices are transformed into layers of chaotic motion of a finite volume, which grows with the parameter *K*. Chaotic layers are formed around every resonance, so they do occur at the resonant frequencies. If the kicking strength exceeds a critical value $K_c \approx 0.97$ the last horizontal curve (the so-called *KAM torus* – see e.g. the book by Ott [12]) brakes down, all chaotic layers become connected, so the trajectories may diffuse in the phase space acquiring arbitrarily high momentum, (the system is kicked and the energy in the system is not conserved).

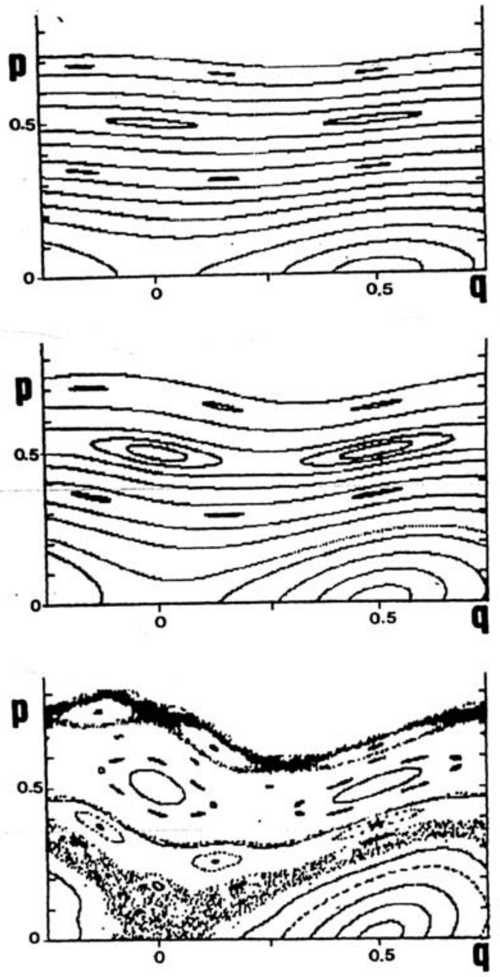

Fig.5. *Phase space diagram for the periodically kicked rotator (1) for kicking strength a) K=0.25; b) K=0.51; c)* K=1.02>K_c. *Note an increase of the volume of the chaotic layer in the phase space.*

Although there are no forces comparable to periodical kicking in the Solar System, as in the above toy model, the interference of the interactions of any body with the third body plays the very same role and is responsible for emergence of chaos. Furthermore, the destabilising role of dynamical resonances may be observed by studying different issues of the Solar System. The famous Kirkwood gaps [13] in the histogram of the density of asteroids plotted as a function of their semi-major axis may be explained by the interaction with Jupiter. This fact is apparent if the same data are used to produce a histogram as a function of the oscillation period: it shows minima at certain frequencies commensurate with the frequency of Jupiter [14,15]. For instance, due to the 3:1 resonant interaction with Jupiter the trajectories of asteroids become unstable, which explains the observed minimum of the asteroids density at the frequency three times larger than the frequency of the motion of Jupiter.

The resonant interaction with Neptune influences the dynamics of bodies in the Kuiper Belt [16] – the group of objects more distant from the Sun than Neptune, including Pluto. On the other hand, the resonant interaction between Saturn and Jupiter could contribute to the destabilisation of the Solar system. In fact the actual frequencies of the both largest planets are close to the 5S:2J resonance (see Fig.6.). As shown by Michtchenko and Ferraz-Mello [3] a relatively small variation of the parameters determining the orbit of Jupiter (say, diminishing its semi-major axis by 4 promile) would increase the role of the resonance, and in consequence would lead to the chaotic dynamics of both giant planets – see Fig. 7.

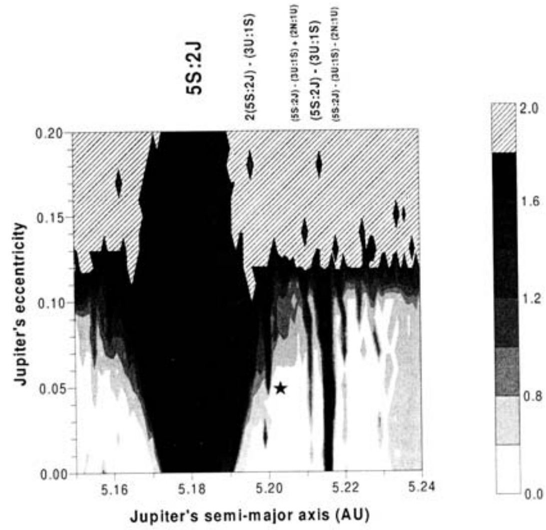

Fig. 6. *Dynamical map of the region around Jupiter: stability of the orbit initiated from a given point in the semi-major axis – eccentricity plane plotted in the grey scale – light (dark) region denotes regular (chaotic) motion while hatched region indicates orbits for which planetary collision occur). Star represents the actual position of the Jupiter, close to the 5:2 resonance [3].*

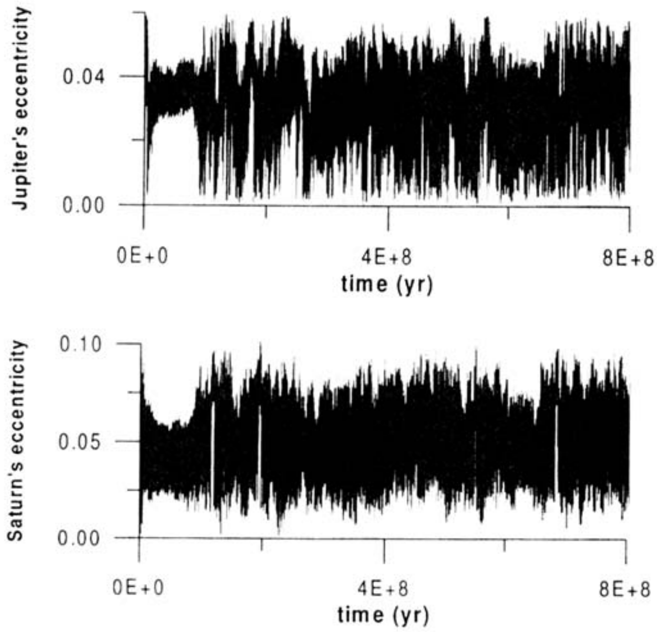

Fig 7. *Chaotic evolution of Jupiter's and Saturn's eccentricity obtained by Michtchenko and S. Ferraz-Mello [3] for modified initial conditions corresponding to 5:2 resonance.*

Entirely different, albeit more complex exemplification of the destabilising role of the dynamical resonances is provided by the rings of Saturn. Existence of the gaps in the rings may be explained by the resonant interaction with Minas and other satellites of Saturn, provided one takes into account the waves which propagate in the matter forming the rings. Thus remnants of chaotic behaviour may be seen at different length scales: as well as analysing the motion of objects circulating around the Sun, as well as studying the dynamics of satellites of large planets. Yet another spectacular example of chaotic motion in the latter scale is afforded by Hyperion. The shape of this small satellite of Saturn is quite far from a sphere – the difference in the length of the axis of the ellipsoid approximating the satellite are responsible for additional coupling between orbital and spin (rotational) motion, and in consequence to chaotic oscillation of the orientation of the largest axis of Hyperion [17].

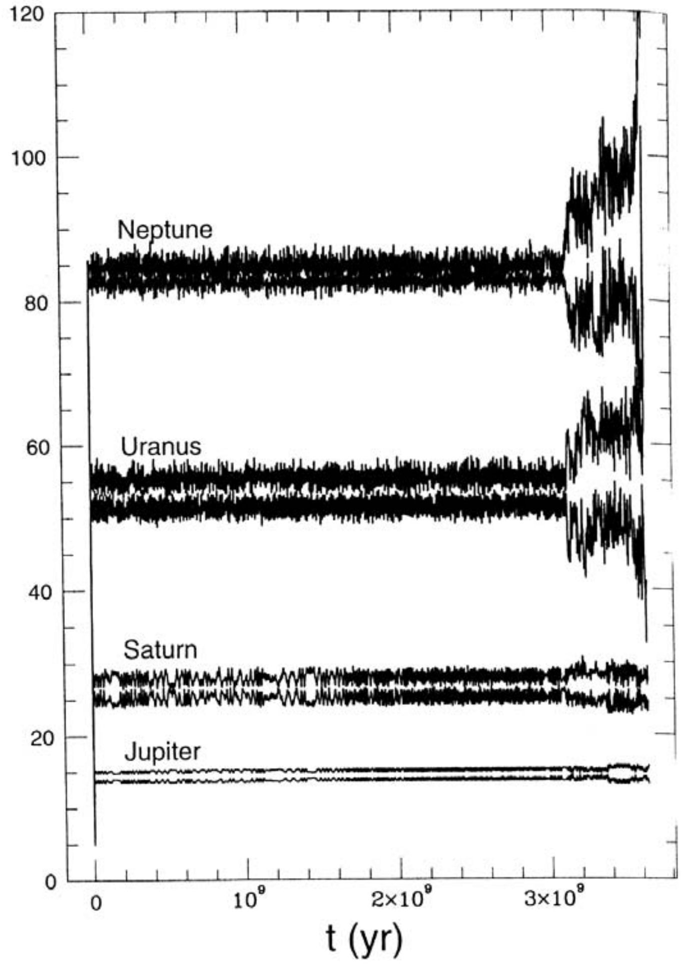

Fig 8. *Chaotic evolution of heliocentric distance for outer planets for a modified Solar System, in which mass of the Sun is decreased to 36% of its true value [18].*

Our understanding of the dynamics of the Solar System has been significantly improved as a result of research performed during the last two decades. Numerical analysis allows definitely to give a positive answer to the question, whether the dynamics of the Solar System is chaotic. Fortunately this fact has no bearing on our daily activities, since the Lapunov time of the Solar System is by many orders of magnitude longer than man's life.

The Solar System is chaotic, and thus unstable, in a sense that one cannot predict the trajectory of Earth for periods of time exceeding, say, 100 Myrs. On the other hand, it is structurally stable, since small variations of the parameters of the planets, comparable with the accuracy of their measurements, lead to different but similar orbits – it is thus unlikely that the Solar System will fall apart during the next billion years. However, this structural stability is limited and the Solar System is fragile: if variations of the parameters were of the order of ten percents, the configuration of the system might suffer crucial qualitative changes. For instance, decreasing the mass of the Sun by half would strongly destabilise dynamics of the System [18] – see Fig. 8.

Dynamics of the Solar System attracts attention of astronomers, mathematicians and physicists for at least 400 (or rather 4 000) years, but is still considered a fascinating field of research. Recent explosion of computing power facilitates more extensive numerical analysis of various aspects of the problem and allows us to expect new interesting results.

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