Random Matrices of Circular Symplectic Ensemble

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Abstract: Random unitary matrices of symplectic ensemble describe statistical properties of time-dependent, periodical quantum systems with a half–integer spin. We present a method of constructing random matrices typical to circular symplectic ensemble and show that the numerically generated unitary symplectic matrices display statistical properties of spectrum and eigenvectors according to the predictions of the random matrix theory.

1 Introduction

Random matrices, often used to describe statistical properties of complicated quantum systems with many degrees of freedom, are also applicable for simple quantum systems with few degrees of freedom, which exhibit chaos in the classical limit [1, 2]. A Hamiltonian of an autonomous quantum system may be represented by a Hermitian matrix of a Gaussian ensemble [3], whereas for a system periodically perturbed in time a more convenient characterization is provided by a unitary matrix representing the evolution operator propagating the wave function of the system over one period of the perturbation. Canonical ensembles of unitary matrices, invariant with respect to orthogonal, unitary or symplectic transformations where introduced by Dyson [4]. Such random matrices are also useful for investigating open scattering systems, described by a unitary S matrix [5].

Depending on the symmetry properties of the system one of the three canonical ensembles should be used in both cases. Systems possessing an antiunitary symmetry (mostly the time reversal invariance) display a linear repulsion of neighbouring energy levels (eigenphases) and are described by orthogonal ensembles. Unitary ensembles, appropriate for systems with the time reversal symmetry broken, exhibit quadratic level repulsion. Systems with a half-integer spin, a time-reversal invariance and no rotational symmetry pertain to the symplectic universality class, which is characterized by a quartic level repulsion [2]. Qualitatively speaking the presence of the Kramers degeneracy makes any additional accidental degeneracy very unlikely. This kind of spectral statistics was found for a periodically kicked top with a half integer spin [6] and an appropriately modified version of the kicked rotator [7, 8], Statistical properties of such time–dependent dynamical systems may be therefore described by random matrices of circular symplectic ensemble (CSE).

It is relatively easy to generate random Hermitian matrices pertaining to different universality classes - the matrix elements of such matrices are statistically independent random variables drawn according to a Gaussian distribution with zero mean. [3]. The only constrains are imposed by the algebraic conditions of symmetry (reality), hermiticity and symplecticity, involving pairs of elements.

Construction of unitary matrices typical of circular ensembles is more complicated, since unitarity imposes correlation between elements of the matrix. Recently we proposed a simple algorithm allowing one to construct random unitary matrices typical of circular unitary ensemble (CUE) and circular orthogonal ensemble (COE). In this work we present a method of generating random matrices characteristic of circular symplectic ensemble and show that obtained matrices conform to the predictions of random matrix theory (RMT).

2 Circular Unitary Ensemble

Circular unitary ensemble is defined by Haar measure in the space of $N \times N$ unitary matrices $U(N)$, invariant under the group of unitary transformations [4]. In order to construct numerically a unitary matrix typical of CUE we apply the parameterization of Hurwitz [9] and use the appropriate generalized Euler angles. An arbitrary unitary transformation U can be composed from elementary unitary transformations in two-dimensional subspaces. The matrix of such an elementary unitary transformation will be denoted by $E^{(i,j)}(\phi, \psi, \chi)$. The only nonzero elements of $E^{(i,j)}$ are

$$\begin{array}{ll} E^{(i,j)}_{kk} = 1 \ , \quad k = 1, \dots, N; k \neq i,j & \\ E^{(i,j)}_{ii} = \cos\phi e^{i\psi} \ , & E^{(i,j)}_{ij} = \sin\phi e^{i\chi} \ , \\ E^{(i,j)}_{ji} = -\sin\phi e^{-i\chi} \ , & E^{(i,j)}_{jj} = \cos\phi e^{-i\psi} \ , \end{array} \tag{1}$$

From the above elementary unitary transformations one constructs the following $N-1$ composite rotations

$$\begin{aligned} E_1 &= E^{(1,2)}(\phi_{12}, \psi_{12}, \chi_{12}) \ , \\ E_2 &= E^{(2,3)}(\phi_{23}, \psi_{23}, 0) E^{(1,3)}(\phi_{13}, \psi_{13}, \chi_{13}) \ , \\ E_3 &= E^{(3,4)}(\phi_{34}, \psi_{34}, 0) E^{(2,4)}(\phi_{24}, \psi_{24}, 0) E^{(1,4)}(\phi_{14}, \psi_{14}, \chi_{14}) \\ &\dots \\ E_{N-1} &= E^{(N-1,N)}(\phi_{N-1,N}, \psi_{N-1,N}, 0) E^{(N-2,N)}(\phi_{N-2,N}, \psi_{N-2,N}, 0) \\ &\quad \dots E^{(1,N)}(\phi_{1N}, \psi_{1N}, \chi_{1N}) \end{aligned} \tag{2}$$

and eventually forms the unitary transformation U as

$$U = e^{i\alpha} E_1 E_2 E_3 \dots E_{N-1}. \tag{3}$$

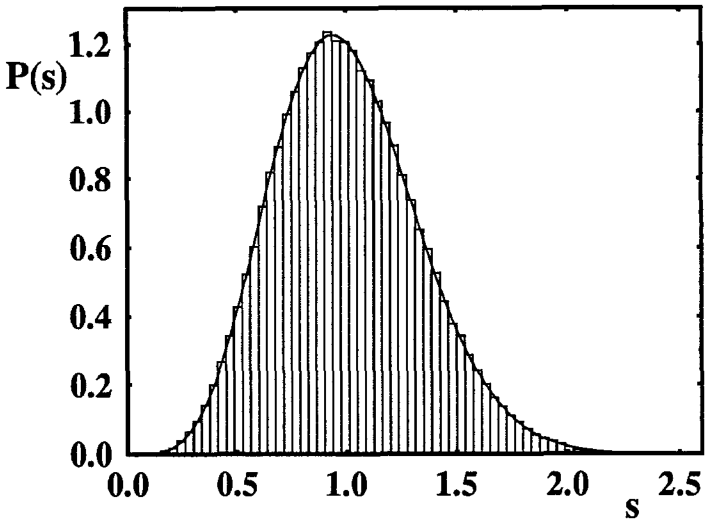

Fig. 1. Nearest neighbours distribution for 5000 self-dual unitary matrices of size $2N = 100$ typical to CSE.

If for a large matrix size N the angles $\alpha, \phi_{rs}, \psi_{rs}$, and χ_{1s} are taken uniformly from the intervals

$$0 \leq \psi_{rs} < 2\pi \ , \quad 0 \leq \chi_{1s} < 2\pi \ , \quad 0 \leq \alpha < 2\pi \ , \tag{4}$$

whereas

$$\phi_{rs} = \arcsin(\xi_{rs}^{1/2r}), \quad r = 1, 2, \ldots, N-1 \ , \tag{5}$$

with ξ_{rs} uniformly distributed in

$$0 \leq \xi_{rs} < 1 \ , \tag{6}$$

then the obtained matrix displays statistical properties characteristic to CUE [10].

3 Circular Symplectic Ensemble

A unitary matrix $S = (S^{\dagger})^{-1}$ of the size $2N$ is called *self-dual* if

$$S^R = -ZS^TZ = S \ , \tag{7}$$

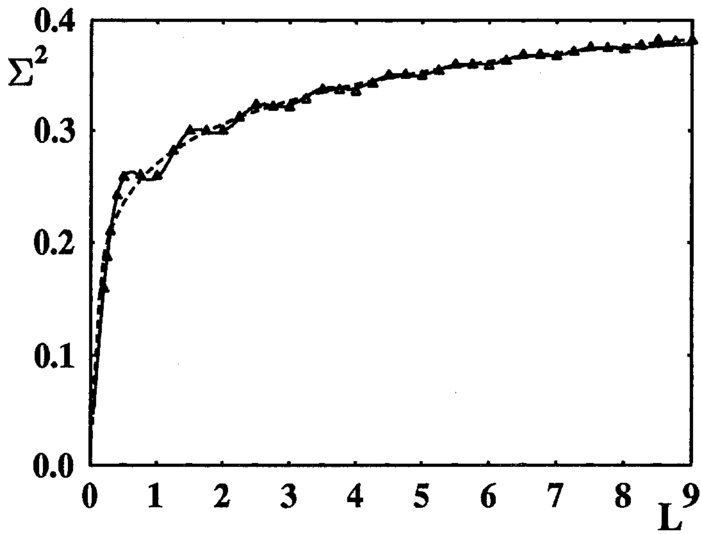

Fig. 2. Number variance $\Sigma^2(L)$ for unitary self-dual matrices ($\triangle$). Solid line stands for the exact result for CSE, while dotted line represents a large L approximation.

where the antisymmetric unitary matrix Z is

$$Z = \begin{pmatrix} 0 & 1 & 0 & 0 \cdots & 0 & 0 \\ -1 & 0 & 0 & 0 \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 \cdots & 0 & 0 \\ 0 & 0 & -1 & 0 \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \ddots & 0 & 0 \\ 0 & 0 & 0 & 0 \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 \cdots & -1 & 0 \end{pmatrix} . \tag{8}$$

For each self dual unitary matrix S there exists a quaternion real (symplectic) matrix B of size N, such that $B^R B = 1$ and $S = B^{-1}EB$, and the diagonal matrix E consists of N complex numbers $[\exp(i\theta_j)]$ on the unit circle, each repeated twice.

The circular symplectic ensemble is uniquely defined in the space of self–dual quaternion matrices by the property of being invariant under every transformation $S \to B^R S B$, where B is any $N \times N$ unitary quaternion matrix [4]. Every self dual unitary matrix can be written as

$$S = U^R U \ , \tag{9}$$

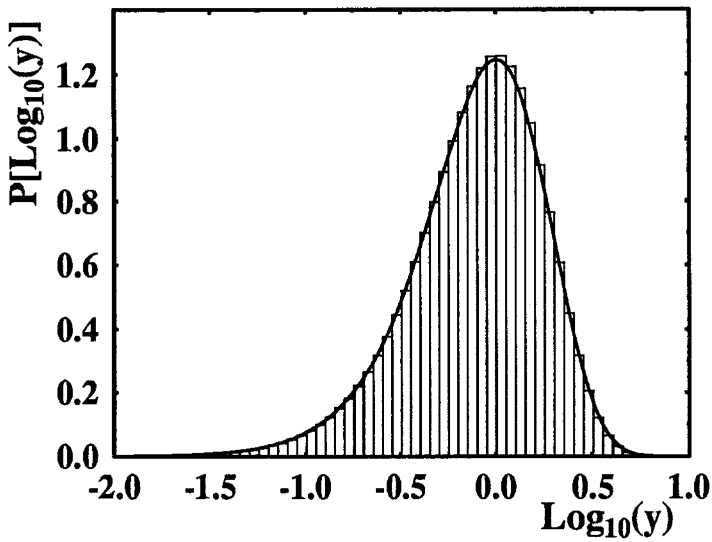

Fig. 3. Eigenvector statistics $P[\log_{10}(y)]$ for 5000 unitary matrices of symplectic ensemble and the corresponding $\chi^2_{\nu=4}$ distribution.

where U is unitary [3]. Note that the matrix U is determined to the extend of transformation $U \to UB$, it is the right multiplication by an arbitrary symplectic matrix B. This property defines the measure of the circular symplectic ensemble. Therefore, if matrices U are taken according to the Haar measure, the self dual matrix S are drawn uniformly with respect to the measure of CSE.

In order to obtain a CSE matrix it is sufficient to generate a $2N \times 2N$ unitary matrix according to equations (1 - 6) and to compute $S = U^R U$. To demonstrate practical importance of this algorithm we have numerically generated several CSE unitary matrices and analyzed the statistical properties of spectrum and eigenvectors. Fig. 1 presents the nearest neighbour distribution $P(s)$ obtained for 5000 matrices of size $2N = 100$. Due to Kramers degeneracy each matrix provides 50 data to the histogram. Note a good coincidence with the approximate Wigner formula [3]

$$P_S(s) = (\frac{64}{9\pi})^3 s^4 \exp\left[-\frac{64s^2}{9\pi}\right] \tag{10}$$

represented by a solid line.

To study the long range correlations of the spectrum we computed the average number of levels $\langle N_S(L)\rangle$ in an interval of the length L, and the number variance $\Sigma^2(L) = \langle N_S^2(L)\rangle - \langle N_S(L)\rangle^2$. Numerical results obtained from the same ensemble of 5000 matrices typical to CSE are represented by triangles in Fig. 2. Often used large L approximation

$$\Sigma^2_{app}(L) = \frac{1}{2\pi^2}\left(\ln(4\pi L) + 1 + \gamma + \pi^2/8\right), \tag{11}$$

is represented by a dotted line, while solid line stands for an exact, albeit more complicated result for number variance [11]

$$\begin{aligned}\Sigma^2_S(L) = \frac{1}{2\pi^2}&\left[\ln(4\pi L) + 1 + \gamma - \cos(4\pi L) - \mathrm{Ci}(4\pi L)\right] \\ &+L\left(1 - \frac{2}{\pi}\mathrm{Si}(4\pi L)\right) + \left(\frac{\mathrm{Si}(2\pi L)}{2\pi}\right)^2 .\end{aligned} \tag{12}$$

Above formula contains integral sine $Si(x)$ and cosine $Ci(x)$ functions and the Euler gamma constant $\gamma \approx 0.577...$. Note that the numerical results are faithful to the predictions of symplectic ensemble to a degree which requires usage of the exact formula (12) with small L corrections.

Diagonalizing self-dual unitary matrices we obtained not only theirs eigenvalues, but also the eigenvectors $|\varphi_l\rangle$, $l = 1, \ldots, N$; each represented in the initial basis by N real quaternions

$$\eta_{\mathbf{lk}} = c^0_{lk}\tau_0 + c^1_{lk}\tau_1 + c^2_{lk}\tau_2 + c^3_{lk}\tau_3 \ , \tag{13}$$

where τ_0 is the 2×2 unit matrix and $\tau_i, i = 1,2,3$ are Pauli matrices. Localization properties of the eigenvectors depend on their components $y_{lk} = \sum_{j=0}^{3}|c^j_{lk}|^2$. Statistical properties of eigenvectors determine the distribution of eigenvector components $P(y)$. It is known [12, 13] that for Gaussian or circular ensembles eigenvector statistics tends in the limit of large N to the χ^2_ν distribution

$$P_\nu(y) = \frac{(\nu/2)^{(\nu/2)}}{\Gamma(\nu/2) < y >}\left(\frac{y}{< y >}\right)^{\nu/2-1} \exp\left[-\frac{\nu y}{2 < y >}\right] , \tag{14}$$

where the number of degrees of freedom ν equals to 1 for the orthogonal, 2 for the unitary and 4 for the symplectic ensemble. Since this distribution is peaked around zero it is convenient to use the logarithmic scale and study $P[\log(y)]$. Figure 3 shows the eigenvector distribution of CSE matrices with the mean value $< y >$ normalized to unity. Observe fine agreement with the distribution (14) (with $\nu = 4$) represented in the figure by a solid line.

We have presented a simple algorithm allowing one to generate a random unitary matrix typical to circular symplectic ensemble. Such matrices can be used to describe statistical properties of quantum dynamical systems with a half–integer spin or to test numerically various hypothesis concerning CSE, which may hardly be verified by means of analytical calculations.

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