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Relatively Random Unitary Operators

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Consider unitary operators $\hat{U}_1$ and $\hat{U}_2$ represented by matrices U_1 and U_2 of size N. We shall call operators $\hat{U}_1$ and $\hat{U}_2$ *relatively random*, if

$$\mu = \frac{\mathrm{Re}[\mathrm{Tr}(U_1^\dagger U_2^\dagger U_1 U_2)]}{N} \ll 1. \tag{1}$$

Let $\hat{U}_1' = \hat{U}_2^\dagger \hat{U}_1 \hat{U}_2$ denotes the image of the operator $\hat{U}_1$ transformed by the unitary operation $\hat{U}_2$. It is easy to see that $\mu = \mathrm{Re}[\langle \hat{U}_1' | \hat{U}_1 \rangle]/N$, where $\langle \hat{A} | \hat{B} \rangle = \mathrm{Tr}(A^\dagger B)$ is the scalar product in the space of operators. In other words, an operator $\hat{U}_2$ is relatively random with respect to $\hat{U}_1$, if $\hat{U}_1$ is orthogonal to its image $\hat{U}_1'$. Moreover, the coefficient μ might be used as a measure of commutativity between $\hat{U}_1$ and $\hat{U}_2$, since the norm of commutator reads

$$||[\hat{U}_1, \hat{U}_2]||^2 = 2N(1-\mu), \tag{2}$$

with the norm $||\hat{A}||^2 = \langle \hat{A} | \hat{A} \rangle$. Relatively random operators do not commute and their eigenbasis are sufficiently different.

The concept of relatively random operators might be used for analysis of quantized chaotic systems. It is well known [1,2] that the statistical properties of quantum chaotic systems are described by ensembles of random matrices [3]. Level spacing distribution which characterizes spectrum of quantum system possessing a generalized time–reversal symmetry is described by the Wigner distribution. Furthermore, according to the theory of random matrices the distribution of components of eigenvectors $y_{ln} = |\langle \psi_l | n \rangle|^2, l = 1, \ldots, N$ of a unitary Floquet operator $\hat{F}$ (or a hermitian Hamiltonian) represented in a suitable basis $|n\rangle, n = 1, \ldots, N$ is given by the Porter–Thomas distribution [4]. We call such a basis *relatively random* with respect to the operator $\hat{F}$.

The distribution of eigenvector components is closely related to the statistics of matrix elements [5,6] of an observable represented in the eigenbasis of the Hamiltonian or the Floquet operator $\hat{F}$. It has been suggested [7] that the statistics of matrix elements of an Hermitian operator $\hat{A}$ is given by the Porter–Thomas distribution, if $\hat{A}$ is relatively random with respect to $\hat{F}$. In this work we conjecture that the statistics of components of eigenvectors of a unitary operator $\hat{F}_1$ describing a chaotic quantum system and represented in the eigenbasis of $\hat{F}_2$, complies to the predictions of random matrices, provided both operators are *relatively random.*

Above mentioned conjecture is supported by a numerical study of the periodically kicked top - a quantum system allowing for chaotic motion [8-10]. Dynamical variables of the system are three components $\hat{J}_l,\ l = 1, 2, 3$ of the angular momentum operator $\hat{J}$. They obey the commutation relation $[\hat{J}_k, \hat{J}_l] = i\epsilon_{kln}\hat{J}_n$. Time evolution of the system is governed by by the Floquet operator

$$\hat{F}(K,p) = \exp(\frac{-iK\hat{J}_x^2}{2j})\exp(-ip\hat{J}_z), \tag{3}$$

where p and K are the parameters of the model. The eigenvalue $j(j+1)$ of the operator $\hat{J}^2$ fixes the dimension of the Hilbert space N as $N = 2j + 1$. It is convenient to analyze the system in the eigenbasis of the operator $\hat{J}_z$, $|j,m\rangle,\ m = -j, \dots, j$.

The perturbation operator $\hat{V}$, quadratic in $\hat{J}_x$, does not couple states $|j,m\rangle$ of different parity and the matrix F breaks down into a block diagonal form of size j and $j+1$. Both subspaces are dynamically independent and numerical calculations can be performed separately for each parity. It has been reported [8] that for $p = 1.4$ and the kicking strength $K > 6$ the classical motion is chaotic and the statistical properties of the Floquet operator $\hat{F}$ corresponding to the quantum model can be described by circular orthogonal ensemble (COE) [3].

We are interested in the statistics of eigenvectors of $\hat{F}_1 = \hat{F}(K_1, p_1)$ represented a given orthonormal basis. This basis can be defined as the eigenbasis of a reference operator $\hat{F}_2 = \hat{F}(K_2, p_2)$. In the standard approach to eigenvector statistics one uses the basis of the unperturbed system [11,12], what corresponds to putting $K_2 = 0$ and $p_2 = p_1$. On the other hand, if $K_2 = K_1$ and $p_2 = p_1$, both operators are equal, the statistics of eigenvectors of $\hat{F}_1$ in its eigenbasis is singular and does not contain any information. We put $p_2 = p_1$ and consider arbitrary values of the parameter K_2 determining reference operator $\hat{F}_2$ and study, how large values of the "rotation parameter" $\Delta = k_2 - k_1$ produces COE–like eigenvector statistics described by Porter–Thomas distribution.

Eigenvector statistics may be characterized by the mean entropy of eigenvectors $\langle H\rangle$ [13]

$$\langle H\rangle = -\frac{1}{N}\sum_{l=1}^{N}\sum_{n=1}^{N} y_{ln}\ln(y_{ln}). \tag{4}$$

This quantity varies from zero for totally localized eigenvectors (one component equal to unity and all others to zero) to $\ln(N)$ for a delocalized eigenvector with all components equal to $1/N$. For random matrices representing a member of the orthogonal ensemble the mean entropy can be found analytically [14] and expressed by means of the Digamma Function Ψ [15]

$$H_{OE} = \Psi(\frac{N+2}{2}) - \Psi(\frac{3}{2}). \tag{5}$$

For convenience we use the scaled entropy $\gamma := \langle H\rangle/H_{OE}$ which is equal to unity for matrices pertaining to the orthogonal ensemble.

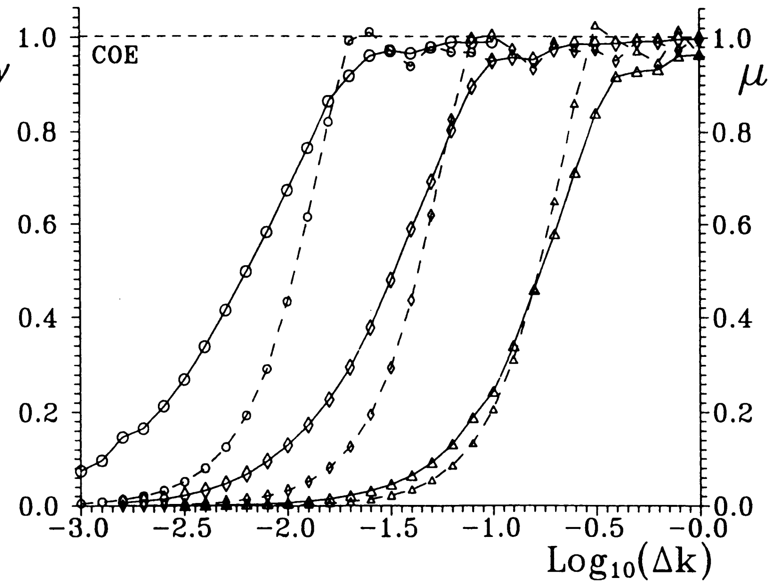

Figure 1. Dependence of scaled entropy of eigenvectors γ on rotation parameter Δ_K for $K = 11.0$, $p = 1.4$ and $j = 400(\circ)$, $j = 100(\diamond)$, $j = 25(\triangle)$. Numerical data are joined by solid lines. Corresponding smaller symbols, connected by dashed lines, represent values of the coefficient μ.

We diagonalized numerically unitary matrices F_1 for values of parameters $p = 1.4$ and $K = 11.0$ corresponding to classically chaotic motion. Obtained eigenvectors where projected onto eigenbasis of $F_2 = F(K + \Delta_K, p)$ and the distribution of eigenvectors was described by the scaled entropy γ. Figure 1 presents the entropy γ as a function of the "rotation parameter" Δ_K (in a logarithmic scale) for $j = 25, 100$ and 400. For small values of Δ_K the reference basis of $\hat{F}_2$ is so close to the eigenbasis of $\hat{F}_1$ that the entropy is negligible. For Δ_K larger than a critical value Δ_c the eigenbasis of F_2 produces eigenvector statistics typical to the orthogonal ensemble and γ achieves unity. Critical value Δ_c is proportional to $1/j$: for larger matrix a smaller value of the rotation parameter Δ_K is sufficient to generate a random basis.

Smaller symbols joined by dashed lines in Figure 1 denote the coefficient μ computed according to equation (1). There are no reasons to expect that for a given spin length j the values of γ and μ would be equal. However, the sudden growth of the coefficient μ coincides, for any j, with the critical value of the rotation parameter Δ_c, for which the scaled entropy tends to unity. Relatively random operator $\hat{F}_2$ generates thus eigenbasis random with respect to the Floquet operator $\hat{F}_1$. Condition (1) might be therefore considered as a simple criterion allowing to select a *random basis*, in which the eigenvector statistics complies with the predictions of ensembles of random matrices.

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