J. Phys. A: Math. Gen. **31** (1998) 9095–9104. Printed in the UK PII: S0305-4470(98)93137-7

The Monge distance between quantum states

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Received 7 April 1998

Abstract. We define a metric in the space of quantum states taking the Monge distance between corresponding Husimi distributions (Q-functions). This quantity fulfils the axioms of a metric and satisfies the following *semiclassical property*: the distance between two coherent states is equal to the Euclidean distance between corresponding points in the classical phase space. We compute analytically distances between certain states (coherent, squeezed, Fock and thermal) and discuss a scheme for numerical computation of Monge distance for two arbitrary quantum states.

1. Introduction

The state space of an n-dimensional quantum system is the set of all $n \times n$ positive semidefinite complex matrices of trace 1 called *density matrices*. The density matrices of rank one (*pure states*) can be identified with nonzero vectors in a complex Hilbert space of dimension n. However, one has to take into account that the same state is described by a vector ψ and $\lambda\psi$, where $\lambda \neq 0$. Hence, pure states are in one-to-one correspondence with rays $\{\lambda\psi : 0 \neq \lambda \in \mathbb{C}\}$. The rays form a smooth manifold called a complex projective space $\mathbb{CP}^{n-1}$. In the infinite-dimensional case we have to consider density operators instead of density matrices, and the space of pure states is the complex projective space over the infinite-dimensional Hilbert space.

The problem of measuring a distance between two quantum states with a suitable metric has attracted much attention in recent years. The Hilbert–Schmidt norm of an operator $\|A\|_2 = \sqrt{\mathrm{Tr}(A^\dagger A)}$ induces a natural distance between two density operators $d_{\mathrm{HS}}(\rho_1, \rho_2) = \sqrt{\mathrm{Tr}[(\rho_1 - \rho_2)^2]}$. This distance has recently been used to describe the dynamics of the field in Jaynes–Cummings model [1] and to characterize the distance between certain states used in quantum optics [2]. Another distance generated by the *trace norm* $\|A\|_1 = \mathrm{Tr}\sqrt{A^\dagger A}$ was used by Hillery [3, 4] to measure the nonclassical properties of quantum states.

A concept of *statistical distance* was introduced by Wootters [5] in the context of measurements which optimally resolve neighbouring pure quantum states. This distance, leading to the Fubini–Study metric in the complex projective space, was later generalized to density matrices by Braunstein and Caves [6] and its dynamics for a two-state system was studied by Braunstein and Milburn [7]. It was shown [6] that for neighbouring

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density matrices the statistical distance is proportional to the distance introduced by Bures in the late 1960s [8]. The latter was studied by Uhlmann [9] and Hübner [10], who found an explicit formula for the Bures distance between two density operators $d_B^2(\rho_1, \rho_2) = 2(1 - \mathrm{tr}[(\rho_1^{1/2}\rho_2\rho_1^{1/2})^{1/2}])$. Note that various Riemannian metrics on the spaces of quantum states were also considered by many other authors (see [11–15]).

In this paper we introduce yet another distance in the space of density operators, which fulfils the following *semiclassical property*: the distance between two coherent states $|\alpha_1\rangle$ and $|\alpha_2\rangle$ localized at points a_1 and a_2 of the classical phase space Ω endowed with a metric d is equal to the distance of these points

$$D(|\alpha\rangle_1\rangle, |\alpha_2\rangle) = d(a_1, a_2). \tag{1.1}$$

This condition is quite natural in the semiclassical regime, where the quasiprobability distribution of a quantum state tends to be strongly localized in the vicinity of the corresponding classical point. A motivation to study such a distance stems from the search for a quantum Lyapunov exponent, where a link between distances in Hilbert space and in the classical phase space is required [16].

In order to find a metric satisfying condition (1.1) it is convenient to represent a quantum state ρ in the Q-representation (also called the Husimi function) [17]

$$H_\rho(\alpha) := \langle\alpha|\rho|\alpha\rangle \tag{1.2}$$

defined with the help of a family of coherent states $|\alpha\rangle, \alpha \in \Omega$, which fulfil the identity resolution $\boldsymbol{I} = \int_\Omega |\alpha\rangle\langle\alpha| \, \mathrm{d}m(\alpha)$, where m is the natural measure on Ω. For a pure state $|\psi\rangle$ one has $H_\psi(\alpha) = |\langle\psi|\alpha\rangle|^2$. The choice of coherent states is somewhat arbitrary and in fact one may work with different systems of coherent states [18]. In this paper we use the classical *harmonic oscillator (field) coherent states*, where $\Omega = \mathbb{C}$ and $\mathrm{d}m(\alpha) = \mathrm{d}^2\alpha/\pi$. For convenience we shall use the renormalized version of the Husimi function putting $H(\alpha) = \langle\alpha|\rho|\alpha\rangle/\pi$ and integrating over $\mathrm{d}^2\alpha$. Note that the Husimi representation of a given density operator determines it uniquely [19]. Since Husimi distributions are non-negative and normalized, i.e. $\int_\Omega H(\alpha)\,\mathrm{d}m(\alpha) = 1$ and $H \geqslant 0$, it follows that a metric in the space of probability densities $Q : \Omega \to R_+$ induces a metric in the state space.

In this work we propose to measure the distance between density operators by the Monge distance between the corresponding Husimi distributions. The original Monge problem consists of finding an optimal way of moving a pile of sand to a new location. The Monge distance between two piles is given by the 'path' travelled by all grains under the optimal transformation [20, 21].

This paper is organized as follows. In section 2 we give a definition of the Monge metric, present an explicit solution for the one-dimensional (1D) case and discuss some methods of tackling the two-dimensional (2D) problem. Section 3 contains some examples of computing the Monge distance between certain states often encountered in quantum optics. Concluding remarks are given in section 4. A variational approach to the Monge problem is briefly presented in the appendix.

2. Monge metric

2.1. Monge transport problem

The original Monge problem, formulated in 1781 [22], emerged from studying the most efficient way of transporting soil [20]:

Split two equally large volumes into infinitely small particles and then associate them with each other so that the sum of products of these paths of the particles to the volume is least.

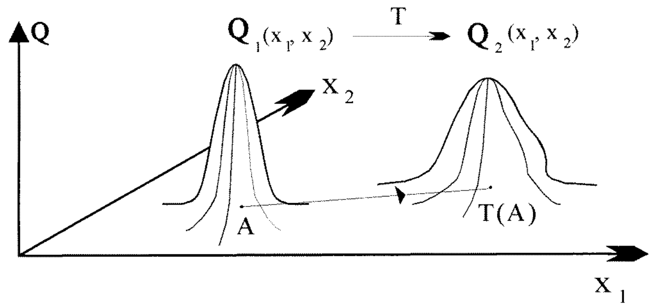

Figure 1. Monge transport problem: how to move a pile of sand $Q_1(x_1, x_2)$ to a new location $Q_2(x_1, x_2)$ minimizing the work done?

Along what paths must the particles be transported and what is the smallest transportation cost?

Figure 1 represents a scheme for this problem. Here we give a general definition of the Monge distance between two smooth probability densities Q_1 and Q_2 defined in $S = R^n$. Let Ω_1 and Ω_2, determined by Q_i, describe the initial and final location of 'sand': $\Omega_i = \{(x, y) \in S \times R^+ : 0 \leqslant y \leqslant Q_i(x)\}$. Due to the normalization of Q_i the integral $\int_{\Omega_i} \mathrm{d}^n x\, \mathrm{d}y$ is equal to unity. Consider that C^1 maps $T : S \to S$ which generate volume-preserving transformations Ω_1 into Ω_2, i.e.

$$Q_1(x) = Q_2(Tx)|T'(x)| \tag{2.1}$$

for all $x \in S$, where $T'(x)$ denotes the Jacobian of the map T at point x. We shall look for a transformation giving the minimal displacement integral and define the Monge distance [20, 21]

$$D_M(Q_1, Q_2) := \inf \int_S |x - T(x)| Q_1(x)\, \mathrm{d}^n x \tag{2.2}$$

where the infimum is taken over all T as above. The optimal transformation (if it exists) T_M is called a *Monge plan*. Note that in this formulation of the problem the 'vertical' component of the sand movement is neglected.

It is easy to see that Monge distance fulfils all the axioms of a metric. This allows us to define a 'classical' distance between two quantum states ρ_1 and ρ_2 via the Monge distance between the corresponding Husimi distributions:

$$D_{cl}(\rho_1, \rho_2) := D_M(H_1(\alpha), H_2(\alpha)). \tag{2.3}$$

The Monge distance satisfies the semiclassical property: it is shown below that the distance between two coherent states, represented by Gaussian Husimi distributions localized at points a_1 and a_2, equals to the classical distance $|a_1 - a_2|$.

It is sometimes useful to generalize the notion of the Monge metric and to define a family of distances D_{M_p} labelled by a continuous index p $(0 < p \leqslant \infty)$ in an analogous way:

$$[D_{M_p}(Q_1, Q_2)]^p := \inf \int_S |x - T(x)|^p Q_1(x)\, \mathrm{d}^m x. \tag{2.4}$$

For $p = 1$ one recovers the Monge distance $D_{M_1} = D_M$, while the Fréchet distance D_{M_2} is obtained for $p = 2$. A more general approach to the transport problem was proposed by Kantorovitch [23] and further developed by Sudakov [24]. In contrast to the definition of

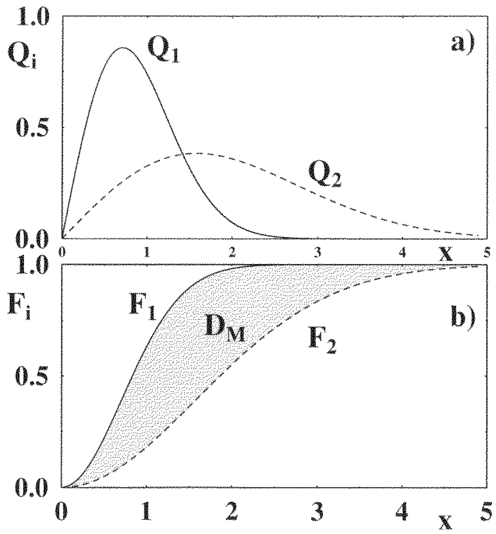

Figure 2. The Monge distance between 1D functions $Q_1(x)$ and $Q_2(x)$ may be represented as the area between graphs of the corresponding distribution functions $F_1(x)$ and $F_2(x)$

Monge discussed in this work, the Kantorovitch distance between Q_1 and Q_2 is explicitly symmetric with respect to exchange of both distributions. For a comprehensive review of metrics in the space of probability measures and other generalizations of the Monge distance see the monograph of Rachev [21].

2.2. Salvemini solution for the 1D problem

For $S = R$ the Monge distance can be expressed explicitly with the help of distribution functions $F_i(x) = \int_{-\infty}^{x} Q_i(t)\,\mathrm{d}t$, $i = 1, 2$. Salvemini [25] and Dall'Aglio [26] obtained the following solution of the problem for $p = 1$.

$$D_M(Q_1, Q_2) = \int_{-\infty}^{+\infty} |F_1(x) - F_2(x)|\,\mathrm{d}x \tag{2.5}$$

represented schematically in figure 2.

This result was generalized in the 1950s to all $p \geqslant 1$ by several authors (see [20, 21]). We have

$$[D_{M_p}(Q_1, Q_2)]^p = \int_0^1 |F_1^{-1}(t) - F_2^{-1}(t)|^p\,\mathrm{d}t. \tag{2.6}$$

2.3. Poisson–Ampere–Monge equation for the 2D case

Consider smooth densities $Q_1, Q_2 : R^2 \to R^+$. We are looking for a transformation field $w = (w_1, w_2) : R^2 \to R^2$ fulfilling $w(x_1, x_2) = T_M(x_1, x_2) - (x_1, x_2)$, where T_M is an optimal Monge transformation minimizing the right-hand side of (2.4). Restricting ourselves to smooth transformations T we may apply the standard variational search for the optimal field w. In the appendix it is shown that $\mathrm{rot}(w) = 0$ if $p = 2$, and the potential $\varphi : w = \mathrm{grad}(\varphi)$ satisfies the following partial differential equation

$$\varphi_{x_1x_1} + \varphi_{x_2x_2} + \varphi_{x_1x_1}\varphi_{x_2x_2} - (\varphi_{x_1x_2})^2 = \frac{Q_1(x_1, x_2)}{Q_2(x_1 + \varphi_{x_1}, x_2 + \varphi_{x_2})} - 1. \tag{2.7}$$

Solving this *Laplace–Ampere–Monge (LAM) equation* for the potential φ we get the Fréchet distance D_{M_2} from (2.4) computing the minimal displacement

$$(D_{M_2}(Q_1, Q_2))^2 = \int_{-\infty}^{+\infty} \mathrm{d}x_1 \int_{-\infty}^{+\infty} \mathrm{d}x_2\, (\varphi_{x_1}^2 + \varphi_{x_2}^2) Q_1(x_1, x_2). \tag{2.8}$$

Although it is hardly possible to solve equation (2.7) in the general case, it might be used to check whether a given transformation can be a solution of the Monge problem. Let us remark that an optimal transformation field w (if it exists) need not be unique. Moreover, (2.7) provides only a sufficient condition for w being optimal. It is important to note that the 2D Monge problem posed 200 years ago has not been solved in the general case [20, 21].

2.4. Estimation of the Monge distance via the transport problem

One of the major tasks of linear programming is the optimization of the following transport problem. Consider N suppliers producing $a_i (i = 1, \ldots, N)$ pieces of a product per time period and M customers requiring $b_j (j = 1, \ldots, M)$ pieces of the product at the same time. Let (c_{ij}) be a $N \times M$ cost matrix, representing for example the distances between sites. Find the optimal transporting scheme, minimizing the total transport costs $C = \sum_{i=1}^{N} \sum_{j=1}^{M} c_{ij} x_{ij}$. The non-negative elements of the unknown matrix (x_{ij}) denote the number of products moved from the ith supplier to the jth customer. The optimization problem is subjected to the following constraints: $\sum_{j=1}^{M} x_{ij} = a_i$ and $\sum_{i=1}^{N} x_{ij} = b_j$. In the simplest case the total supply equals the total demand and $\sum_{i=1}^{N} a_i = \sum_{j=1}^{M} b_j$.

The *transport problem* described above gave, with a related *assignment problem*, an impulse to develop methods of linear programming more than 50 years ago [23, 27]. Since then several algorithms for solving the transport problem numerically have been proposed. Some of them, as the *northwest corner procedure* and *Vogel's approximation* [28], are available in specialized software packages. It is worth adding that the transport methods are widely used to solve a variety of problems in business and economy such as, for instance, market distribution, production planning, plant location and scheduling problems.

It is easy to see that the transport problem is a discretized version of the Monge problem defined for continuous distribution functions. One can, therefore, approximate the Monge distance between two distributions $Q_1(x)$ and $Q_2(x)$ (where x stands for 2D vector), by solving the transport problem for delta peaks approximation of the continuous distributions: $q_1 = \sum_{i=1}^{N} Q_1(x_i)\delta(x - x_i)$ and $q_2 = \sum_{j=1}^{M} Q_2(x_j)\delta(x - x_j)$. The quality of this approximation depends on the numbers N and M of peaks representing the initial and final distribution, respectively, and also on their location with respect to the shape of both distributions. The numerical study performed with the *northwest corner* algorithm for some analytically soluble examples of the 2D Monge problem shows [29] that for reasonably smooth distributions one obtains Monge distance with fair accuracy for a number of peaks of the order of hundreds.

3. Monge distance between some states of quantum optics

In this section we compute Monge distances between certain quantum states. Even though our considerations are valid in the general framework of quantum mechanics, for the sake of concreteness we will use the language of quantum optics. Let us recall briefly the necessary definitions and properties.

Let a and $a^\dagger$ be the annihilation and creation operators satisfying the commutation relation $[a, a^\dagger] = 1$. The '*vacuum*' state $|0\rangle$ is distinguished by the relation $a|0\rangle = 0$.

Standard harmonic-oscillator *coherent states* $|\alpha\rangle$ can be defined as the eigenstates of the annihilation operator $a|\alpha\rangle = \alpha|\alpha\rangle$ or by the Glauber translation operator $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ as $|\alpha\rangle = D(\alpha)|0\rangle$. Coherent states, determined by an arbitrary complex number $\alpha = x_1 + \mathrm{i}x_2$, enjoy a minimum uncertainty property. They are not orthogonal and do overlap. The Husimi distribution of a coherent state $|\beta\rangle$ is Gaussian

$$H_\beta(\alpha) = \frac{1}{\pi}|\langle\beta|\alpha\rangle|^2 = \frac{1}{\pi}\exp(-|\alpha-\beta|^2). \tag{3.1}$$

Squeezed states $|\gamma,\alpha\rangle$ also minimize the uncertainty relation, however, the variances of both canonically coupled variables are not equal. They are defined as

$$|\gamma,\alpha\rangle := D(\alpha)S(\gamma)|0\rangle \tag{3.2}$$

where the squeezing operator is $S(\gamma) = \exp[\frac{1}{2}(\gamma^* a^2 - \gamma a^{\dagger 2})]$. The modulus g of the complex number $\gamma = g\mathrm{e}^{2\mathrm{i}\theta}$ determines the strength of squeezing, $s = \mathrm{e}^g - 1$, while the angle θ orients the squeezing axis. The Husimi distribution of a squeezed state $|\gamma,\beta\rangle$ is a nonsymmetric Gaussian localized at point β and for $\theta = 0$ reads

$$H_{\gamma,\beta}(x_1,x_2) = \frac{1}{\pi}\exp[-(\mathrm{Re}\,(\beta) - x_1)^2/(s+1)^2 - (\mathrm{Im}\,(\beta) - x_2)^2(s+1)^2]. \tag{3.3}$$

Each pure state can be expressed in the Fock basis consisting of n-photon states $|n\rangle$, $n = 0, 1, 2, \ldots$. Each coherent state can be expanded in the Fock basis as

$$|\alpha\rangle = \mathrm{e}^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{3.4}$$

The known scalar product $\langle\alpha|n\rangle$ allows one to write the Husimi function of a *Fock state*

$$H_{|n\rangle}(\alpha) = \frac{1}{\pi}\frac{|\alpha|^{2n}}{n!}\mathrm{e}^{-|\alpha|^2}. \tag{3.5}$$

The *vacuum state* $|0\rangle$ can be thus regarded as the single Fock state being simultaneously coherent.

In contrast to the above-mentioned pure states, the *thermal* mixture of states with the mean number of photon equal to $\bar{n}$ is represented by the density operator

$$\rho_{\bar{n}} = \sum_{n=0}^{\infty}\frac{\bar{n}^n}{(\bar{n}+1)^{n+1}}|n\rangle\langle n|. \tag{3.6}$$

Its Husimi distribution is the Gaussian centred at zero with the width depending on the mean photon number,

$$H_{\rho_{\bar{n}}}(\alpha) = \frac{1}{\pi(\bar{n}+1)}\exp\left(-\frac{|\alpha|^2}{\bar{n}+1}\right). \tag{3.7}$$

3.1. Two coherent states

Let us consider an arbitrary 2D distribution function $Q_1(x_1,x_2)$ and a shifted one $Q_2(x_1,x_2) = Q_1(x_1 - x_1^*, x_2 - x_2^*)$. It is intuitive to expect that the simple translation given by $w(x_1,x_2) = \text{constant} = (x_1^*, x_2^*)$ solves the corresponding Monge problem. Since for the respective potential $\phi(x_1,x_2) = x_1^* x_1 + x_2^* x_2$, the second derivatives vanish, then both sides of the LAM equation (2.7) are equal to zero and the maximization condition is fulfilled.

It follows from this observation that for two coherent states defined on the complex plane the Monge plan reduces to translation. The integration in (2.2) is trivial and provides the Monge distance between two arbitrary coherent states $|\alpha\rangle$ and $|\beta\rangle$

$$D_{cl}(|\alpha\rangle, |\beta\rangle) = |\alpha - \beta|. \tag{3.8}$$

This is exactly the *semiclassical property* we demanded from the metric in the state space. The distance of the coherent state $|\alpha\rangle$ from the vacuum state $|0\rangle$ is equal to $|\alpha|$, which is simply the square root of the mean number of photons in the state $|\alpha\rangle$. The classical property is fulfilled by the generalized Monge metric D_{M_p} for any positive parameter p.

3.2. Coherent and squeezed states

We compute the distance between a coherent state $|\alpha\rangle$ and the corresponding squeezed state $|\gamma, \alpha\rangle$. Due to the invariance of the Monge optimization with respect to translations, this distance is equal to the distance between vacuum $|0\rangle$ and the squeezed vacuum $|\gamma, 0\rangle$. For simplicity we will assume that squeezing is performed along the x_1-axis, i.e. the complex squeezing parameter is real $\gamma = g \in R$.

The corresponding Monge problem consists of finding the optimal transformation of the symmetric Gaussian $Q_1(x_1, x_2) = \exp(-x_1^2 - x_2^2)/\pi$ into an asymmetric one $Q_2(x_1, x_2) = \exp(-x_1^2/(s+1)^2 - x_2^2(s+1)^2/\pi$. Considering contours of the Husimi distribution, often used to represent a state in quantum optics, a circle has to be transformed into an ellipse. If $p = 2$ then the following affine transformation $T(x_1, x_2) = (x_1/(s+1), x_2(s+1))$ corresponds to the irrotational transport field $w(x_1, x_2) = (-sx_1/(s+1), sx_2)$. It can be obtained as the gradient of the potential $\varphi(x_1, x_2) = -sx_1^2/(2s+2) + sx_2^2/2$, for which both sides of the LAM equation (2.7) vanish. Hence, field w provides a Monge plan for this problem and the Fréchet distance D_{M_2} is given by

$$\begin{aligned}[D_{M_2}(|0\rangle, |g, 0\rangle)]^2 &= \frac{s^2}{\pi} \int_{-\infty}^{\infty} \mathrm{d}x_1 \int_{-\infty}^{\infty} \mathrm{d}x_2 \, \exp(-x_1^2 - x_2^2)(x_1^2/(s+1)^2 + x_2^2) \\ &= \frac{s^2}{2}\left(1 + \frac{1}{(s+1)^2}\right).\end{aligned} \tag{3.9}$$

3.3. Vacuum and thermal states

Since Husimi distributions of both states is rotationally invariant, it is convenient to use radial components of the distribution $R_i(r) = 2\pi r H_i(r, \phi)$. One may then reduce the problem to one dimension and find the Monge distance via radial distribution functions $F_i(r) = \int_0^r R_i(r')\,\mathrm{d}r'$. Taking Husimi distributions (3.1) and (3.7) we get the corresponding distribution functions $F_1(r) = 1 - \exp(-r^2)$ and $F_2(r) = 1 - \exp(-r^2/(\bar{n}+1))$. Using the Salvemini formula (2.5) we obtain

$$D_{cl}(|0\rangle\langle 0|, \rho_{\bar{n}}) = \int_0^{\infty} |F_1(r) - F_2(r)|\,\mathrm{d}r = \frac{\sqrt{\pi}}{2}(\sqrt{\bar{n}+1} - 1). \tag{3.10}$$

Applying formula (2.6) we get a formula for the generalized Monge distance between two thermal states

$$D_{M_p}(\rho_{\bar{n}_1}, \rho_{\bar{n}_2}) = \left(\Gamma\left(1 + \frac{p}{2}\right)\right)^{1/p} \sqrt{\bar{n}_1 + 1} - \sqrt{\bar{n}_2 + 1}|. \tag{3.11}$$

This agrees with an observation that $p_1 < p_2$ implies $D_{M_{p1}} \leqslant D_{M_{p2}}$ [21].

3.4. Two Fock states

As in the previous example, the rotational symmetry of the Fock states allows us to use the 1D formula. Integrating (3.5) for a Fock state $|n\rangle$ we can express the radial distribution function in terms of the incomplete Gamma function $\Gamma(x, r)$ as

$$F_n(r) = 1 - \frac{\Gamma(n+1, r^2)}{\Gamma(n+1)}. \tag{3.12}$$

Since for different Fock states the distribution functions do not cross, applying Salvemini formula (2.5) we obtain the Monge distance

$$D_{cl}(|m\rangle, |n\rangle) = \left| \int_0^\infty F_n(r)\,\mathrm{d}r - \int_0^\infty F_m(r)\,\mathrm{d}r \right| = \sqrt{\pi}|C_m - C_n| \tag{3.13}$$

where $C_0 = \frac{1}{2}$; $C_{k+1} - C_k = \binom{2k+1}{k}\frac{1}{4^{k+1}} \sim \frac{1}{2\sqrt{\pi k}}$ for $k = 0, 1, 2 \ldots$.

4. Discussion

We have presented a definition of distance between quantum states (i.e. elements of a Hilbert space) which possesses a certain classical property, natural for investigation of the semiclassical limit of quantum mechanics. The Monge distance between the corresponding Husimi functions fulfils the axioms of a metric and induces a 'classical' topology in the Hilbert space. It is worth emphasizing that the Monge distance between two given density matrices depends on the topology of the corresponding classical phase space.

Consider a quantum state prepared as a superposition of two coherent states separated in the phase space by x. It is known [30–32] that such a state interacting with an environment evolves quickly towards a mixture of the two localized states. The decoherence time decreases with the classical distance x, just equal to the Monge distance between both coherent states. We expect therefore that the Monge distance between two arbitrary quantum states might be useful to determine the rate with which the superposition of these two states suffers the decoherence.

It is possible to generalize our approach in several directions. Instead of the standard Husimi distributions used throughout this paper, one may study Monge distances between generalized Husimi distributions $\tilde{H}_\rho(\alpha) = \langle\tilde{\alpha}|\rho|\tilde{\alpha}\rangle$, where $|\tilde{\alpha}\rangle$ are generalized coherent states [33]. For example, one may use for this purpose squeezed states [34], or the spin coherent states [35, 36], if the classical space is the 2D sphere.

Moreover, our considerations based on the Husimi representation of quantum states, may be in fact extended to the Wigner function. Despite the fact that the Wigner function can take on negative values, it is normalized and the Monge problem of finding an optimal way to transport one Wigner function into another might also be considered. The concept of the classical distance between two Husimi (Wigner) functions is not only of theoretical interest, since novel methods of measuring Husimi and Wigner distributions have recently been developed [37–39].

Acknowledgments

We thank Harald Widemannn for fruitful cooperation and Fritz Haake for valuable remarks. KŻ is grateful to the Isaac Newton Institute for Mathematical Sciences in Cambridge, where a part of this work was done and KŻ also acknowledges the Fulbright Fellowship. Financial support by the Polish KBN grant P03B 060 13 is gratefully acknowledged.

Appendix. Variational approach to Monge problem

1D case

Let Q_1 and Q_2 be smooth densities. In the 1D case there is only one map T fulfilling (2.1). It can be described with the aid of distribution functions as $T(x) = F_2^{-1}(F_1(x))$, where $F_i(x) = \int_{-\infty}^{x} Q_i(t)\,\mathrm{d}t$ for $x \in R$. This allows us to express the generalized Monge distance as an integral (2.4)

$$[D_{M_p}(Q_1, Q_2)]^p := \int_{-\infty}^{\infty} Q_1(x)|F_2^{-1}[F_1(x)] - x|^p\,\mathrm{d}x. \qquad \text{(A.1)}$$

In the simplest case $p = 1$ (A1) reduces to the Salvemini formula (2.5) and for $p > 1$ to formula (2.6).

2D case

Consider two smooth densities $Q_1(x_1, x_2)$ and $Q_2(x_1, x_2)$. We are looking for a map $T(x_1, x_2) = (x_1 + w_1(x_1, x_2), x_2 + w_2(x_1, x_2))$ transforming Q_1 into Q_2 (i.e. such that (2.1) is fulfilled) and minimizing the quantity

$$I_p = \int_{-\infty}^{+\infty} Q_1(x_1, x_2)|w_1^2(x_1, x_2) + w_2^2(x_1, x_2)|^{p/2}\,\mathrm{d}x_1\,\mathrm{d}x_2. \qquad \text{(A.2)}$$

The index p, labelling the generalized distance, is equal to one for the Monge metric and to two for the Fréchet metric. Introducing a Lagrange factor λ we write the Lagrange function in the form

$$L_p = Q_1(w_1^2 + w_2^2)^{p/2} + \lambda(Q_1 - Q_2(T))[(1 + w_{1x_1})(1 + w_{2x_2}) - w_{1x_2}w_{2x_1}]. \qquad \text{(A.3)}$$

The Lagrange–Euler equations for variations of L_p allow us to obtain the partial derivatives of λ

$$\begin{aligned} \lambda_{x_1} &= 2pC_p(w_1(1 + w_{1x_1}) + w_2 w_{2x_1}) \\ \lambda_{x_2} &= 2pC_p(w_2(1 + w_{2x_2}) + w_1 w_{1x_2}) \end{aligned} \qquad \text{(A.4)}$$

where $C_p = (w_1^2 + w_2^2)^{(p-2)/2}$. Using the equality $\lambda_{x_1x_2} = \lambda_{x_2x_1}$ we get

$$w_{1x_2}(w_1^2(p-1) + w_2^2) - w_{2x_1}(w_2^2(p-1) + w_1^2) + (p-2)(w_{2x_2} - w_{1x_1})w_1w_2 = 0. \qquad \text{(A.5)}$$

If $p = 2$ we deduce from (A5) that $w_{1x_2} = w_{2x_1}$, i.e. $\mathrm{rot}(w) = 0$. Taking the potential $\varphi : w = \mathrm{grad}(\varphi)$ we see that φ fulfils (2.7) and formula (2.8) holds.

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