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Scaling laws of complex band random matrices

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Abstract. We study a model of complex band random matrices capable of describing the transitions between three different ensembles of Hermitian matrices: Gaussian orthogonal, Gaussian unitary and Poissonian. Analyzing numerical data we observe new scaling relations based on the generalized localization length of eigenvectors. We show that during transitions between canonical ensembles of random matrices the changes of statistical properties of eigenvalues and eigenvectors are correlated.

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1. Introduction

Energy levels of generic quantum systems, the classical analogues of which are integrable, tend to cluster and the nearest neighbour statistics $P(S)$ is given by Poisson distribution [1, 2]. This distribution is characteristic for a diagonal matrix build of random Gaussian numbers. On the other hand, if the classical dynamics behaves chaotically the levels seem to repel each other [2–4]. In this case the statistical properties of the spectra of quantum dynamical systems are well described by ensembles of random matrices [5–7]. Depending on the symmetry properties of the dynamical system one of the three universality classes should be applied: orthogonal, unitary or symplectic.

Variation of the perturbation strength or other external parameter of a given quantum system changes its dynamical properties and corresponds to the transition between different ensembles of random matrices. Since the matrix representation of Hamiltonians describing some quantum systems have a band structure [8,9], several ensembles of band random matrices were analyzed [8,10-12]. The simplest model of $N \times N$ real symmetric random matrices with the zero mean and the band width b was analyzed by Casati et al. [10,13]. The limiting cases of $b = 1$ and $b = N$ correspond to the Poisson and Gaussian Orthogonal Ensemble (GOE), respectively. It was shown by means of a numerical investigation that this model possesses the property of scaling: the level spacing distribution of eigenvalues [13], the mean localization length [10] and the eigenvector statistics [14] depend only on the scaling parameter $x = b^2/N$. Recently also some analytical results concerning the property of scaling [15] in band random matrices and the semicircle Wigner law of the level density distribution [16, 17] have been obtained.

The transition from regular to chaotic motion in a quantized dynamical system corresponds to an increase of the band width b, or rather the scaling parameter x, characterizing the band random matrix. On the other hand, the physical effect of the time reversal symmetry breaking in a dynamical system, corresponding to a transition from Gaussian orthogonal to Gaussian unitary ensemble (GUE), cannot be represented by real band random matrices.

The aim of this paper is to propose and analyze an ensemble of complex Hermitian band random matrices useful to represent dynamical systems with no generalized time reversal symmetry. The discussed ensemble is a simple generalization of the ensemble of real matrices [10] and is capable of describing the transitions GOE – GUE and Poisson – GUE. Apart from the band width b our model is parametrized by a variable c determining the relative size of the imaginary part of the matrix elements. In the case of the full random matrix ($b = N$) the parameter c is connected with the degree of level repulsion: one limiting case $c = 1$ corresponds to GOE, while GUE is obtained for $c = 2$.

The paper will be organized as follows. In the second section we describe our model and in the next one we review known results concerning the transition Poisson – GOE. In the subsequent sections the transitions Poisson – GUE and GOE – GUE are discussed and some new results are presented. The general case of the model, which covers all intermediate cases interpolating between three canonical ensembles, is described in Sect. VI. We examine the properties of eigenvalues of matrices (the spectrum) and of their eigenvectors (information entropy) and demonstrate the correlations between them. Defining the generalized localization length of eigenvectors we show the scaling laws of the model. The last section contains a summary and concluding remarks.

2. The model

We propose a generalization of the ensemble of band random matrices analyzed by Casati et al. [10]. The complex Hermitian band matrix $A_{ij} = A^{\dagger}_{ij}$ is defined by [18, 19]:

$$A_{ij} = (\xi_{ij} + i\zeta_{ij})\,\Theta(b - |i-j|), \quad i,j = 1, \ldots, N \tag{1}$$

where b is the bandwidth and $\Theta(.)$ denotes the unit step function vanishing at the origin. The symbols ξ_{ij} and ζ_{ij} represent statistically independent random variables distributed according to the Gaussian distribution with the zero mean and the variances $(\sigma^R_{ij})^2$ and $(\sigma^I_{ij})^2$, respectively. The parameter c measuring the effective size of the imaginary part of the off diagonal matrix elements is determined by the condition

$$c = \frac{(\sigma^R_{ij})^2 + (\sigma^I_{ij})^2}{(\sigma^R_{ij})^2}, \quad i \neq j. \tag{2}$$

In order to keep all eigenvalues in the constrained energy range the matrices are normalized as $\mathrm{Tr}(A^2) = N + 1$. This condition allows to express the variances of the real and the imaginary part of matrix elements by the parameters N, b and c

$$(\sigma^R_{ij})^2 = \frac{(N+1)}{2N + c(2N-b)\;(b-1)}(1 + \delta_{ij}) \tag{3}$$

and

$$(\sigma^I_{ij})^2 = \frac{(c-1)\;(N+1)}{2N + c(2N-b)(b-1)}(1 - \delta_{ij}) \tag{4}$$

A slightly different normalization previously used in [18,19] does not influences the statistical properties of the model studied. For a diagonal random matrix ($b = 1$) the density of eigenvalues is Gaussian and the level spacings are distributed according to the Poisson distribution, independently of the parameter c. This limiting case of the model studied will be called Poissonian. In the opposite limiting case of the full matrix ($b = N$) variations of the parameter c correspond to the process of the time reversal symmetry breaking in a dynamical system and control the transition between orthogonal and unitary ensembles of random matrices.

For $c = 1$ the variances of the imaginary part of the matrix elements σ^I vanish. Therfore a real symmetric matrix A belongs to GOE and displays the linear level repulsion. For larger values of the parameter c the matrix elements are complex but the variance of the real part is greater than the variance of the imaginary part; the probability distribution of the matrix elements on the complex plane is represented by an ellipse. An increase of the parameter c corresponds to the squeezing along the larger axis of this ellipse. In the limiting case of $c = 2$ the variances of the real and imaginary parts of the off diagonal elements are equal and the complex matrix A pertains to GUE and exhibits quadratic level repulsion. Thus in both limiting cases the value of the parameter c coincides with the degree of level repulsion.

The effect of the time reversal symmetry breaking in autonomous quantized chaotic system can be described by a model of the additive random matrices [20–22] defined by

$$A = n(A_{GOE} + \lambda A_{GUE}), \tag{5}$$

where the normalization constant $n = (1 + \lambda^2)^{1/2}$ was introduced in order to keep all eigenvalues on a finite support. The transition between orthogonal and unitary matrices of size N is governed by the scaling parameter $v = \lambda^2 N$ [20]. Putting $b = N$ into definition (3) and (4) of our model and varying the parameter c from 1 to 2 we obtain the additive model (5) with $\lambda = \sqrt{2(c-1)/(2-c)}$. The scaling parameter equals therefore

$$v = 2\,\frac{N(c-1)}{2-c}. \tag{6}$$

Model discussed in this work may be thus considered as a generalization of the model of real band random matrices describing the transition Poisson–GOE, and simultaneously, as a generalization of the additive model representing the transition GOE–GUE.

3. Transition Poisson – GOE

The transition Poisson – GOE is obtained in the model discussed by varying the band width b and keeping the constant c equal to unity. Since different features of this transition were analyzed in the work [10] of Casati et al. and further discussed in other recent papers [13, 14, 23, 24], we will give in this section a brief survey of the results obtained. The level spacing distribution for band random matrices $P(s)$ might be approximated by the Brody distribution [6] or by a phenomenological formula proposed recently by Izrailev [25]

$$P_\beta(s) = As^\beta(1 + sB\beta)^{f(\beta)}\,\exp\left[\frac{-\pi^2\beta s^2}{16} - \frac{\pi}{4}(2-\beta)s\right], \tag{7}$$

where

$$f(\beta) = \frac{2^\beta(1-\beta/2)}{\beta} - 0.16874.$$

The constants A and B are determined by the normalization of the distribution $\int_0^\infty P_\beta(s)\mathrm{d}s = 1$ and the mean value normalized to unity $\int_0^\infty sP_\beta(s)\mathrm{d}s = 1$. Formula (7) gives Poisson distribution for $\beta = 0$ and for $\beta = 1$ and 2 provides fair approximations to the exact GOE and GUE distributions, respectively. Usefulness of this distribution has been demonstrated by a numerical study of the level spacing of real band random matrices [13]. It has been shown that the fitted value of the parameter β depends only on the scaling parameter $x = b^2/N$.

A similar scaling law governs the statistical properties of eigenvectors. Consider the eigenvectors $\{\Phi_k, k = 1, \ldots, N\}$ of the matrix A, each described by N complex components $\{c_k^l, l = 1, \ldots, N\}$. The distribution of the squared moduli $y_k^l = |c_k^l|^2$, called eigenvector statistics $P(y)$, enables to distinguish between different universality classes [5, 6]. In the limit of large N this distribution tends to the χ_ν^2 distribution with the number of degrees of freedom ν equal to 1, 2 or 4 for the orthogonal, unitary and symplectic ensemble, respectively. Moreover, it was shown that the χ^2 distribution is appropriate to describe the eigenvector statistics of dynamical systems, pertaining to different universality classes [26]. A family of distributions $P_x(y)$ giving a fair approximation of the eigenvector statistics for real band random matrices has been proposed [14]. Theoretical results concerning statistical properties of eigenvectors have recently been obtained by Fyodorov and Mirlin [27].

The distribution of the components of a single eigenvector Φ_k may be described by the information entropy H_k of eigenvector [10]

$$H_k = -\sum_{l=1}^{N} y_k^l \ln(y_k^l). \tag{8}$$

The entire matrix A is characterized by the mean entropy $\langle H \rangle = (\sum_{k=1}^N H_k)/N$. For random matrices representing a member of a Gaussian ensemble the mean information entropy can be found analytically [28] and expressed by means of the Digamma Function Ψ [29]

$$\bar{H}_\nu = \Psi\left(\frac{\nu N}{2} + 1\right) - \Psi\left(\frac{\nu}{2} + 1\right), \tag{9}$$

with $\nu = 1$ and 2 for GOE and GUE, respectively.

Comparing the mean entropy $\langle H \rangle$ of real band matrices ($v = 0$) with the GOE entropy $\bar{H}_1$ we define the averaged scaled localization length of eigenvectors

$$\eta_1 = \exp(\langle H \rangle - \bar{H}_1). \tag{10}$$

This quantity does not depend on the size of matrix N and takes values between 0 for a diagonal matrix and 1 for GOE. It was demonstrated [10] that for real band random matrices the average scaled localization length η_1 depends only on the scaling parameter $x = b^2/N$ and a simple approximate relation, valid for $x < 10$ was proposed.

$$\eta_1 = \frac{\gamma_1 x}{1 + \gamma_1 x}, \gamma_1 \approx 1.4. \tag{11}$$

A first theoretical explanation of this relation has been given by Fyodorov and Mirlin [30]. Note that the same scaling parameter x describes statistical properties of the eigenvalues of band random matrices (level spacing distribution), as well as the properties of eigenvectors (localization length).

4. Transition Poisson – GUE

Fixing the parameter c to 2 and varying the band size b in the discussed ensemble of random matrices we could study the transition Poisson - GUE. We constructed random Hermitian matrices according to the definition (3) and (4) of the model and diagonalized them numerically. In order to obtain a relevant statistical information we collected data from several independent matrices produced for same sets of parameters. The number of matrices was fixed by a requirement to accumulate at least 25 thousands eigenvalues or tree millions eigenvector components in each sample. The size of matrices varied from 100 to 1000.

4.1. Eigenvalues

Increasing the band width b of the matrix we observed a transition of the level density $\rho(E)$ from Gaussian to semicircle, typical to GUE [5,6]. This transition is completed considerably faster than the transitions of the statistical properties of the spectrum measured by the level spacing distribution or the spectral rigidity. A similar behaviour was already reported for real band random matrices [13].

Since the level density is not uniform, we applied the standard unfolding technique [6,31] in order to analyze the statistical properties of the spectrum. The level spacing statistics $P(s)$ might be approximated by the distribution (7) and displays a Poisson – GUE transition. Fitted value of β depends only on the one parameter $x = b^2/N$, just as in the case of real matrices [13]. In other words the scaling property found for the ensemble of real symmetric band random matrices holds also for its complex Hermitian analogue. As in the case of the real band matrices the transition of the spectral properties is completed for the values of the parameter x of the order of 10. This fact allows to understand recent numerical results of Camarda [32], who observed the GUE level statistics for complex band matrices with $N = 3100$ and $b = 310$, i.e. with $x = 31$.

Comparing the level spacing distribution $P(s)$ of real symmetric band matrices ($c = 1$) and complex Hermitian matrices ($c = 2$) constructed for the same value of the scaling parameter x we observed an interesting feature. The value of the fitted parameter β_2 obtained in the latter case was, up to 3%, two times larger than the value β_1 received for the real band matrices. Such a comparison is depicted in Fig. 1 representing the level statistics $P(s)$ for real symmetric and complex Hermitian band matrices constructed for the same value of the scaling parameter $x = 0.25$. The best fit of the distribution (7), denoted by the solid lines gave $\beta_1 = 0.29$ and $\beta_2 = 0.57$. The similar behaviour observed for several other band widths confirms the relation proposed $\beta_2(x) = 2\beta_1(x)$, valid in a wide range of x. The spectral properties of real and complex band matrices are therefore correlated: The scaling parameter x controls in a same manner the transition Poisson - GOE (fitting parameter β_1 changes from 0 to 1) and the transition Poisson – GUE (parameter β_2 changes from 0 to 2).

To study the long range correlations of the spectrum we analyzed the spectral rigidity $\Delta_3(L)$. This quantity is defined as the average deviation of the straight line fit of the level staircase at the distance L [6, 7]. Figure 2 shows the dependence $\Delta_3(L)$ drawn for several values of the band width b. Observe a transition from the linear relation characteristic to the Poisson spectrum ($b = 1$) to the

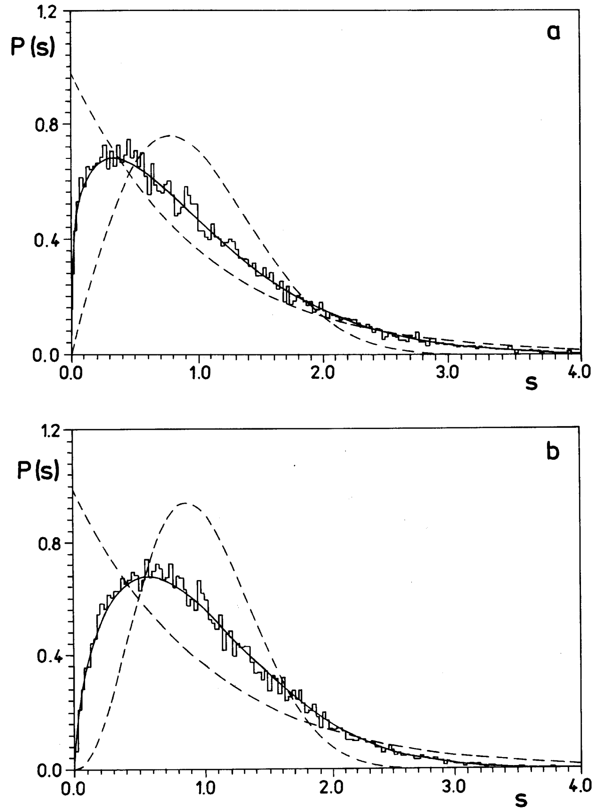

Fig. 1. Nearest neighbour distribution $P(s)$ for **a** real symmetric matrix ($c = 1$, $\beta = 0.29$), **b** complex Hermitian matrix ($c = 2.0$, $\beta = 0.57$) . Size of matrices $N = 100$, band width $b = 5$. Dashed lines stand for Poisson and GOE/GUE distributions

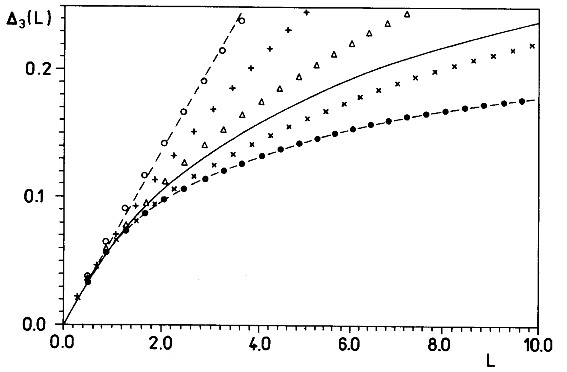

Fig. 2. Spectral rigidity $\Delta_3(L)$ drawn for several values of the scaling parameter x during the transition Poisson – GUE. $N = 100$, $c = 2.0$, $b = 1, 3, 5, 10, 100$. Solid line represents the rigidity typical of GOE

logarithmic dependence typical to the GUE ($b = N$) drawn as a lower dashed line. For an intermediate case (x close to 0.5) the spectral rigidity is similar the GOE (represented by a solid line on the picture), at least for small values of L. On the other hand, for an optimal value of the scaling parameter x, at larger distance L the spectrum analyzed is more rigid that this characteristic of GOE. The system–dependent influence of classical periodic orbits for the rigidity is different, since it causes that the spectral rigidity at large distance L becomes lower than the GOE universal dependence. In other words, even if the level spacing distribution is described by a GOE-like spectrum, the long–range correlation allow us to distinguish between GOE ensemble and an appropriately chosen intermediate state of the Poisson–GUE transition.

4.2. Eigenvectors

Under the change of the control parameter x the eigenvector statistics $P(y)$ displays the transition from the normalized $1/y$ distribution, typical to the classically regular systems [14,26], to the χ^2_ν distribution with $\nu = 2$ degrees of freedom. In order to describe the properties of eigenvectors during the transition Poisson – GUE by a single quantity we introduce the generalized localization length η_2 comparing the averaged entropy of eigenvectors $\langle H \rangle$ with the entropy $\bar{H}_2$ of GUE

$$\eta_2 = \exp(\langle H \rangle - \bar{H}_2). \tag{12}$$

Throughout the transition Poisson – GUE the localization length $\eta_2 = \eta_2(N, b)$ varies form zero to unity. It depends on the same scaling variable $x = b^2/N$, and the relation η_2 on the parameter x is similar to (11). The corresponding constant γ_2 is approximately equal in this case to 2.5, and is larger than γ_1, in agreement to [30].

Properties of eigenvectors change during the transition parallel with the changes of the spectra. Fig. 3 presents a dependence of the fitting parameter β of the Izrailev distribution (7) on the localization length η for Poisson – GOE transition ($\beta \in [0, 1]$), and Poisson – GUE transition ($\beta \in [0, 2]$). For both transitions the symbols denoting size of matrices (100, 200 and 400) are localized close to a single line. This scaling behaviour may serve as an additional argument that the properties of spectrum and eigenvectors of band random matrices are correlated.

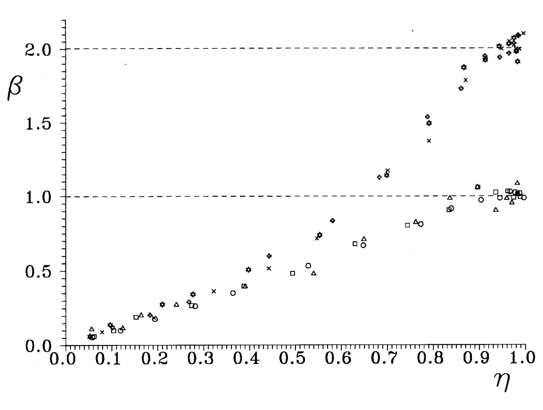

Fig. 3. Level spacing parameter β represented as a function of the localization length η for Poisson – GOE transition ($c = 1$, lower line) and for Poisson–GUE transition ($c = 2$, upper line). Size of matrices $N = 100, 200$ and 400

5. Transition GOE – GUE

Fixing the width b of the band to N and varying the parameter c of the model from 1 to 2 we may study the transition from the orthogonal to the unitary ensemble. This transition corresponds to breaking of a generalized time-reversal symmetry in a dynamical system [2, 6] and has recently been investigated in context of mesoscopic systems in magnetic field [33, 34].

In agreement to the analysis of Pandey and Mehta [20] there exist a scaling parameter $v = \lambda^2 N$, given in our notation by (6). Level spacing distribution is well described by the Lenz–Haake formula [22] derived for 2×2 matrices

$$P_\lambda(s) = \left(\frac{2+\lambda^2}{2}\right)^{1/2} [D(\lambda)]^2 s \exp\left(\frac{-s^2[D(\lambda)]^2}{2}\right) \times \operatorname{erf}\left[\frac{sD(\lambda)}{\lambda}\right], \quad (13)$$

where

$$D(\lambda) = \left(\frac{\pi(\lambda^2+2)}{4}\right)^{1/2} \left(1 - \frac{2}{\pi}\left[\arctan\left[\frac{\lambda}{\sqrt{2}}\right] - \frac{\sqrt{2}\lambda}{2+\lambda^2}\right]\right). \quad (14)$$

Since the transition speed depends on the level density [20], it will be different for the center of the spectra and for the wings. The fitting parameter λ_f of (13), which describes the level spacing distribution, depends not only on the parameters of the matrix ($b = N, c$), but also on the position in the spectrum of eigenlevels included into computation. Analysis of the nearest neighbour distribution in a given energy window from the spectra allows us to establish the property of scaling: fitted value λ_f is a function of the scaling parameter v (different in each energy window). An example of the level spacing distribution for the GOE–GUE transition is shown in Fig. 4.

A similar property of scaling concerns also the eigenvectors. The eigenvector statistics might be, in a first approximation characterized by the parameter $\nu \in [1, 2]$ of the χ^2_ν distribution. Better approximation for the distribution $P(y)$ was proposed in [35]. Recently Sommers and

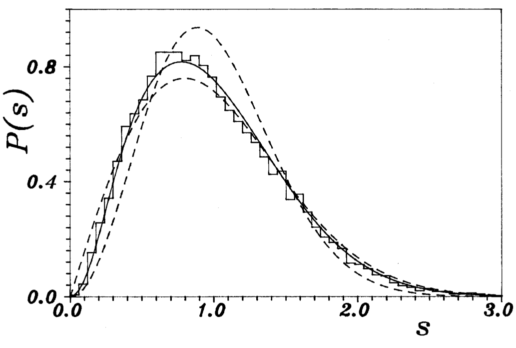

Fig. 4. Nearest level spacing distribution for GOE-GUE transition; $N = b = 100$, $c = 1.002$, $v \approx 0.2$, best fit of the distribution (11) gives $\lambda_f = 0.41$ represented by solid line. Dashed lines represent GOE and GUE distributions

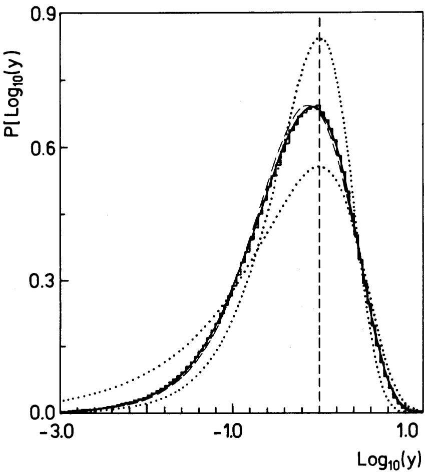

Fig. 5. Eigenvector statistics for GOE-GUE transition; $N = b = 100$, $c = 1.006$, $v \approx 0.6$. Dashed lines denote GOE and GUE statistics while narrow and bold solid lines represent distributions derived in [35] and [36], respectively

Iida [36] derived another distribution for eigenvector statistics for GOE-GUE transition which gives the best agreement to the numerical data. Fig. 5 presents an exemplary numerical data obtained for $v = 0.6$ and both theoretical distributions.

6. General case of the model

The model of Hermitian band matrices is described (for a constant matrix size N) by two independent parameters: the band width b and the parameter c governing the relative size of imaginary part of the matrix. We shall analyze the general case of the model by allowing one of the parameters to change, while the other one remains fixed.

The transition from GOE to GUE is controlled by the parameter v given in (6), which varies form zero (GOE) to infinity (GUE). In order to define the generalized localization length analogous to (10) and (12) it is convenient to introduce a parameter $\omega = (2v + 1)/(v + 1)$. Due to relation (6) this scaling parameter is

$$\omega = 2 - \frac{2-c}{2N(c-1)+2-c}, \quad (15)$$

and changes from 1 (GOE) to 2 (GUE). Comparison of the mean entropy of eigenvectors of a given band matrix with the mean entropy of eigenvectors of the full matrix ($b = N$) by constant parameter ω leads to the definition of the generalized localization length

$$\eta_\omega = \exp[\langle H(N, b, c(N, \omega))\rangle - \langle H(N, b = N, c(N, \omega))\rangle]. \quad (16)$$

Note that in the particular cases of $\omega = 1, 2$ the generalized scaled localization length η_ω reduces to the previous definitions (10) and (12), applicable to the cases of orthogonal and unitary ensembles respectively. Our numerical analysis shows that for any value of ω the transition from

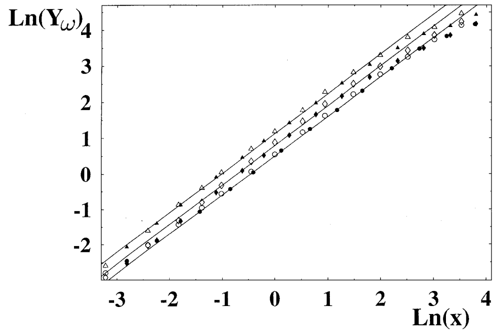

Fig. 6. Scaling variable $Y_\omega = \eta_\omega/(1-\eta_\omega)$ depicted in log–log scale as a function of the parameter x for $N = 100$ (open symbols) and $N = 150$ (closed symbols) and $\omega = 1.0(\bigcirc), 1.95(\Diamond), 2.0(\triangle)$. Linear fits made of data for $x < 10$

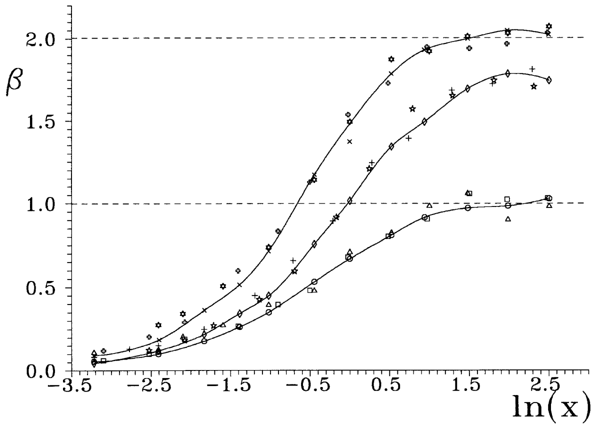

Fig. 7. Fitting parameter β of the distribution (6) drawn as a function of the scaling parameter x for $\omega = 1.0, 1.66, 2.0$ for $N = 100, 200, 400$

diagonal to full matrices is controlled by the same scaling parameter $x = b^2/N$. Moreover, the dependence of η_ω on x might be approximated by relation

$$\frac{\eta_\omega}{1-\eta_\omega} = \gamma_\omega x, \tag{17}$$

where the constant γ_ω is equal approximately to 1.4, 1.8. 2.2, and 2.8 for $\omega = 1.0, 1.7,$ 1.9, and 2.0, respectively. For $\omega = 1$ the above formula is equivalent to (11).

Fig. 6 presents the dependence of $Y_\omega = \eta_\omega/(1-\eta_\omega)$ on the scaling parameter x in a log – log scale for two different values of the matrix size N. The scaling relation (17) seems not to be valid for large values of x exceeding 10. The same restriction concerning the scaling law (11) has been reported for the case of real symmetrical matrices [10]. Standard model of band random matrices corresponds to a one dimensional chain of N sites coupled together by an interaction with the range $2b$. The chain may be closed into a circle by adding into the model non-zero elements into corners of size b of the matrix. Recent study of periodic band random matrices [37] shows a scaling relation in form of (17) valid for arbitrary large values of x. The discrepancy from this relation visible in Fig. 6 for $x > 10$ might be, therefore, associated with the "edge effects" caused by existence of two ends of the chain.

Scaling parameter x describes for arbitrary value of ω also the properties of the spectra. Fig. 7 shows the dependence of the fitting parameter β of the Izrailev distribution (7) as a function of x. For each value of the parameter $\omega = 1.0$, 1.66, 2.0 the data resulting of matrices of different size (100, 200, 400) are localized close to a single line, which are plotted to guide the eye. This observation supports the conjecture that the statistical properties of spectra and eigenvectors of random matrices are correlated.

A complementary information about the scaling properties of the model can be obtained by comparing the mean entropy of eigenvectors of a given ensemble of

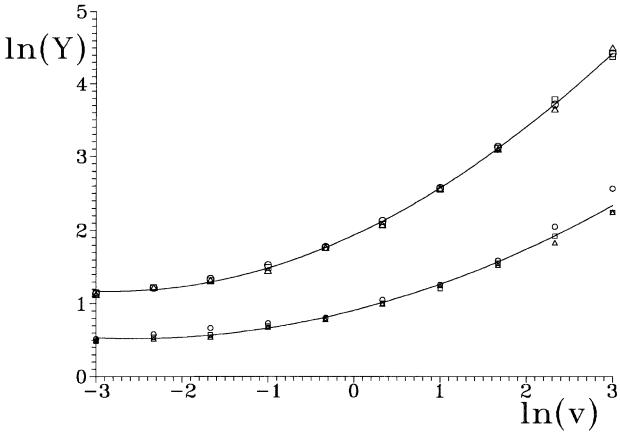

Fig. 8. Variable $Y = \xi/(1-\xi)$, representing the relative localization length ξ, plotted in logarithmical scale as a function of the scaling parameter v for a full matrices $(b = N, x \to \infty)$-larger symbols and b band matrices with $x = 1$ – smaller symbols. Matrix size $N = 100(\bigcirc), 200(\square), 400(\triangle)$. Solid lines are drawn to guide the eye

matrices characterized by $\{N, b, c\}$ with the entropy of matrices $\{N, b, c = 2\}$. In a way analogous to (16) we define a relative localization length ξ_x

$$\xi_x = \exp[\langle H(N, b = \sqrt{xN}, c)\rangle - \bar{H}(N, b = \sqrt{xN}, c = 2)]. \tag{17}$$

For full matrices $(b = N)$ the quantity ξ_∞ describes the GOE–GUE transition. For large matrices it depends only on the scaling parameter v and varies from $2/e \approx 0.736$ (GOE) to unity (GUE). Our numerical results show that a similar scaling law hold for Hermitian band matrices described by arbitrary values of x. Let $Y_x = \xi_x/(1-\xi_x)$. Fig. 8 shows such scaling for full matrices Y_∞ (upper line starting with $\ln[2/(e-2)]$, larger symbols) and band matrices Y_1 (lower line, smaller symbols) for three different sizes of matrices.

7. Concluding remarks

We have studied a general model of Hermitian band random matrices allowing us to study three transitions: Poisson–GOE, Posisson – GUE and GOE–GUE. During each transition the statistical properties of the spectra of random matrices are correlated with the properties of their eigenvectors. There exist two scaling parameters in the model: $x = b^2/N$ controlling the transition from localization to delocalization and $v = 2N(c-1)/(2-c)$ describing the time reversal symmetry breaking. For a fixed value of one the scaling parameters $\{x, v\}$ the properties of spectrum and eigenvectors depend only on the other parameter.

It is hardly possible to find a family of interpolating distributions for the level spacings distribution $P(s)$ or the eigenvector statistics $P(y)$ to describe the properties of the Hermitian random matrices in the general case. We have found, however, that for the case of almost full matrices ($x \gg 1$) the formulae derived in [22] and [36], apply for level spacing and eigenvectors distribution, respectively. For narrower band matrices ($x < 10$) a reasonable approximation to numerical results is provided by Izrailev distribution (7) for $P(s)$ and an eigenvector distribution $P_a(y)$ proposed in [14].

An analogous process of the time reversal symmetry breaking in time–dependent, periodic dynamical systems described by circular ensembles of unitary matrices corresponds to the COE–CUE transition. This transition has been recently studied on the simple dynamical models of the periodically kicked rotator [38–41] and the periodically kicked top [42]. A simple algorithm allowing to construct numerically random unitary matrices pertaining to both circular ensembles has been proposed [43]. It will be interesting to study a transition between them.

Much attention has recently been paid to analyze the properties of the Wigner model [12, 44–46]. A simple generalization of the standard band random matrices allows one to describe the effect of an external electric field by choosing a non-zero mean of the diagonal elements of the matrix. It may be instructive to investigate the consequences of the time reversal symmetry breaking in such a model and to study the properties of Hermitian Wigner model defined in an analogy to (2–4).

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