Square Root Voting System, Optimal Threshold and π

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1 Introduction

Recent political debate on the voting system used in the Council of Ministers of the European Union stimulated research in the theory of indirect voting, see e.g. (Felsenthal and Machover 2001; Leech 2002; Andjiga et al. 2003; Pajala and Widgrén 2004; Beisbart et al. 2005). The *double majority* voting system, adopted for the Council by The Treaty of Lisbon in December 2007 is based on two criteria: 'per capita' and 'per state'. This system apparently reflects the principles of equality of Member States and that of equality of citizens. However, as recently analyzed by various authors Baldwin and Widgrén (2004), Ade (2003), Słomczyński and Życzkowski (2006), Algaba et al. (2007), Hosli (2008), Bârsan-Pipu and Tache (2009), Kirsch (2010), Moberg (2010), Leech and Aziz (2010), Pukelsheim (2010), Słomczyński and Życzkowski (2010), in such a system the large states gain a lot of power from the direct link to population, while the smallest states derive disproportionate power from the other criterion.

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The combined effect saps influence away from all medium-sized countries. Ironically, a similar conclusion follows from a book by Lionel Penrose, who wrote already in 1952 (Penrose 1952):

If two votings were required for every decision, one on a per capita basis and the other upon the basis of a single vote for each country, this system would be inaccurate in that it would tend to favor large countries.

To quantify the notion of voting power, mathematicians introduced the concept of power index of a member of the voting body, which measures the probability that his vote will be decisive in a hypothetical ballot: Should this member decide to change his vote, the winning coalition would fail to satisfy the qualified majority condition. Without any further information about the voting body it is natural to assume that all potential coalitions are equally likely. This very assumption leads to the concept of Banzhaf(-Penrose) index called so after John Banzhaf, an American attorney, who introduced this index independently in 1965 (Banzhaf 1965).

Note that this approach is purely normative, not descriptive: we are interested in the potential voting power arising from the voting procedure itself. Calculation of the voting power based on the counting of majority coalitions is applicable while analyzing institutions in which alliances are not permanent, but change depending upon the nature of the matter under consideration.

To design a representative voting system, i.e. the system based on the democratic principle, that the vote of any citizen of any Member State is of equal worth, one needs to use a weighted voting system. Consider elections of the government in a state with population of size N. It is easy to imagine that an average German citizen has smaller influence on the election of his government than, for example, a citizen of the neighboring Luxembourg. Analyzing this problem in the context of voting in the United Nations just after the World War II Penrose showed, under some natural assumptions, that in such elections the voting power of a single citizen decays as one over square root of N. Thus, the system of indirect voting applied to the Council is representative, if the voting power of each country is proportional to the square root of N, so that both factors cancel out. This statement is known in the literature under the name of the *Penrose square root law* (Penrose 1946; Felsenthal and Machover 1998). It implies that the voting power of each member of the EU Council should behave as $\sqrt{N}$ and such voting systems have been analyzed in this context by several experts since late 1990s (Felsenthal and Machover 1997; Laruelle and Widgrén 1998).

It is challenging to explain this fact in a way accessible to a wide audience (Życzkowski et al. 2006; Kirsch et al. 2007; Pukelsheim 2007; Pöppe 2007). A slightly paradoxical nonlinearity in the result of Penrose is due to the fact that voting in the Council should be considered as a two-tier voting system: Each member state elects a government, which delegates its representative to the Council. Any representative has to say *'Yes'* or *'No'* on behalf of his state in every voting organized in the Council. The key point is that in such a voting each member of the Council cannot split his vote. Making an idealistic assumption that the vote of a Minster in the Council represents the will of the majority of the citizens of the state he represents, his vote '*Yes*' means only that a majority of the population of his state supports this decision, but does not reflect the presence of a minority.

Consider an exemplary issue to be voted in the Council and assume that the preferences of the voters in each state are known. Assume hypothetically that a majority of population of Malta says '*Yes*' on a certain issue, the votes in Italy split as 30 millions '*Yes*' and 29 millions '*No*', while all 43 millions of citizens of Spain say '*No*'. A member of the Council from Malta follows the will of the majority in his state and votes '*Yes*'. So does the representative of Italy. According to the double majority voting system his vote is counted on behalf of the total number of 59 millions of the population of Italy. Thus these voting rules allow 30 millions of voters in Italy to over-vote not only the minority of 29 millions in their state (which is fine), but also, with the help of less than half a million of people from Malta, to over-vote 43 millions of Spaniards.

This pedagogical example allows one to conclude that the double majority voting system would work perfectly, if all voters in each member state had the same opinion on every issue. Obviously such an assumption is not realistic, especially in the case of the European states, in which the citizens can nowadays afford the luxury of an independent point of view. In general, if a member of the Council votes '*Yes*' on a certain issue, in an ideal case one may assume that the number of the citizens of his state which support this decision varies from 50 to 100 % of the total population. In practice, no concrete numbers for each state are known, so to estimate the total number of European citizens supporting a given decision of the Council one has to rely on statistical reasoning.

To construct the voting system in the Council with voting powers proportional to the square root of populations one can consider the situation, where voting weights are proportional to the square root of populations and the Council takes its decision according to the principle of a *qualified majority*. In other words, the voting in the Council yields acceptance, if the sum of the voting weights of all Ministers voting '*Yes*' exceeds a fixed quota q, set for the qualified majority. From this perspective the quota q can be treated as a free parameter (Leech and Machover 2003; Machover 2010), which may be optimized in such a way that the mean discrepancy Δ between the voting power (measured by the Banzhaf index) and the voting weight of each member state is minimal.

In the case of the population in the EU consisting of 25 member states it was shown (Słomczyński and Życzkowski 2004; Życzkowski et al. 2006) that the value of the optimal quota q_* for qualified majority in the Penrose's square root system is equal to 62 %, while for EU-27 this number drops down to 61.5 % (Słomczyński and Życzkowski 2006, 2007). Furthermore, the optimal quota can be called *critical*, since in this case the mean discrepancy $\Delta(q_*)$ is very close to zero and thus the voting power of every citizen in each member state of the Union is practically equal. This simple scheme of voting in the EU Council based on the square root law of Penrose supplemented by a rule setting the optimal quota to q_* happens to give larger voting powers to the largest EU than the Treaty of Nice, but smaller ones than the Treaty of Lisbon. Therefore this voting system has been dubbed by the media as the *Jagiellonian Compromise*.

It is known that the existence of the critical quota q_*, is not restricted to the particular distribution of the population in the European Union, but it is also characteristic of a generic distribution of the population (Słomczyński and Życzkowski 2004; Chang et al. 2006; Słomczyński and Życzkowski 2006). The value of the critical quota depends on the particular distribution of the population in the 'union', but even more importantly, it varies considerably with the number M of member states. An explicit approximate formula for the critical quota was derived in Słomczyński and Życzkowski (2007). It is valid in the case of a relatively large number of the members of the 'union' and in the asymptotic limit, $M \to \infty$, the critical quota tends to 50 %, in consistence with the so-called *Penrose limit theorem* (Lindner and Machover 2004).

On one hand it is straightforward to apply this explicit formula for the current population of all member states of the existing European Union, as well as to take into account various possible scenarios of a possible extension of the Union. On the other hand, if the number of member states is fixed, while their populations vary in time, continuous update of the optimal value for the qualified majority may be cumbersome and unpractical. Hence one may try to neglect the dependence on the particular distribution of the population by selecting for the quota the mean value of $\langle q \rangle$, where the average is taken over a sample of random population distributions, distributed uniformly in the allowed space of M-point probability distributions. In this work we perform such a task and derive an explicit, though approximate, formula for the average critical quota.

This chapter is organized as follows. In Sect. 2 devoted to the one-tier voting system, we recall the definition of Banzhaf index and review the Penrose square root law. In Sect. 3, which concerns the two-tier voting systems, we describe the square root voting system and analyze the average number of misrepresented voters. Section 4 is devoted to the problem of finding the optimal quota for the qualified majority. It contains the key result of this paper: derivation of a simple approximate formula for the average optimal quota, which depends only on the number M of the member states and is obtained by averaging over an ensemble of random distributions of the population of the 'union'.

2 One Tier Voting

Consider a voting body consisting of M voters voting according to the qualified majority rule. Assume that the weights of the votes need not to be equal, which is typical e.g. in the case of an assembly of stockholders of a company: the weight of the vote of a stockholder depends on the number of shares he or she possesses. It is worth to stress that, generally, the voting weights do not directly give the voting power.

To quantify the a priori voting power of any member of a given voting body game theorists introduced the notion of a power index. It measures the probability that a member's vote will be decisive in a hypothetical ballot: should this player decide to change its vote, the winning coalition would fail to satisfy the qualified majority condition. In the game theory approach to voting such a player is called *pivotal*.

The assumption that all potential coalitions of voters are equally likely leads to the concept of the Banzhaf index (Penrose 1946; Banzhaf 1965). To compute this power index for a concrete case one needs to enumerate all possible coalitions, identify all winning coalitions, and for each player find the number of cases in which his vote is decisive.

Let M denote the number of voters and ω the total number of all winning coalitions, that satisfy the qualified majority condition. Assume that ω_k denotes the number of winning coalitions that include the kth player; where $k = 1, \ldots, M$. Then the Banzhaf index of the kth voter reads

$$\psi_k \;:=\; \frac{\omega_k - (\omega - \omega_k)}{2^{M-1}} = \frac{2\omega_k - \omega}{2^{M-1}}. \tag{1}$$

To compare these indices for decision bodies consisting of different number of players, it is convenient to define the *normalized Banzhaf (-Penrose) index*:

$$\beta_k \;:=\; \frac{\psi_k}{\sum_{i=1}^{M} \psi_i} \tag{2}$$

such that $\sum_{i=1}^{M} \beta_i = 1$.

In the case of a small voting body such a calculation is straightforward, while for a larger number of voters one has to use a suitable computer program.

2.1 Square Root Law of Penrose

Consider now the case of N members of the voting body, each given a single vote. Assume that the body votes according to the standard majority rule. On one hand, since the weights of each voter are equal, so must be their voting powers. On the other hand, we may ask, what happens if the size N of the voting body changes, for instance, if the number of eligible voters gets doubled, how does this fact influence the voting power of each voter?

For simplicity assume for a while that the number of voters is odd, $N = 2j + 1$. Following original arguments of Penrose we conclude that a given voter will be able to effectively influence the outcome of the voting only if the votes split half and half: If the vote of j players would be '*Yes*' while the remaining j players vote '*No*', the role of the voter we analyze will be decisive.

Basing upon the assumption that all coalitions are equally likely one can ask, how often such a case will occur? In mathematical language the model in which this assumption is satisfied is equivalent to the *Bernoulli scheme*. The probability that out of $2j$ independent trials we obtain k successes reads

$$P_k := \binom{2j}{k} p^k (1-p)^{2j-k}, \tag{3}$$

where p denotes the probability of success in each event. In the simplest symmetric case we set $p = 1 - p = 1/2$ and obtain

$$P_j \;=\; \left(\frac{1}{2}\right)^{2j} \frac{(2j)!}{(j!)^2}. \tag{4}$$

For large N we may use the Stirling approximation for the factorial and obtain the probability ψ that the vote of a given voter is decisive

$$\psi = P_j \;\sim\; 2^{-2j} \frac{(2j/e)^{2j}\sqrt{4\pi j}}{\left[(j/e)^j\sqrt{2\pi j}\right]^2} \;=\; \frac{1}{\sqrt{\pi j}} \;\sim\; \sqrt{\frac{2}{\pi N}}. \tag{5}$$

For N even we get the same approximation. In this way one can show that the voting power of any member of the voting body depends on its size as $1/\sqrt{N}$, which is the *Penrose square root law*. The above result is obtained under the assumption that the votes of all citizens are uncorrelated. A sound mathematical investigation of the influence of possible correlations between the voting behavior of individual citizens for their voting power has been recently presented by Kirsch (2007). It is easy to see that due to strong correlations certain deviations from the square root law have to occur, since in the limiting case of unanimous voting in each state (perfect correlations), the voting power of a single citizen from a state with population N will be inversely proportional to N.

The issue that the assumptions leading to the Penrose law are not exactly satisfied in reality was raised many times in the literature, see, e.g. (Gelman et al. 2002, 2004), also in the context of the voting in the Council of the European Union (Laruelle and Valenciano 2008). However, it seems not to be easy to design a rival model voting system which correctly takes into account the essential correlations, varying from case to case and evolving in time. Furthermore, it was argued (Kirsch 2007) that the strength of the correlations between the voters tend to decrease in time. Thus, if one is to design a voting system to be used in the future in the Council of the European Union, it is reasonable to consider the idealistic case of no correlations between individual voters. We will follow this strategy and in the sequel rely on the square root law of Penrose.

2.2 Pivotal Voter and the Return Probability in a Random Walk

It is worth to emphasize that the square root function appearing in the above derivation is typical to several other reasonings in mathematics, statistics and physics. For instance, in the analyzed case of a large voting body, the probability distribution P_k in the Bernoulli scheme can be approximated by the Gaussian distribution with the standard deviation being proportional to $1/\sqrt{N}$. It is also instructive to compare the above voting problem with a simple model of a random walk on the one dimensional lattice.

Assume that a particle subject to external influences in each step jumps a unit distance left or right with probability one half. What is the probability that it returns to the initial position after N steps? It is easy to see that the probability scales as $1/\sqrt{N}$, since the answer is provided by exactly the same reasoning as for the Penrose law.

Consider an ensemble of particles localized initially at the zero point and performing such a random walk on the lattice. If the position of a particle at time n differs from zero, in half of all cases it will jump towards zero, while in the remaining half of cases it will move in the opposite direction. Hence the *mean* distance $\langle D \rangle$ of the particle from zero will not change. On the other hand, if at time n the particle happened to return to the initial position, in the next step it would certainly jump away from it, so the mean distance from zero would increase by one.

To compute the mean distance form zero for an ensemble of random particles performing N steps, we need to sum over all the cases, when the particle returns to the initial point. Making use of the previous result, that the return probability $P(n)$ at time n behaves as $1/\sqrt{n}$, we infer that during the time N the mean distance behaves as

$$\langle D(N) \rangle \approx \sum_{n=1}^{N} P(n) \approx \sum_{n=1}^{N} \frac{1}{\sqrt{n}} \sim \sqrt{N}. \tag{6}$$

This is just one formulation of the *diffusion law*. As shown, the square root of Penrose is closely related with some well known results from mathematics and physics, including the Gaussian approximation of binomial distribution and the diffusion law.

3 Two Tier Voting

In a two-tier voting system each voter has the right to elect his representative, who votes on his behalf in the upper chamber. The key assumption is that, on one hand, he should represent the will of the population of his state as best he can, but, on the other hand, he is obliged to vote '*Yes*' or '*No*' in each ballot and cannot split his vote. This is just the case of voting in the Council of the EU, since citizens in each member state choose their government, which sends its Minister to represent the entire state in the Council.

These days one uses in the Council the triple majority system adopted in 2001 in the *Treaty of Nice*. The Treaty assigned to each state a certain number of 'weights', distributed in an ad hoc fashion. The decision of the Council is taken if the coalition voting in favour of it satisfies three conditions:

(a) it is formed by the standard majority of the member states;
(b) states forming the coalition represent more then 62 % of the entire population of the Union;
(c) the total number of weights of the '*Yes*' votes exceeds a quota equal to approximately 73.9 % of all weights.

Although all three requirements have to be fulfilled simultaneously, detailed analysis shows that condition (c) plays a decisive role in this case: if it is satisfied, the two others will be satisfied with a great likelihood as well (Felsenthal and Machover 2001; Leech 2002).

Therefore, the voting weights in the Nice system play a crucial role. However, the experts agree (Felsenthal and Machover 2001; Pajala and Widgrén 2004) that the choice of the weights adopted is far from being optimal. For instance the voting power of some states (including e.g. Germany and Romania) is significantly smaller than in the square root system. This observation is consistent with the fact that Germany was directly interested to abandon the Nice system and push toward another solution that would shift the balance of power in favor of the largest states.

In the double majority voting system, adopted in December 2007 in Lisbon, one gave up the voting weights used to specify the requirement (c) and decided to preserve the remaining two conditions with modified majority quotas. A coalition is winning if:

(a') it is formed by at least 55 % of the members states;
(b') it represents at least 65 % of the population of the Union;

Additionally, every coalition consisting of all but three (or less) countries is winning even if it represents less than 65 % of the population of the Union.

The double majority system will be used in the Council starting from the year 2014. However, a detailed analysis by Moberg (2010) shows that in this concrete case the 'double majority' system is not really double, as the per capita criterion (b') plays the dominant role here. In comparison with the Treaty of Nice, the voting power index will increase for the four largest states of the Union (Germany, France, the United Kingdom and Italy) and also for the smallest states. To understand this effect we shall analyze the voting system in which the voting weight of a given state is directly proportional to its population.

3.1 Voting Systems with Per Capita Criterion

The idea 'one citizen–one vote' looks so natural and appealing, that in several political debates one often did not care to analyze in detail its assumptions and all its consequences. It is somehow obvious that a minister representing a larger (if population is considered) state should have a larger weight during each voting in the EU Council. On the other hand, one needs to examine whether the voting weights of a minister in the Council should be proportional to the population he represents. It is clear that this would be very much the case, if one could assume that all citizens in each member state share the very same opinion in each case.

However, this assumption is obviously false, and nowadays we enjoy in Europe the freedom to express various opinions on every issue. Let us then formulate the question, how many citizens from his state each minister actually represents in an exemplary voting in the Council? Or to be more precise, how many voters from a given state with population N share in a certain case the opinion of their representative? We do not know!

Under the idealizing assumption that the minister always votes according to the will of the majority of citizens in his state, the answer can vary from $N/2$ to N. Therefore, the difference between the number of the citizens supporting the vote of their minister and the number of those who are against it can vary from 0 to N. In fact it will vary from case to case in this range, so an assumption that it is always proportional to N is false. This crucial issue, often overlooked in popular debates, causes problems with representativeness of a voting system based on the 'per capita' criterion.

There is no better way to tackle the problem as to rely on certain statistical assumptions and estimate the *average* number of 'satisfied citizens'. As such an analysis is performed later in this paper, we shall review here various arguments showing that a system with voting weights directly proportional to the population is advantageous to the largest states of the union.

Consider first a realistic example of a union of nine states: a large state A, with 80 millions of citizens and eight small states from B to I, with 10 millions each. Assume now that in a certain case the distribution of the opinion in the entire union is exactly polarized: in each state approximately 50 % of the population support the vote '*Yes*', while the other half is against. Assume now that the government of the large state is in position to establish exactly the will of the majority of citizens in their state (say it is the vote '*Yes*') and order its minister to vote accordingly. Thus the vote of this minister in the council will then be counted as a vote of 80 millions of citizens.

On the other hand, in the remaining states the probability that the majority of citizens support '*Yes*' is close to 50 %. Hence it is most likely that the votes of the ministers from the smaller states split as 4:4. Other outcomes: 5:3, 6:2, or 7:1 are less probable, but all of them result in the majority of the representative of the large state A. The outcome 8:0 is much less likely, so if we sum the votes of all nine ministers we see that the vote of the minister from the largest state will be decisive. Hence we have shown that the voting power of all citizens of the nine small states is negligible, and the decision for this model union is practically taken by the half of its population belonging to the largest state A. Even though in this example we concentrated on the 'per capita' criterion and did not take into account the other criterion, it is not difficult to come up with analogous examples which show that the largest states are privileged also in the double majority system. Similarly, the smallest states of the union benefit from the 'per state' criterion.

Let us have a look at the position of the minority in large states. In the above example the minority in the 80 million state can be as large as 40 million citizens, but their opinion will not influence the outcome of the voting, independently of the polarization of opinion in the remaining eight states. Thus one may conclude that in the voting system based on the 'per capita' criterion, the influence of the politicians representing the majority in a large state is enhanced at the expense of the minority in this state and the politicians representing the smaller states.

Last but not least, let us compare the maximal sizes of the minority, which can arise during any voting in an EU member state. In Luxembourg, with its population of about 400,000 people, the minority cannot exceed 200,000 citizens. On the other hand, in Germany, which is a much larger country, it is possible that the minority exceeds 41 millions of citizens, since the total population exceeds 82 millions. It is then fair to say, that, due to elections in smaller states, we know the opinion of citizens in these states with a better accuracy, than in larger members of the union. Thus, as in smaller states the number of misrepresented citizens is smaller, their votes in the EU Council should be weighted by larger weights than the vote of the largest states. This very idea is realized in the weighted voting system advocated by Penrose.

3.2 Square Root Voting System of Penrose

The Penrose system for the two-tier voting is based on the square root law reviewed in Sect. 2.1. Since the voting power of a citizen in state k with population N_k scales as $1/\sqrt{N_k}$, this factor will be compensated, if the voting power of each representative in the upper chamber will behave as $\sqrt{N_k}$. Only in this way the voting power of each citizen in every state of a union consisting of M states will be equal.

Although we know that the voting power of a minister in the Council needs not coincide with the weight of his vote, as a rough approximation let us put his weights w_k proportional to the square root of the population he represents, that is $w_k = \sqrt{N_k}/\sum_{i=1}^{M}\sqrt{N_i}$.

To see a possible impact of the change of the weights let us now return to the previous example of a union of one big state and eight small ones. As the state A is 8 times as large as each of the remaining states, its weight in the Penrose system will be $w_A = \sqrt{8}w_B$. As $\sqrt{8}$ exceeds 2 and is smaller then 3, we see that accepting the Penrose system will increase the role of the minority in the large state and the voting power of all smaller states. For instance, if the large state votes '*Yes*' and the votes in the eight states split as 2:6 or 1:7 in favor for '*No*', the decision will not be taken by the council, in contrast to the simple system with one 'per capita' criterion. There, we have assumed that the standard majority of weights is sufficient to form a winning coalition. If the threshold for the qualified majority is increased to 54 %, also the outcome 3:5 in favor for '*No*' in the smaller states suffices to block the decision taken in the large state.

This simple example shows that varying the quota for the qualified majority considerably influences the voting power, see also (Leech and Machover 2003; Machover 2010). The issue of the selection of the optimal quota will be analyzed in detail in the subsequent section. At this point, it is sufficient to add that in general it is possible to find such a level of the quota for which the voting power β_k of the kth state is proportional to $\sqrt{N_k}$ and, in consequence, the Penrose law is almost exactly fulfilled (Słomczyński and Życzkowski 2004, 2006).

Applying the square root voting system of Penrose combined with the optimal quota to the problem of the Council, one obtains a fair solution, in which every citizen in each member state of the Union has the same voting power, hence the same influence on the decisions taken by the Council. In this case, the voting power of each European state measured by the Banzhaf index scales as the square root of its population. This weighted voting system happens to give a larger voting power to the largest EU states (including Germany) than the Treaty of Nice but smaller than the double majority system. On the other hand, this system is more favorable to all middle size states then the double majority, so it is fair to consider it as a compromise solution. The square root voting system of Penrose is simple (one criterion only), transparent and efficient—the probability of forming a winning coalition is reasonably high. Furthermore, as discussed later, it can be easily adopted to any possible extension of the Union.

3.3 The Second Square Root Law of Morris

To provide an additional argument in favour of the square root weights of Felsenthal and Machover (1999), consider a model state of N citizens, of which a certain number k support a given legislation to be voted in the council. Assume that the representative of this state knows the opinion of his people and, according to the will of the majority, he votes '*Yes*' in the council if $k \geq N/2$. Then the number of citizens satisfied with his decision is k. The number $N-k$ of disappointed citizens compensates the same number of yes-votes, so the vote of the minister should effectively represent the *difference* between them, $w = k - (N-k) = 2k - N$. By our assumption concerning the majority this number is positive, but in general the effective weight of the vote of the representative should be $w = |2k - N|$.

Assume now that the votes of any of N citizens of the state are independent, and that both decisions are equally likely, so that $p = 1 - p = 1/2$. Thus, for the statistical analysis, we can use the Bernoulli scheme (3) and estimate the weight of the vote of the minister by the average using the Stirling approximation:

$$\langle w_N \rangle = \sum_{k=0}^{N} P_k |2k - N| = \sum_{k=0}^{N} \binom{N}{k} \frac{1}{2^N} |2k - N|$$
$$= \frac{\lfloor N/2 \rfloor + 1}{2^{N-1}} \binom{N}{\lfloor N/2 \rfloor + 1} \sim \sqrt{\frac{2N}{\pi}}. \tag{7}$$

Here $\lfloor x \rfloor$ denotes the largest integer not greater than x. This result provides another argument in favor of the weighted voting system of Penrose: Counting all citizens of a given state, we would attribute the weights of the representative proportionally to the population N he is supposed to represent. On the other hand, if we take into account the obvious fact that not all citizens in this state share the opinion of the government on a concrete issue and consider the average number of the majority of citizens which support his decision one should weight his vote proportionally to $\sqrt{N}$. From this fact one can deduce the *second square root law of Morriss* (Morriss 1987; Felsenthal and Machover 1998, 1999; Laruelle and Valenciano 2008) that states that the average number of misrepresented voters in the union is smallest if the weights are proportional to the square root of the population and quota is equal to 50 %, provided that the population of each member state is large enough. Simultaneously, in this situation, the total voting power of the union measured by the sum of the Banzhaf indices of all citizens in the union is maximal.

To illustrate the result consider a model union consisting of one large state with population of 49 millions, three medium states with 16 million each and three small with 1 million citizens. For simplicity assume that the double majority system and the Penrose system are based on the standard majority of 50 %. If the polarization of opinion in each state on a given issue is as in the table below, only 39 % of the population of the union is in favor of the legislative. However, under the rules of the double majority system the decision is taken (against the will of the vast majority!), what is not the case in the Penrose system, for which the coalition gains only 10 votes out of 22, so it fails to gather the required quota (see Table 1).

To qualitatively understand this result, consider the minister representing the largest country G with a population of 49 millions. In the double majority system

Table 1 *Case study* voting in the council of a model union of 7 members under a hypothetical distribution of population and voting preferences

State	*A*	*B*	*C*	*D*	*E*	*F*	*G*	Total	
Population (Million)	1	1	1	16	16	16	49	100	
Votes: Yes (Million)	2/3	2/3	2/3	4	4	4	25	39	
Votes: No (Million)	1/3	1/3	1/3	12	12	12	24	61	
State votes	1	1	1	0	0	0	1	4/7	Y
Minister's votes	1	1	1	0	0	0	49	52/100	Y
Square root weights	1	1	1	4	4	4	7	22	
Square root votes	1	1	1	0	0	0	7	10/22	N

Although 61 % of the total population of the union is against a legislative it will be taken by the council, if the rules of the double majority are used. The outcome of the voting according to the weighted voting system of Penrose correctly reflects the will of the majority in the union he uses his 49 votes against the will of 24 millions of inhabitants. By contrast, the minister of the small state A will misrepresent at most one half of the million of his compatriots. In other words, the precision in determining the will of all the citizens is largest in the smaller states, so the vote of their ministers should gain a higher weight than proportional to population, which is the case in the Penrose system.

4 Optimal Quota for Qualified Majority

Designing a voting system for the Council one needs to set the threshold for the qualified majority. In general, this quota can be treated as a free parameter of the system and is often considered as a number to be negotiated. For political reasons one usually requires that the voting system should be *moderately conservative*, so one considers the quota in the wide range from 55 to 75 %.

However, designing the voting system based on the theory of Penrose, one can find a way to obtain a single number as the optimal value of the quota. In order to assure that the voting powers of all citizens in the 'union' are equal one has to impose the requirement that the voting power of each member state should be proportional to the square root of the population of each state.

Let us analyze the problem of M members of the voting body, each representing a state with population N_i, $i = 1, \ldots, M$. Denote by w_i the voting weight attributed to each representative. We work with renormalized quantities, such that $\sum_{i=1}^{M} w_i = 1$. Assume that the decision of the voting body is taken, if the sum of the weights w_i of all members of the coalition exceeds the given quota q.

In the Penrose voting system one sets the voting weights proportional to the square root of the population of each state, $w_i \sim \sqrt{N_i}$ for $i = 1, , M$. For any level of the quota q one may compute numerically the power indices β_i. To characterize the overall representativeness of the voting system one may use various indices designed to quantify the resulting inequality in the distribution of power among citizens (Laruelle and Valenciano 2002). Analyzing the influence of the quota q for the average inequality of the voting power we are going to use the mean discrepancy Δ, defined as:

$$\Delta := \sqrt{\frac{1}{M}\sum_{i=1}^{M}(\beta_i - w_i)^2}, \tag{8}$$

If the discrepancy Δ is equal to zero, the voting power of each state is proportional to the square root of its population. Under the assumption that the Penrose law is fulfilled, in such a case the voting power of any citizen in each state is the same.

In practice, the coefficient Δ will not be exactly equal to zero, but one may try to minimize this quantity. The optimal quota q_* can be defined as the quota for which the discrepancy Δ is minimal. Let us note, however, that this definition works fine for the Banzhaf index, while the dependence of the Shapley–Shubik index (Shapley and Shubik 1954) on the quota does not exhibit such a minimum.

Table 2 Optimal quota q_n for the Council of the European Union of M member states compared with predictions q_{av} of the approximate formula (16) and the lower bound q_{min} given in (10)

M	25	27	28	29	...	$M \to \infty$
$q_n(\%)$	62.16	61.58	61.38	61.32	...	50.0
$q_{av}(\%)$	61.28	60.86	60.66	60.48	...	50.0
$q_{min}(\%)$	60.00	59.62	59.45	59.28	...	50.0

The calculations of the optimal quotas for the EU were based upon the Eurostat data on the distribution of population for the EU-25 (2004) and the EU-27 (2010). The extended variant EU-28 contains EU-27 and Croatia, while EU-29 includes also Iceland

Studying the problem for a concrete distribution of the population in the European Union, it was found (Słomczyński and Życzkowski 2004) that in these cases all M ratios β_i/w_i for $i = 1, \ldots, M$, plotted as a function of the quota q, cross approximately near a single point. In other words, the discrepancy Δ at this critical point q_* is negligible. Numerical analysis allows one to conclude that this optimal quota is approximately equal to 62 % for the EU-25 (Słomczyński and Życzkowski 2004). At this very level of the quota the voting system can be considered as optimal, since the voting power of all citizens becomes equal. Performing detailed calculations one needs to care to approximate the square root function with a sufficient accuracy, since the rounding effects may play a significant role (Kurth 2007).

It is worth to emphasize that in general the value of the optimal quota decreases with the number of member states. For instance, in the case of the EU-27 is is equal to 61.5 % (Życzkowski et al. 2006; Słomczyński and Życzkowski 2007), see Table 2. The optimal quota was also found for other voting bodies including various scenarios for an EU enlargement—see Leech and Aziz (2010). Note that the above results belong to the range of values of the quota for qualified majority, which are used in practice or recommended by experts.

4.1 Large Number of Member States and a Statistical Approximation

Further investigation has confirmed that the existence of such a critical point is not restricted to the concrete distribution of the population in European Union. On the contrary, it was reported for a model union containing M states with a random distribution of population (Słomczyński and Życzkowski 2004; Chang et al. 2006; Słomczyński and Życzkowski 2006). However, it seems unlikely that we can obtain an analytical expression for the optimal quota in such a general case. If the number of member states is large enough one may assume that the distribution of the sum of the weights is approximately Gaussian (Owen 1975; Feix et al. 2007; Słomczyński and Życzkowski 2007). Such an assumption allowed us to derive an explicit approximate formula for the optimal quota for the Penrose square root voting system (Słomczyński and Życzkowski 2007)

$$q_n \ := \ \frac{1}{2}\left(1+\frac{\sqrt{\sum_{i=1}^{M} N_i}}{\sum_{i=1}^{M}\sqrt{N_i}}\right), \tag{9}$$

where N_i denotes the population of the i–th state. In practice it occurs that already for $M = 25$ this approximation works fine and in the case of the EU-25 gives the optimal quota with an accuracy much better than one percent. Although the value of the optimal quota changes with M, the efficiency of the system, measured by the probability of forming the winning coalition, does not decrease if the union is enlarged. It was shown in Słomczyński and Życzkowski (2007) that, according to the central limit theorem, the efficiency of this system tends to approximately $15.9\,\%$ if $M \to \infty$.

It is not difficult to prove that for any fixed M the above expression attains its minimum if the population of each member state is the same, $N_i = const(i)$. In this way one obtains a lower bound for the optimal quota as a function of the number of states (Słomczyński and Życzkowski 2007):

$$q_{min} \ := \ \frac{1}{2}\left(1+\frac{1}{\sqrt{M}}\right). \tag{10}$$

Note that the above bound decreases with the number of the states forming the union as $1/\sqrt{M}$ to $50\,\%$. Such a behavior, reported in numerical analysis of the problem (Słomczyński and Życzkowski 2004; Chang et al. 2006; Słomczyński and Życzkowski 2006) is consistent with the so-called Penrose limit theorem—see Lindner and Machover (2004).

4.2 Optimal Quota Averaged over an Ensemble of Random States

Concrete values of the optimal quota obtained by finding numerically the minimum of the discrepancy (8) for the EU-25 and the EU-27 Słomczyński and Życzkowski (2004, 2006, 2010) are consistent, with an accuracy up to two per cent, with the data obtained numerically by averaging over a sample of random distribution of the populations of a fictitious union. This observation suggests that one can derive analytically an approximate formula for the optimal quota by averaging the explicit expression (9) over an ensemble of random populations N_i.

To perform such a task let us denote by x_i the relative population of a given state, $x_i = N_i/\sum_{i=1}^{M} N_i$. Since $\sqrt{N_i}/\sqrt{\sum_{i=1}^{M} N_i} = \sqrt{x_i}$ one can rewrite expression (9) in the new variables to obtain

$$q_n(\overrightarrow{x}) = \frac{1}{2}\left(1+\frac{1}{\sum_{i=1}^{M}\sqrt{N_i}/\sqrt{\sum_{i=1}^{M}N_i}}\right) = \frac{1}{2}\left(1+\frac{1}{\sum_{i=1}^{M}\sqrt{x_i}}\right). \qquad (11)$$

By construction, $\overrightarrow{x} = (x_1, \ldots, x_M)$ forms a probability vector with $x_i \geq 0$ and $\sum_{i=1}^{M} x_i = 1$. Hence the entire distribution of the population of the union is characterized by the M-point probability vector $\overrightarrow{x}$, which lives in an $(M-1)$ dimensional simplex Δ_M. Without any additional knowledge about this vector we can assume that it is distributed uniformly on the simplex,

$$P_D(x_1, \ldots, x_M) = \frac{1}{(M-1)!}\,\delta\left(1-\sum_{i=1}^{M}x_i\right). \qquad (12)$$

Technically it is a particular case of the *Dirichlet distribution*, written $P_D(\overrightarrow{x})$, with the Dirichlet parameter set to unity.

In order to get a concrete result one should then average expression (11) with the flat probability distribution (12). Result of such a calculation can be roughly approximated by substituting M-fold mean value over the Dirichlet measure, $M\langle\sqrt{x}\rangle_D$, instead of the sum into the denominator of the correction term in (11),

$$q_{av}(M) := \langle q_n\rangle_D \approx \frac{1}{2}\left(1+\frac{1}{M\langle\sqrt{x}\rangle_D}\right). \qquad (13)$$

The mean square root of a component of the vector $\overrightarrow{x}$ is given by an integral with respect to the Dirichlet distribution

$$\langle\sqrt{x}\rangle_D = \int_{\Delta_M} \sqrt{x_1}\,P_D(x_1, \ldots, x_M)\,dx_1 \cdots dx_M. \qquad (14)$$

Instead of evaluating this integral directly, we shall rely on some simple fact from the physical literature. It is well known that the distribution of the squared absolute values of an expansion of a random state in an M-dimensional complex Hilbert space is given just by the flat Dirichlet distribution (see e.g. Bengtsson and Życzkowski 2006). In general, all moments of such a distribution where computed by Jones (1991). The average square root is obtained by taking his expression from Jones and setting $d = M$, $l = 1$, $v = 2$ and $\beta = 1/2$. This gives the required average

$$\langle\sqrt{x}\rangle_D = \frac{\Gamma(M)\,\Gamma(3/2)}{\Gamma(M+1/2)} \sim \frac{\sqrt{\pi}}{2\sqrt{M}}. \qquad (15)$$

Here Γ denotes the Euler gamma function and the last step follows from its Stirling approximation. Substituting the average $\langle\sqrt{x}\rangle_D$ into (13) we arrive at a compact expression

$$q_{av}(M) \approx \frac{1}{2} + \frac{1}{\sqrt{\pi M}} = \frac{1}{2}\left(1 + \frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{M}}\right). \tag{16}$$

This approximate formula for the mean optimal quota for the Penrose voting system in a union of M random states constitutes the central result of this work. Note that this expression is averaged over all possible distributions of populations in the union, so it depends only on the size M of the union and on the form of averaging. The formula has a similar structure as the lower bound (10), but the correction term is enhanced by the factor $2/\sqrt{\pi} \approx 1.128$. In some analogy to the famous *Buffon's needle (or noodle) problem* (Ramaley 1969), the final result contains the number π—it appears in (16) as a consequence of using the normal approximation. The key advantage of the result (16) is due to its simplicity. Therefore, it can be useful in a practical case, if the size M of the voting body is fixed, but the weights of the voters (e.g. the populations in the EU) vary.

5 Concluding Remarks

In this work we review various arguments leading to the weighted voting system based upon the square root law of Penrose. However, the key result consists in an approximate formula for the mean optimal threshold of the qualified majority. It depends only on the number M of the states in the union, since the actual distribution of the population is averaged out.

Making use of this result we are in a position to propose a simplified voting system. The system consists of a single criterion only and is determined by the following two rules:

(1) Each member of the voting body of size M is attributed his voting weight proportional to the square root of the population he represents.
(2) The decision of the voting body is taken if the sum of the weights of members of a coalition exceeds the critical quota $q = 1/2 + 1/\sqrt{\pi M}$.

This voting system is based on a single criterion. Furthermore, the quota depends on the number of players only, but not on the particular distribution of weights of the individual players. This feature can be considered as an advantage in a realistic case, if the distribution of the population changes in time. The system proposed is objective and it cannot a priori handicap a given member of the voting body. The quota for qualified majority is considerably larger than 50 % for any size of the voting body of a practical interest. Thus the voting system is moderately conservative, as it should be. If the distribution of the population is known and one may assume that it is invariant in time, one may use a modified rule (2') and set the optimal quota according to the more precise formula (9).

Furthermore, the system is transparent: the voting power of each member of the voting body is up to a high accuracy proportional to his voting weight. However, as a crucial advantage of the proposed voting system we would like to emphasize its extendibility: if the size M of the voting body changes, all one needs to do is to set the voting weights according to the square root law and adjust the quota q according to the rule (2'). Moreover, for a fixed number of players, the system does not depend on the particular distribution of weights. This feature is specially relevant for voting bodies in corporate management for which the voting weights may vary frequently.

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