Truncations of random unitary matrices

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Abstract. We analyse properties of non-Hermitian matrices of size M constructed as square submatrices of unitary (orthogonal) random matrices of size $N > M$, distributed according to the Haar measure. In this way we define ensembles of random matrices and study the statistical properties of the spectrum located inside the unit circle. In the limit of large matrices, this ensemble is characterized by the ratio M/N. For the truncated CUE we analytically derive the joint density of eigenvalues and all correlation functions. In the strongly non-unitary case universal Ginibre behaviour is found. For $N - M$ fixed and N to ∞ the universal resonance-width distribution with $N - M$ open channels is recovered.

1. Introduction

Random unitary matrices may be applied to describe chaotic scattering [1], conductance in mesoscopic systems [2] or statistical properties of periodically driven quantum systems (see [3] and references therein). They can be defined by circular ensembles of unitary matrices introduced by Dyson [4]. He defined circular orthogonal, unitary or symplectic ensembles (COE, CUE and CSE), which display different transformation properties [5]. For these ensembles the distribution of matrix elements and their correlations are known [6–8].

In this paper we discuss properties of non-Hermitian matrices defined as square submatrices of size M of unitary (orthogonal) matrices of size N, where $N > M$. These matrices may be considered as unitary (orthogonal) matrices with $N - M$ bottom rows and $N - M$ last columns truncated. Let $U_{[N,M]}$ denote such a $M \times M$ matrix obtained from a unitary matrix, while $O_{[N,M]}$ is obtained by truncating an orthogonal matrix. The truncated matrices are non-unitary by construction, and their eigenvalues are located inside the unit circle.

Motivation for such a study stems from the problems of chaotic scattering. Consider a mesoscopic device coupled to two leads, each of which supports $N/2$ open channels. The process of scattering can be described by a unitary S-matrix of size N. In the diffusive regime the scattering matrix pertains to an appropriate circular ensemble [2]. The reflection (transmission) matrix of size $M = N/2$ may just be considered as a truncation of the unitary S-matrix. The random matrix approach to resonances in chaotic scattering was recently presented in [9]. In particular, the distribution of width of resonance in the presence of L open channels was derived in the weakly non-Hermitian limit for broken time-reversal symmetry.

In recent papers [10, 11] the authors introduce $N \times N$ unitary matrices enlarged in an asymmetric way to the size $(N + L) \times (N + L)$ by adding L *upper* rows and L *last* columns with all elements equal to zero. These matrices are used to describe the chaotic scattering in

a 1D crystal electron model in ac and dc fields. It is easy to see that the spectrum of such an *enlarged* matrix consists of $2L$ zeros and $M = N - L$ complex eigenvalues of the *truncated* matrix $U_{[N,N-L]}$. Our results are therefore directly applicable to the problems analysed in those papers.

Related problems arise by analysing the time evolution of periodically perturbed systems. The model of the kicked rotator with absorbing boundaries was studied in [12–14]. In this case the presence of the absorbing boundaries corresponds to the truncation of the infinite evolution matrix. Another line of research is related to the Frobenius–Perron (FP) operators describing the evolution of densities under a classical map. If the map is area preserving, the FP operator is represented by an infinite unitary matrix, which for practical purposes is truncated to a finite size. Properties of the spectra of such truncated matrices have recently attracted considerable attention [15, 16].

This paper is organized as follows. In section 2 we analyse truncations of orthogonal matrices and show a geometric interpretation of this problem. We derive the probability distributions of the radii of points uniformly covering a given hypersphere and projected into a smaller space. Section 3 is devoted to truncations of random unitary matrices. We demonstrate a link between the distributions studied and the eigenvector statistics. In section 4 we present numerical results concerning the distribution of the complex eigenvalues of the truncated matrices. We show to what extent the ratio $\mu = M/N$ determines the properties of the truncated matrix. In section 5 we analytically derive the joint density of eigenvalues for truncated matrices of CUE. From this a kernel is derived which determines all correlation functions. In particular, the averaged density of eigenvalues is obtained for arbitrary dimensions and truncations. In the strongly non-unitary limit $M \to \infty$, N/M *fixed* the correlations of the Ginibre ensemble [24] are obtained rescaled by the local mean level distance, which are thus revealed as universal. The weakly non-unitary limit $N - M$ fixed and M to ∞ recovers the universal distribution of resonance widths in the weakly non-Hermitian case for broken time-reversal symmetry [9] and the corresponding correlations [29]. The truncations of symmetric matrices of COE are briefly discussed in section 6. The convolution properties of the derived distributions are presented in the appendix.

2. Submatrices of random orthogonal matrices

Let us start the discussion by considering a simple geometric exercise. Random points uniformly cover a hypersphere S^{N-1} of radius 1 embedded in R^N. After an orthogonal projection into R^M, where $M < N$, they are localized inside the hypersphere S^{M-1} or at its surface. What is the radial probability distribution $P_{N,M}(t)$, where t denotes the distance of a projected point from the origin?

It is helpful to analyse first the most intuitive case $N = 3$, $M = 2$. The surface element of the sphere S^2 in spherical coordinates reads $d\Omega_2 = \sin \theta \, d\theta \, d\phi$. The orthogonal projection maps the points of the sphere into a plane. Their distance from the origin is $t = \sin \theta$, which allows us to find the required distribution

$$P_{3,2}(t) = \frac{t}{2\sqrt{1 - t^2}}.$$ (1)

Analogously we get $P_{3,1}(t) = 1$ for $t \in [0, 1]$ and $P_{2,1}(t) = 1/(2\pi \sqrt{1 - t^2})$.

The general formula for $P_{N,M}(t)$ may be obtained in a similar way from the element of the hypersphere S^{N-1}:

$$d\Omega_{N-1} = d\varphi \, \Pi_{k=1}^{N-2} \sin^k \theta_k \, d\theta_k.$$ (2)
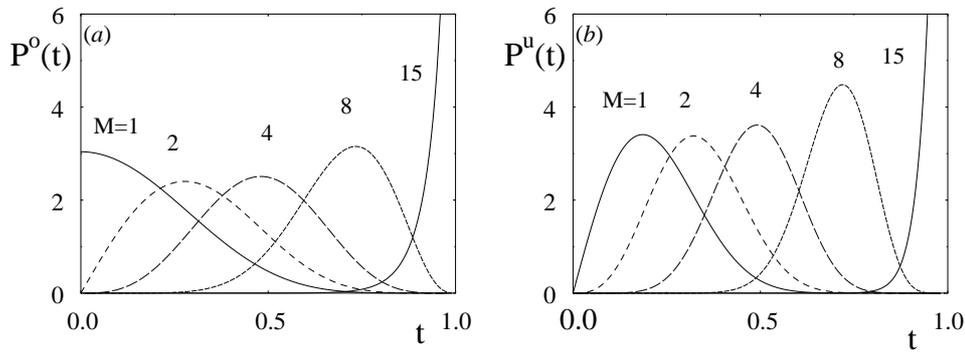
Figure 1. (*a*) Probability distribution of the radii t after projection from R^{16} to R^M with $M = 1, 2, 4, 8, 15$. The variable t^2 is equal to the sum of M squared elements of a random orthogonal matrix distributed according to the Haar measure on $O(16)$. (*b*) Analogous distributions $P^u(t)$ obtained from random unitary matrices pertaining to $U(16)$.

Integrating out $N - M$ variables we obtain

$$P^o_{N,M}(t) = c_{N,M} t^{M-1} (1 - t^2)^{(N-M-2)/2} \tag{3}$$

where the normalization constant can be expressed by the Euler beta function $B(x, y)$ [17]

$$c_{N,M} = \frac{2}{B(\frac{M}{2}, \frac{N-M}{2})} = \frac{2\Gamma(\frac{N}{2})}{\Gamma(\frac{M}{2})\Gamma(\frac{N-M}{2})}. \tag{4}$$

Convolution relations between the distributions $P^o_{N,M}(t)$ are demonstrated in the appendix.

Consider an orthogonal matrix O of size N. Its first column can be interpreted as a vector $x_k = O_{k1}$ of coordinates determining a point on the hypersphere S^{N-1}. Let us call by $O_{[N,M]}$ the upper left submatrix of O of size $M < N$. The total length of the vector represented by the first column of $O_{[N,M]}$ and given by $t = \sqrt{\sum_{k=1}^{M} x_k^2}$ is just equal to the defined above distance of a point projected from the hypersphere S^{N-1} into the interior of S^{M-1} from the origin. If O are distributed uniformly with respect to the Haar measure on $O(N)$, than the points x uniformly cover the hypersphere. The distributions $P^o_{N,M}(t)$ are then given by equation (3).

Figure 1(*a*) shows these distributions for $N = 16$ and $M = 1, 2, 4, 8$ and 15. With increasing M the probability distribution is shifted to the right. For $M = N$ the matrix remains unitary and $P_{N,N}(t) = \delta(t - 1)$. Let us now consider an ensemble $O_{[N,\mu N]}$ by increasing the dimension N and keeping the ratio $\mu = M/N$ fixed, where $\mu < 1$. Straightforward integration allows us to compute the expectation value of t for this ensemble:

$$\langle t \rangle_{N,\mu N} = \frac{\Gamma(\frac{N}{2})\Gamma(\frac{\mu N}{2} + \frac{1}{2})}{\Gamma(\frac{N}{2} + \frac{1}{2})\Gamma(\frac{\mu N}{2})} \tag{5}$$

which in the limit $N \to \infty$ tends to $\sqrt{\mu}$. The second moment reads $\langle t^2 \rangle_{N,\mu N} = \mu$, thus the variance tends to zero in the limit of large matrices. This result is quite intuitive in view of the central limit theorem.

3. Submatrices of random unitary matrices

Let $U_{[N,M]}$ denote the $M \times M$ matrix obtained by truncation of the $N \times N$ unitary matrix U. In a similar way we define $t = \sqrt{\sum_{k=1}^{M} |U_{k1}|^2}$. In this case, to find the distribution $P^u_{N,M}(t)$

it is useful to represent a unitary matrix $U(N)$ as a product (the normal product ($\times$) or the twisted product ($\ltimes$)) of the hyperspheres [18]

$$U(N) \sim S^1 \times S^3 \ltimes \cdots \ltimes S^{2N-3} \ltimes S^{2N-1}. \tag{6}$$

Truncation of the dimension of a unitary matrix by one corresponds to the projection from S^{2N-1} to S^{2N-3}, which is equivalent to the truncation of the matrix $O(2N)$ by two. The same argument works for any size M of the truncated matrix. Therefore $P^u_{N,M}(t) = P^o_{2N,2M}(t)$ and

$$P^u_{N,M}(t) = c_{2N,2M} t^{2M-1}(1 - t^2)^{N-M-1} \tag{7}$$

with the normalization constants given by (4). Some of these distributions for $N = 16$ are plotted in figure 1(*b*). Expectation values $\langle t \rangle$ are asymptotically the same for both ensembles, while the variance is smaller for the ensemble of truncated unitary matrices $U_{[N,M]}$.

For a fixed value of N we defined the ensembles of truncated matrices for each integer value of $M \in [1, N]$. However, to study the evolution of a spectrum of a given matrix it is convenient to define an ensemble depending on a continuous parameter. This can be achieved in several different ways. For example, one may multiply the last column and the last vector of the matrix $U_{[N,M]}$ by a parameter $p \in [0, 1]$, which mimics a continuous transition from M to $M - 1$ [16].

Taking $M = 1$ the variable t is just the absolute value of the first element of a matrix $|U_{11}|$. It is known [5, 19] that a unitary matrix of eigenvectors of a CUE matrix is distributed according to the Haar measure on $U(N)$, while the orthogonal matrix of eigenvectors of a matrix typical of COE is distributed according to the Haar measure on $O(N)$. To establish a link with the eigenvector statistics let us set $M = 1$ and consider the distributions $P^o_{N,1}(t)$ and $P^u_{N,1}(t) = P^o_{2N,2}(t)$. Putting $y = t^2$ and changing the variable we arrive at the known formulae

$$P^o_N(y) = \frac{\Gamma(\frac{N}{2})}{\Gamma(\frac{N-1}{2})} \frac{(1 - y)^{(N-3)/2}}{\sqrt{\pi y}} \tag{8}$$

and

$$P^u_N(y) = (N - 1)(1 - y)^{N-2} \tag{9}$$

which describe the eigenvector statistics for the orthogonal and the unitary ensemble [20, 21]. In the limit $N \to \infty$ they converge to the χ^2_ν distributions with the number of degrees of freedom ν equal to 1 and 2, respectively. The former case is often known in the literature as the Porter–Thomas distribution.

4. Distribution of eigenvalues

Consider spectra of the truncated orthogonal matrices $O_{[N,M]}$ and truncated unitary matrices $U_{[N,M]}$. In both cases there exist M complex eigenvalues $z_j = r_j \exp(i\phi_j)$ localized inside (or at) the unit circle. This is due to the fact that the norm of the truncation is smaller than or equal to the norm of the initial matrix. For the truncations of random matrices of CUE there exist an rotational symmetry, $U \to U \exp(i\alpha)$. Therefore, $P(\phi) = \text{const}$, consequently we will study the radial distribution $P(r)$. Sometimes it is convenient to write $r = e^{(-\gamma/2)}$ and to study the distribution $P(\gamma)$ of the 'level widths' γ [12]. For any fixed N the limiting cases are known: for $M = 1$ the eigenvalues are trivial, $r = t$, so for both ensembles $P_{N,1}(r) = P_{N,1}(t)$. For $M = N$ the matrix is unitary and thus $P_{N,N}(r) = \delta(r - 1)$ or, in other variables, $P_{N,N}(\gamma) = \delta(\gamma)$.

Figure 2 presents 2000 eigenvalues of the matrices truncated out of CUE matrices of size 5. For $M = 4$ there exist several eigenvalues close to the unit circle, while for stronger truncation ($M = 2$) the eigenvalues are clustered closer to the origin.
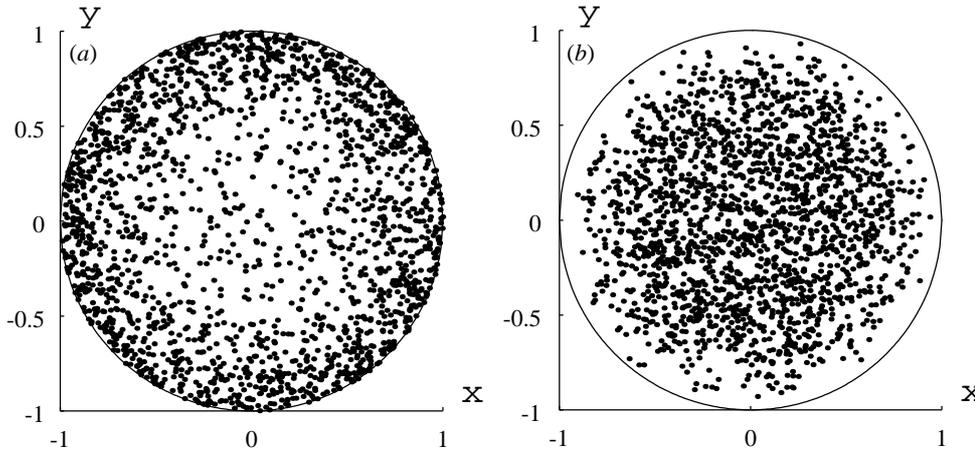
Figure 2. Complex eigenvalues of random matrices pertaining to the ensembles (*a*) $U_{[5,4]}$ and (*b*) $U_{[5,2]}$.
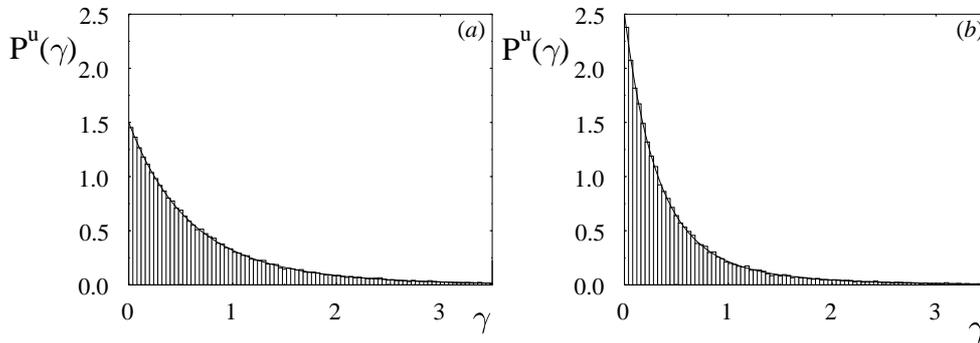

Figure 3. Distribution of decay times $P(\gamma)$ for random matrices pertaining to the ensembles (*a*) $U_{[3,2]}$ and (*b*) $U_{[5,4]}$.

In the simplest interesting case, $N = 3$ and $M = 2$, the data for the truncations of unitary matrices conform to the distribution $P_{3,2}^u(r) = r + 2r^3$. For comparison with the results of [12] we present the numerical data as the distribution $P(\gamma)$. The above distribution, derived in the following section, corresponds to the biexponential decay

$$P_{3,2}^u(\gamma) = \tfrac{1}{2} \exp(-\gamma) + \exp(-2\gamma) \tag{10}$$

represented by a solid curve in figure 3(*a*). Numerical data obtained for the ensemble $U_{[5,4]}$, shown in figure 3(*b*), are compared with the distribution $P_{5,4}^u(\gamma) = \tfrac{1}{4} \exp(-\gamma) + \tfrac{1}{2} \exp(-2\gamma) + \tfrac{3}{4} \exp(-3\gamma) + \exp(-4\gamma)$, which corresponds to $P_{5,4}^u(r)$ discussed below.

Distributions $P(r)$ for both ensembles obtained with $N = 16$ and some intermediate values of M are presented in figure 4. The histograms are performed out of 10^4 random unitary (orthogonal) matrices constructed according to the algorithm given in [22]†. The statistics

† In [22] a misprint occurred in the algorithm for generating random matrices typical of CUE. The corrected version (the indices in appendix B changed according to $r \to r + 1$) can be found in [23].
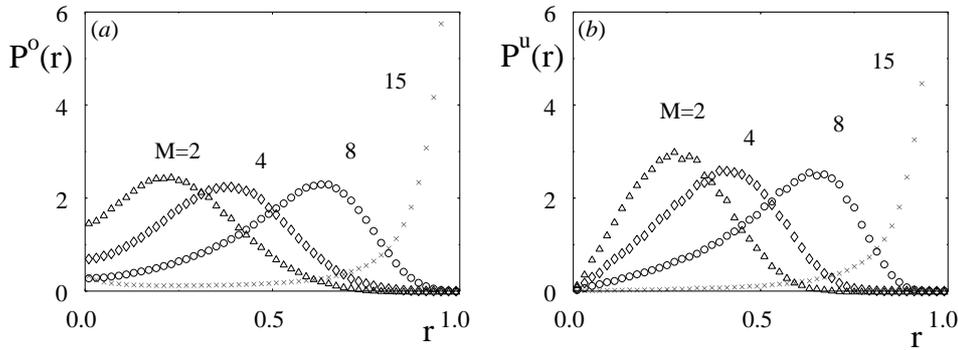
Figure 4. (*a*) Probability distribution of the radii *r* of the complex eigenvalues of the $M \times M$ sub-blocks of random matrices of $O(16)$. (*b*) Analogous distributions $P^u(r)$ obtained from random unitary matrices pertaining to $U(16)$.

obtained do not depend on which columns and rows of the initially unitary (orthogonal) matrix are removed during the truncation. This is due to the fact that the Haar measure on the unitary (or orthogonal) group is invariant with respect to multiplication by the permutation matrices, which change the order of the columns and vectors.

With increase of M the distribution $P(r)$ extends to the larger values of r. In contrast with the distributions $P(t)$, for any M there exists a non-zero probability of finding small values of r. For small values r the distribution $P^u(r)$ grows linearly with r. This is a purely geometric factor (we analyse the distribution at the complex plane), which corresponds to the uniform density of eigenvalues close to the origin.

The data collected for large matrices reveal a scaling behaviour: the distribution $P_{N,M}(r)$ depends only on the ratio $\mu = M/N$. Figure 5 shows the distributions $P^u_{N,N/2}(r)$ and $P^u_{N,N/4}(r)$ obtained from ensembles of random unitary matrices of different sizes. The larger value of N, the sharper is the cut-off of the probability at the critical radius $r_\mu = \sqrt{\mu}$. In analogy to the properties of the Ginibre ensemble one expects an infinitely sharp edge of the distribution in the limit $N \to \infty$. In the case of large matrices the spectrum covers the entire circle of radius r_μ, while the density is largest close to the rim.

For $\mu \ll 1$ (and N large) the radial distribution may be approximated by a linear function $P^u_\mu(r) \sim 2r/\mu$ with a cut-off at r_μ. This property is characteristic of the Ginibre ensemble [24], constructed of non-Hermitian random matrices with no correlations between their elements. It is thus intuitive to expect, that for large N the constraints stemming from the unitarity of $U(N)$ do not induce very strong correlations between elements of a much smaller matrix of size M.

Eigenvalues of several truncations of random orthogonal matrices are shown in figure 6. Since the truncated matrix is real the eigenvalues are real or appear in complex conjugate pairs, $re^{i\phi}$, $re^{-i\phi}$. Therefore these spectra exhibit the symmetry along the real line. One can observe a clustering of eigenvalues along this line. The fraction of real eigenvalues equals approximately 0.65, 0.38 and 0.68 for the ensembles $O_{[3,2]}$, $O_{[5,4]}$ and $O_{[5,2]}$, respectively. This fact explains a positive probability $P^o(r)$ for $r = 0$ visible in figure 4(*a*).

If the ratio μ is kept constant, the relative number of real eigenvalues decreases with the matrix size. A similar effect is known in the theory of random polynomials. Kac considered the random polynomials of order M with real coefficients, being independent random variables drawn according to the normal distribution. He showed [25] that the fraction of real roots
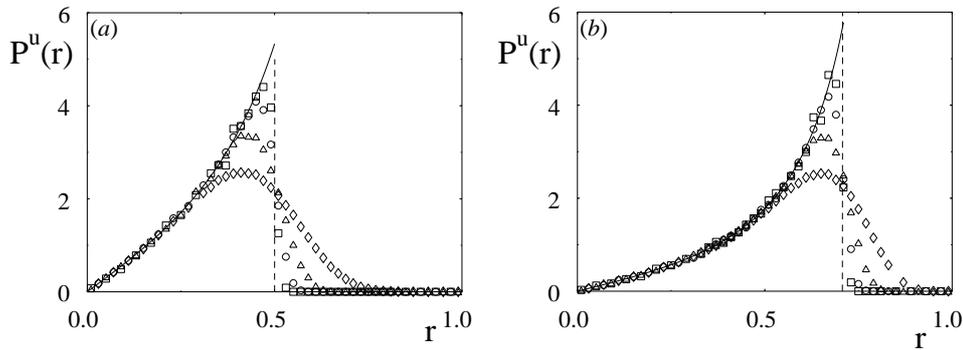
Figure 5. Probability distribution $P^u(r)$ of radii of eigenvalues of matrices $U_{[N,M]}$ constructed from unitary matrices of size $N = 16(\lozenge)$, $64(\triangle)$, $256(\bigcirc)$ and $1024(\square)$ with (*a*) $M = N/4$ and (*b*) $M = N/2$. Dashed lines represent the asymptotic cut-off at $r = r_\mu$ and the solid curves denote the distribution (19).
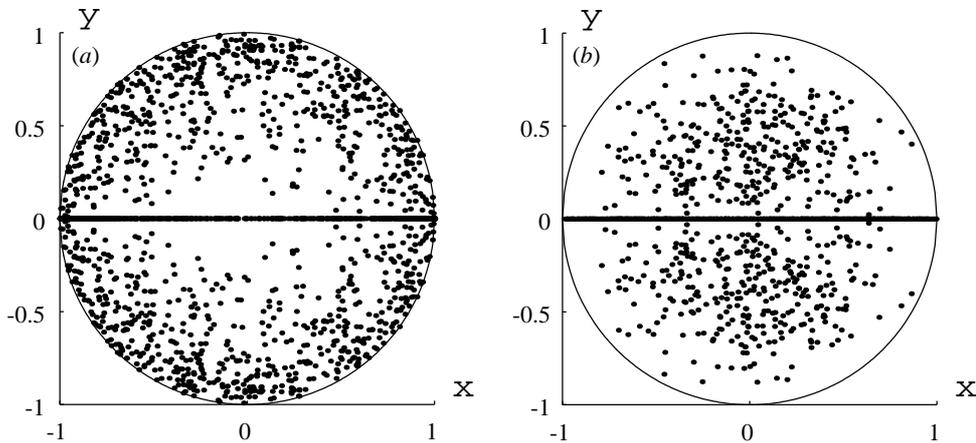

Figure 6. As in figure 2 for truncations of random orthogonal matrices (*a*) $O_{[5,4]}$ and (*b*) $O_{[5,2]}$. Note clustering of eigenvalues at the real axis.

decreases as $(\ln M)/M$. Our problem is not exactly the same since the real coefficients of the secular polynomial of the truncated random matrix are not Gaussian, nor independent random variables. In spite of this fact, our numerical results suggest that the fraction of the real eigenvalues of truncations of orthogonal matrices $O_{[2M,M]}$ decreases as $(\ln(M)/M)$. A recent discussion of properties of random polynomials and their applications to quantum chaos may be found in [26]. The issue of clustering of zeros of random polynomials along a given curve and its relation to the time-reversal symmetry is discussed in [27].

In the limit of large matrices the relative strength of the clustering of the complex eigenvalues along the real axis decreases and the distribution $P(\phi)$ becomes uniform. Moreover, the radial distribution $P^o(r)$ becomes close to the distribution (19) derived for truncations of unitary matrices. Although for $N = 16$ the differences between the distributions $P^o_\mu(r)$ and $P^u_\mu(r)$ are significant, especially for small values of r, for large N the data for both ensembles seem to converge.

5. Analytical results for truncations of CUE

Now we derive analytical results for the truncated circular unitary ensemble. Let

$$U = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

be an $N \times N$ matrix from this ensemble and A a subunitary $M \times M$ matrix. The joint density of elements of U can be written as

$$P(U) \propto \delta(A^\dagger A + C^\dagger C - 1)\delta(A^\dagger B + C^\dagger D)\delta(B^\dagger B + D^\dagger D - 1) \tag{11}$$

with appropriate matrix δ-functions. Integrating out B and D we obtain as joint density of elements of A

$$P(A) \propto \int dC\delta(A^\dagger A + C^\dagger C - 1) \tag{12}$$

with a $2M(N - M)$-dimensional integration over the complex parameters C. The matrix A may be brought to upper triangular form by a unitary transformation T: $A = T(z + \Delta)T^{-1}$, where z is a diagonal matrix consisting of the complex eigenvalues of A and Δ is strictly upper triangular (Schur decomposition). The transformation can be made unique restricting T to a certain cosetspace. The Jacobian of this transformation is given by the square of the Vandermonde determinant [28]

$$|V|^2 = \prod_{i<j} |z_i - z_j|^2$$

such that after integrating out the unitary transformations T the joint density of eigenvalues is given by

$$P(z) \propto |V|^2 \int dC \int d\Delta\delta((z^\dagger + \Delta^\dagger)(z + \Delta) + C^\dagger C - 1). \tag{13}$$

In the following we first integrate out the $M(M - 1)/2$ complex parameters Δ_{ij} and then the $M(N - M)$ complex parameters C written as complex vectors C_i. For $i < j$ we have the hierarchical equations

$$z_i^* \Delta_{ij} + C_i^\dagger C_j + \sum_{k<i} \Delta_{ki}^* \Delta_{kj} = 0. \tag{14}$$

Integration over Δ yields the Jacobian $\prod_{i<j} |z_i|^{-2}$ and a product of δ-functions:

$$\prod_i \delta(|z_i|^2 + C_i^\dagger X_i C_i - 1) \tag{15}$$

where X_i denotes an $(N - M) \times (N - M)$ matrix defined by the quadratic form $C_i^\dagger X_i C_i$ which is given by

$$C_i^\dagger C_i + \sum_{k<i} \Delta_{ki}^* \Delta_{ki}$$

containing the solution Δ_{ki} of equation (14) and depending otherwise on C_k only for $k < i$. The integration over C_i can now be done successively starting from C_M and yields the factor

$$\prod_{i=1}^M (1 - |z_i|^2)^{N-M-1}\Theta(1 - |z_i|^2)/\det(X_i)$$

where $\Theta(.)$ denotes the Heaviside step function, $\Theta(x) = 1$ for $x > 0$ and zero otherwise.

For $\det(X_i)$ we can derive from the equations (14) for Δ, and using implicitly the δ-functions (15) the recursive relation

$$\det(X_{i+1}) = \det(X_i)/|z_i|^2$$

with $\det(X_1) = 1$. Thus the previous Jacobian $\prod_{i<j} |z_i|^{-2}$ will be compensated and the final very simple and important result is

$$P(z) \propto \prod_{i<j}^{1...M} |z_i - z_j|^2 \prod_{i=1}^{M} (1 - |z_i|^2)^{N-M-1} \Theta(1 - |z_i|^2).$$ (16)

This result is completely analogous to the Ginibre ensemble and we immediately know all the correlation functions by the method of orthogonal polynomials [5]. Here the powers z^{n-1} are already orthogonal. An equivalent method is to consider the joint density $P(z)$ as the absolute square of a Slater determinant of normalized wavefunctions

$$\phi_n(z) = z^{n-1} w(|z|^2) / \sqrt{N_n}$$

with

$$w(x) = (1 - x)^{(N-M-1)/2} \Theta(1 - x)$$

where N_n stands for an normalization factor.

The kernel, which determines all correlation functions is [5, 28]

$$K(z_1, z_2^*) = \sum_{n=1}^{M} (z_1 z_2^*)^{n-1} w(|z_1|^2) w(|z_2|^2) / N_n.$$

For example, the cluster function is given by $Y(z_1, z_2) = |K(z_1, z_2^*)|^2$ and the averaged density of eigenvalues z normalized to 1 is given by

$$\rho(z) = K(z, z^*)/M = \frac{1}{M} \sum_{n=1}^{M} |z|^{2n-2} w^2(|z|^2) / N_n.$$ (17)

The normalization factor N_n is easily calculated as

$$N_n = \pi (n - 1)! (N - M - 1)! / (N - M + n - 1)!.$$

For example, with $r^2 = |z|^2$ we obtain for the distribution of r with $M = N - 1$

$$P(r) = \frac{2}{M} (r + 2r^3 + 3r^5 + \cdots + M r^{2M-1})$$

and in general with $x = r^2$

$$P(r) = \frac{2r}{M} \frac{(1-x)^{N-M-1}}{(N-M-1)!} \left(\frac{d}{dx} \right)^{N-M} \frac{(1-x^N)}{1-x}.$$ (18)

There are two important limiting cases for large M: $\mu = M/N$ fixed and $L = N - M$ fixed. For fixed μ and M to ∞ we find the mentioned scaling behaviour:

$$P(r) = \left(\frac{1}{\mu} - 1 \right) \frac{2r}{(1 - r^2)^2}$$ (19)

for $r^2 < \mu$ and $P(r) = 0$ otherwise. The distribution shows a gap near the unit circle. This gap resembles the one obtained for resonances in the chaotic scattering problem for large number of channels [30]. In this strongly non-unitary limit we are also able to simplify the cluster function:

$$Y(z, z + \delta) = (M\rho(z))^2 \exp(-\pi M \rho(z) |\delta|^2)$$ (20)

which is just the Ginibre behaviour [5, 23] with the distance δ rescaled by the local mean level distance $1/\sqrt{M\rho(z)}$ given by equation (19) through $\rho(z) = P(r)/2\pi r$. The same can be shown for the nearest-neighbour distance distribution obtained by Grobe *et al* [32] and applied to a damped chaotic kicked top.
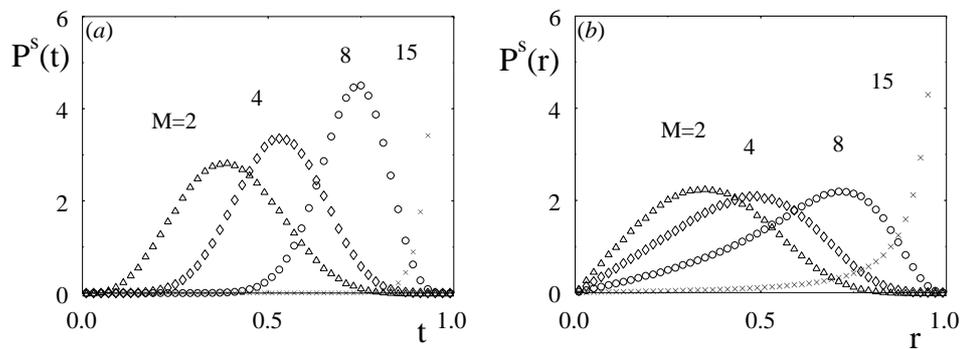
Figure 7. Probability distributions (*a*) $P(t)$ and (*b*) $P(r)$ for complex symmetric matrices $W_{[16,M]}$ with $M = 2, 4, 8, 15$.

In the other limit of fixed $L = N - M$ and M to ∞, which may be considered as weakly non-unitary, we recover exactly the universal resonance-width distribution [9] for perfect coupling to L channels with $y = N(1 - r)$

$$\rho(y) = \frac{y^{L-1}}{(L-1)!} \left(\frac{-\mathrm{d}}{\mathrm{d}y} \right)^L \frac{1 - \mathrm{e}^{-2y}}{2y}. \tag{21}$$

Similarly, the cluster function obtained in this limit can be shown to coincide with the one obtained by Fyodorov and Khoruzhenko [29] for chaotic scattering with a finite number of perfectly coupled channels. The statistics (21) has also been found by Kottos and Smilansky [33] for chaotic scattering on graphs and by Glück *et al* [10, 11] for a model of crystal electron in the presence of dc and ac fields. In both of these works the *S*-matrix is reduced to the resolvent of a subunitary matrix as is investigated in this paper.

6. Submatrices of unitary symmetric matrices

For several applications one uses symmetric unitary matrices typical for the circular orthogonal ensemble. This case is relevant if the physical system possesses time-reversal symmetry, or any generalized anti-unitary symmetry [3]. Let U be a random unitary matrix typical of CUE. It is easy to prove that the symmetric matrix $W := UU^T$ is typical to COE [5]. We shall thus define the ensemble of truncated symmetric unitary matrices $W_{[N,M]}$. In the definition of this ensemble the position of the submatrix is crucial. We take the left upper part of the symmetric matrix W, thus the truncated matrices $W_{[N,M]}$ are symmetric.

The distributions $P(t)$ and $P(r)$ for the symmetric matrices generated out of COE matrices of size $N = 16$ are shown in figure 7. Each plot contains data from 10^4 symmetric random unitary matrices. Note the differences between these figures and the corresponding data for orthogonal and unitary matrices presented in figures 1 and 4. If the truncation of the matrix W is performed asymmetrically (e.g. we take the lower left submatrix), the distribution $P(r)$ becomes closer to that corresponding to the truncations of random unitary matrices $U_{[N,M]}$.

In the asymptotic limit the properties of the ensemble of the truncations of symmetric matrices $W_{[N,M]}$ depends on the same scaling parameter $\mu = M/N$. Moreover, the distribution $P(r)$ becomes close to the corresponding one for the unitary ensemble described by the distribution (19). Therefore we may conjecture that the distribution $P_\mu(r)$, which describes the distribution of moduli of eigenvalues of truncated matrices in the limit of large N, is universal

and does not depend on the initial ensemble of random matrices, provided M/N is fixed. This corresponds to results for resonances in the limit of L/N fixed and N to ∞ [30]. In contrast, there are differences to be expected in the limit of weakly non-unitary matrices: $N - M$ fixed and M to ∞ [29, 31].

7. Concluding remarks

Three families of ensembles of random matrices are proposed. They are defined by cutting an M-dimensional submatrix of an initially N-dimensional unitary matrix, pertaining to a given ensemble of unitary, unitary symmetric or orthogonal matrices. Using a link between the truncation of an orthogonal matrix and the projection of a hypersphere into a smaller dimensional space we found the probability distributions of the lengths t of the projected vectors.

Truncated matrices are not unitary and their complex eigenvalues are located inside the unit circle. We derived an analytical formula for the distribution $P(r)$ of moduli of eigenvalues of truncations of the CUE matrices. It takes a particularly simple form for small values of N and M. In the asymptotic limit $N \to \infty$ this distribution depends only on the scaling parameter $\mu = M/N$, provided M/N is not very close to 1. For small r the distribution $P_\mu(r)$ grows linearly, later displays a nonlinear behaviour and eventually suffers a sudden cut-off at $r_\mu = \sqrt{\mu}$. For $N \gg 1$ the probability distribution $P(r)$ does not depend on whether the initial matrices are orthogonal, unitary or unitary symmetric—again, provided that M/N is not very close to 1. In this strongly non-unitary limit, at least for the case of broken time reversal, symmetry correlations are shown to coincide with those obtained from the Ginibre ensemble of general complex matrices, provided distances are rescaled by the local mean level distance. In the weakly non-unitary limit $N - M$ fixed and M to ∞, once again, the eigenvalue distribution for broken time-reversal symmetry is shown to coincide with the universal resonance widths distribution in the weakly non-Hermitian limit. The same is true for the two-point cluster function.

Acknowledgments

Appendix A. Convolution properties of the distributions $P(t)$

In this appendix we demonstrate the convolution properties of the distributions $P_{N,M}^o(t)$ which might be used to derive the formula (3). We start thus with a random orthogonal matrix $O(N)$ or with random points covering uniformly the hypersphere S^{N-1} of radius 1. Their distribution in the polar coordinates is given by equation (2). For simplicity we will denote the distance from the origin of a point projected into R^M by $t_{N,M}$. It is just the argument of the distribution $P_{NM}^o(t)$. Due to the definition of the polar coordinates $t_{N,N-1} = \sin\theta_{N-2}$, $t_{N,N-2} = \sin\theta_{N-2}\sin\theta_{N-3}, \ldots$, and $t_{N,1} = \cos\theta_{N-2}$. Therefore all variables $t_{N,M}$ may be

rewritten as the product consisting of $L = N - M$ factors

$$t_{N,M} = \prod_{k=1}^{N-M} t_{N-k+1,N-k}. \tag{A.1}$$

This factorization allows us to find the distributions (3).

Probability distribution of a sum of two independent random variables $z = x + y$ is given by the standard convolution $P_x \circ P_y := P(z) = \int_{-\infty}^{\infty} P_x(x)P_y(z-x)\,\mathrm{d}x$. In a similar way, the distribution of the product of two independent random variables, $z = xy$, is given by the product convolution

$$P_x \star P_y := P(z) = \int_z^1 P_x(x)P_y\left(\frac{z}{x}\right)\frac{1}{|x|}\,\mathrm{d}x. \tag{A.2}$$

In the general case the integration should be performed over the entire real axis, but in our case the integration is restricted to $[z, 1]$, since all arguments $t \in [0, 1]$.

Factorization (A.1) allows us to write convolution relations between probability distributions $P_{NM}^o(t)$. For example

$$P_{31}^o(t) = P_{32}^o \star P_{21}^o \tag{A.3}$$
$$P_{42}^o(t) = P_{43}^o \star P_{32}^o \tag{A.4}$$
$$P_{41}^o(t) = P_{43}^o \star P_{32}^o \star P_{21}^o. \tag{A.5}$$

In general, we obtain a convolution relation

$$P_{N,M}^o(t) = P_{N,N-1}^o \star P_{N-1,N-2}^o \star \cdots \star P_{M+1,M}^o \tag{A.6}$$

which might be used to derive formula (3).

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