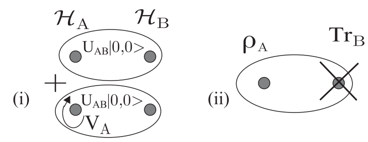

FIG. 5. To generate states according to the Bures distribution one (i) constructs a superposition of a random bi-partite state $|\psi_1\rangle = U_{AB}|0,0\rangle$ with a locally transformed random state $|\psi_2\rangle = (V_A \otimes \mathbb{1})|\psi_1\rangle$, and (ii) performs partial trace over an auxiliary subsystem B.

support the claim that without any prior knowledge on a certain state acting on $\mathcal{H}_N$, the optimal way to mimic it is to generate a random density operator with respect to the Bures measure.

The Bures measure is characterized by the following joint probability of eigenvalues:[13]

$$P_B(\lambda_1, \ldots, \lambda_N) = C_N^B \prod_i \lambda_i^{-1/2} \prod_{i<j}^{1\ldots N} \frac{(\lambda_i - \lambda_j)^2}{\lambda_i + \lambda_j}, \tag{20}$$

where all eigenvalues are non-negative, $\lambda_i \geq 0$ and they sum to unity, $\sum_i \lambda_i = 1$. The normalization constant for this measure

$$C_N^B = 2^{N^2-N} \frac{\Gamma(N^2/2)}{\pi^{N/2} \prod_{j=1}^N \Gamma(j+1)} \tag{21}$$

was obtained in Refs. 13 and 14 for small N and in Ref. 29 for the general case.

To generate random states with respect to the Bures ensemble one can proceed according to the following algorithm[15] illustrated in Fig. 5.

(i) Take a complex random matrix G of size N pertaining to the Ginibre ensemble and a random unitary matrix U distributed according to the Haar measure on $U(N)$.[30]

(ii) Write down the random matrix

$$\rho_B = \frac{(\mathbb{1}+U)GG^\dagger(\mathbb{1}+U^\dagger)}{\mathrm{Tr}[(\mathbb{1}+U)GG^\dagger(\mathbb{1}+U^\dagger)]} \tag{22}$$

which is distributed according to the Bures measure. In the analogy to the Hilbert-Schmidt ensemble we can write this state as reduction of a pure state on the composed system,

$$\rho_B = \frac{\mathrm{Tr}_N|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle}, \quad \text{where} \quad |\phi\rangle := [(\mathbb{1}+V_A)\otimes\mathbb{1}]|\psi_1\rangle. \tag{23}$$

Here $|\psi_1\rangle = U_{AB}|0,0\rangle$ is a random state on the bipartite system used in Eq. (11) and $V_A \in U(N)$.

The asymptotic probability distribution for the rescaled eigenvalue $x = N\lambda$ of a random density matrix generated according to the Bures ensemble reads[20]

$$P_B(x) = C\left[\left(\frac{a}{x}+\sqrt{\left(\frac{a}{x}\right)^2-1}\right)^{2/3} - \left(\frac{a}{x}-\sqrt{\left(\frac{a}{x}\right)^2-1}\right)^{2/3}\right], \tag{24}$$

where $C = 1/4\pi\sqrt{3}$ and $a = 3\sqrt{3}$. This distribution is defined on a support larger than the standard MP distribution, $x \in [0, a]$ and it diverges for $x \to 0$ as $x^{-2/3}$.

The average entropy of a random state from the Bures ensemble reads[20]

$$\langle S(\rho)\rangle_B = \ln N - \ln 2 + O\left(\frac{\ln N}{N}\right). \tag{25}$$

This value is smaller than the average entropy (18), which shows that the Bures states are typically less mixed than the states from the Hilbert–Schmidt ensemble. A similar conclusion follows from the comparison of average purity for the Bures ensemble, $\langle \mathrm{Tr}\rho^2\rangle_B \approx 5/2N$,[20] with the average purity for the HS measure.

By considering random matrices of the form $W = (a\mathbb{1} + (1-a)U)G$, one may construct a continuous family of measures interpolating between the Bures and the Hilbert–Schmidt ensembles[15] and labeled by a real parameter $a \in [0, 1/2]$. A more general class of interpolating ensembles is proposed in Sec. VII.

V. PROJECTION ONTO MAXIMALLY ENTANGLED STATES

A. Four-partite systems and measurements in a maximally entangled basis

Consider a system consisting of four subsystems, labeled as A, B, C, and D. For simplicity assume here that the dimensions of all subsystems are equal, $N_1 = N_2 = N_3 = N_4 = N$. Consider an arbitrary four-partite product state, say $|\psi_0\rangle = |0\rangle_A \otimes |0\rangle_B \otimes |0\rangle_C \otimes |0\rangle_D =: |0, 0, 0, 0\rangle$. Taking two independent random unitary matrices U_{AB} and U_{CD} of size N^2, which act on the first and the second pair of subsystems, respectively, we define a random state $|\psi\rangle = U_{AB} \otimes U_{CD}|\psi_0\rangle$. By construction, it is a product state with respect to the partition into two parts: (A, B) and (C, D). In the analogy to (2) it can be expanded in the product basis:

$$|\psi\rangle = \sum_{i,j=1}^{N} \sum_{k,l=1}^{N} G_{ij} E_{kl} \; |i\rangle_A \otimes |j\rangle_B \otimes |k\rangle_C \otimes |l\rangle_D. \tag{26}$$

Consider now a maximally entangled state on the second and the third subsystem:

$$|\Psi^+_{BC}\rangle = \frac{1}{\sqrt{N}} \sum_{\mu=1}^{N} |\mu\rangle_B \otimes |\mu\rangle_C, \tag{27}$$

and the corresponding projector $P_{BC} := |\Psi^+_{BC}\rangle\langle\Psi^+_{BC}|$. One can extend it into a four-partite operator and define

$$P := \mathbb{1}_A \otimes \Big(\frac{1}{N} \sum_{\mu,\nu} |\mu, \mu\rangle_{BC} \langle \nu, \nu|\Big) \otimes \mathbb{1}_D \tag{28}$$

Let us assume that the random pure state $|\psi\rangle$ defined in (26) is subjected to a projective measurement performed onto the second and third subsystem, and that the result is post-selected to be associated to the projector P. This leads to a non-normalized pure state $|\phi\rangle$ describing the remaining two subsystems,

$$|\phi\rangle = P|\psi\rangle = \frac{1}{N} \sum_{i=1}^{N} \sum_{l=1}^{N} \sum_{k=1}^{N} G_{ik} E_{kl} \; |i\rangle_A \otimes |l\rangle_D. \tag{29}$$

Note that the projection on the entangled state $|\Psi^+_{BC}\rangle$ introduces a coupling between the subsystems B and C. A similar idea was used in analysis of entanglement swapping,[34] and in studies of "matrix product states"[31] and "projected entangled pair states."[32,33] Constructing the matrix product states an $N \times N$ entangled state is projected down into a subspace of an arbitrary dimension d, while in our approach a projection onto the maximally entangled state of BC takes place, (which formally corresponds to $d = 1$), and only two edge systems labeled by A and D survive the projection (see Fig. 6).

Normalizing the resulting state $|\phi\rangle$ and performing the partial trace over the fourth subsystem D we arrive at a compact expression for the resulting mixed state on the first subsystem A,

$$\rho \;=\; \frac{\mathrm{Tr}_D |\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{G E E^\dagger G^\dagger}{\mathrm{Tr}\, G E E^\dagger G^\dagger}. \tag{30}$$

For any matrices G and E this expression provides a valid quantum state, normalized and positive. If initial pure states are random, then the matrices G and E belong to the Ginibre ensemble and the spectrum of a random state ρ consists of squared singular values of the product GE of two random Ginibre matrices. Random states with the same statistical properties were recently found in ensembles associated with certain graphs.[5]

Observe that the statistical properties of the ensemble defined will change if the projection is performed with respect to an arbitrary maximally entangled state, $|\Psi^+_{BC}\rangle' = (U_B \otimes U_C)|\Psi^+_{BC}\rangle$, as

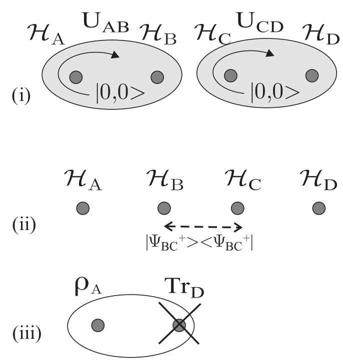

FIG. 6. To generate a random mixed state with spectral density $\pi^{(2)}$ one has to (i) take a product random pure state $|\psi_{AB}\rangle \otimes |\psi_{CD}\rangle$ on a four particle system A, B, C, D, (ii) measure subsystems B, C by a projection onto the maximally entangled state $|\Psi^+_{BC}\rangle$, (iii) average over the subsystem D.

the local unitaries U_B and U_C can be absorbed in the definition of random Ginibre matrices, G and E, respectively.

Furthermore, one may chose N^2 unitary matrices $U_i \in U(N)$, which are orthogonal in the sense of the Hilbert–Schmidt scalar product, Tr $U_i U_j^\dagger = N\delta_{ij}$. Then the set of N^2 maximally entangled states, $|\Psi_i^+\rangle = (U_i \otimes \mathbb{1})|\Psi^+_{BC}\rangle$, forms a maximally entangled basis, which are known to exist in any dimension.[35] Thus, one may consider another setup in which a selective measurement on subsystems B and C is performed in the maximally entangled basis. The outcome state on subsystems AD depends on the result of the quantum measurement of subsystems BC. However, these results are equivalent up to a unitary transformation, which again can be absorbed into the definition of the Ginibre ensemble. Thus, the statistical properties of the random state (30) on subsystem A obtained in consequence of the measurement in the maximally entangled basis in BC followed by the partial trace over subsystem D do not depend on the outcome of the measurement.

Furthermore, the same construction holds in a more general setup in which some dimensions of four subsystems are different. To use the maximally entangled state $|\Psi^+_{BC}\rangle$ we set $N_B = N_C$, but the dimensions $N = N_A$ and N_D can be different. This leads to two rectangular random Ginibre matrices, G of size $N \times N_B$ and E of size $N_B \times N_D$. As in the previous case, formula (30) provides a density matrix ρ of size N, but now the model is a function of two parameters: dimensions N_B and N_D. It is sometimes convenient to use two ratios, $c_1 = N_B/N$ and $c_2 = N_D/N$, so the standard version of the model corresponds to putting $c_1 = c_2 = 1$.

B. Multi-partite systems

Another possibility to generalize the model is to consider a larger system consisting of an even number $2s$ of subsystems. For simplicity assume first that their dimensions are set to N. In analogy to (26) we use s independent unitaries of size N^2 to generate a random pure state $|\psi\rangle$.

To work with an arbitrary even number of subsystems it is convenient to modify the notation and label the subsystems by integers $1, 2, \ldots, 2s$ (see Fig. 7). Consider an arbitrary product state of a $2s$ – particle system, $|\psi_0\rangle = |0\rangle_1 \otimes \cdots \otimes |0\rangle_{2s} =: |0, \ldots, 0\rangle$. Taking s independent Haar random unitary matrices $U_{1,2}, U_{3,4}, \ldots, U_{2s-1,2s}$ of size N^2, we define a random state $|\psi\rangle = U_{1,2} \otimes \cdots U_{2s-1,2s}|\psi_0\rangle$. Expanding this state in the product basis one obtains

$$|\psi\rangle = \big(U_{1,2} \otimes \cdots U_{2s-1,2s}\big)|0, \ldots, 0\rangle \qquad (31)$$

$$= \sum_{i_1,\ldots,i_{2s}} (G_1)_{i_1,i_2} \cdots (G_s)_{i_{2s-1},i_{2s}} |i_1, \ldots, i_{2s}\rangle.$$

Performing a projection onto a product of $s - 1$ maximally entangled states,

$$P_s := \mathbb{1}_1 \otimes |\Psi^+_{2,3}\rangle\langle\Psi^+_{2,3}| \otimes \cdots \otimes |\Psi^+_{2s-2,2s-1}\rangle\langle\Psi^+_{2s-2,2s-1}| \otimes \mathbb{1}_{2s}, \qquad (32)$$

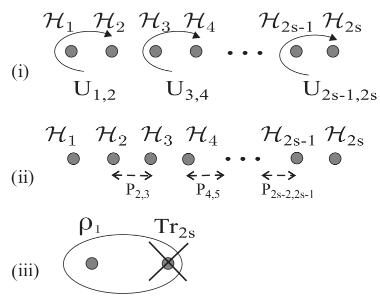

FIG. 7. To obtain a random mixed state with spectral density $\pi^{(s)}$ use a system consisting of $2s$ subsystems, $1, \dots, 2s$ of size N: (i) take the product of s bi-partite random pure states generated by random unitary matrices $U_{1,2}, U_{3,4}, \dots, U_{2s-1,2s} \in U(N^2)$, (ii) measure subsystems $2, \dots, 2s-1$ by a projection onto the product of $s-1$ maximally entangled states, $P_{23} \otimes P_{4,5} \otimes \cdots \otimes P_{2s-2,2s-1}$, where $P_{i,j} = |\Psi^+_{i,j}\rangle\langle\Psi^+_{i,j}|$, (iii) perform partial trace average over the subsystem $2s$.

we obtain a pure state $|\phi\rangle$ describing the remaining two subsystems,

$$|\phi\rangle = P|\psi\rangle = N^{1-s}\sum_{i,j}\Big(G_1G_2\cdots G_s\Big)_{ij}|i\rangle_1 \otimes |j\rangle_{2s}. \tag{33}$$

Normalizing this state and performing the partial trace over the last subsystem we obtain an explicit expression for the resulting mixed state on the first subsystem:

$$\rho \;=\; \frac{\mathrm{Tr}_{2s}|\phi\rangle\langle\phi|}{\langle\phi|\phi\rangle} = \frac{G_1G_2\cdots G_s(G_1G_2\cdots G_s)^\dagger}{\mathrm{Tr}\,[G_1G_2\cdots G_s(G_1G_2\cdots G_s)^\dagger]}. \tag{34}$$

Alternatively, one may assume that this state is obtained as a result of an orthogonal measurement into the product of $s-1$ maximally entangled bases. The first entangled basis correlates subsystem 2 with subsystem 3, the next one couples subsystem 4 with 5, while due to the last one the subsystem $2s-2$ is correlated with $2s-1$. Thus, eigenvalues of a random state generated in this way coincide with squared singular values of the product of s independent random Ginibre matrices. Their statistical properties will be analyzed in Sec. VI. In general, the Ginibre matrices need not to have the same dimension so the model can be generalized. Assuming that a rectangular matrix G_i has dimensions $N_i \times M_i$ one has to put $M_i = N_{i+1}$ for $i = 1, \dots, s-1$, so that the product (34) is well defined. Setting $N_1 = N$ and defining the ratios $c_i = M_i/N$ for $i = 1, \dots, s$ one obtains ensemble of random states parametrized by the vector of coefficients, $\mathbf{c} := \{c_1, \dots, c_s\}$.

VI. PRODUCT OF GINIBRE MATRICES AND FUSS-CATALAN DISTRIBUTION

For any integer number s, there exists a probability measure $\pi^{(s)}$, called the Fuss-Catalan distribution of order s, whose moments are the generalized Fuss-Catalan numbers[10,12] given in terms of the binomial symbol,

$$\int_0^{b(s)} x^m \pi^{(s)}(x)dx = \frac{1}{sm+1}\binom{sm+m}{m} =: \mathrm{FC}^{(s)}_m\,. \tag{35}$$

The measure $\pi^{(s)}$ has no atoms, it is supported on $[0, b(s)]$ where $b(s) = (s+1)^{s+1}/s^s$, its density is analytic on $(0, b(s))$, and bounded at $x = b(s)$, with asymptotic behavior $\sim 1/(\pi x^{s/(s+1)})$ at $x \to 0$. This distribution arises in random matrix theory as one studies the product of s independent random square Ginibre matrices, $W = \prod_{j=1}^s G_j$. In this case squared singular values of W (i.e., eigenvalues of $WW^\dagger$) have asymptotic distribution $\pi^{(s)}$. The same Fuss–Catalan distribution describes asymptotically the statistics of singular values of sth power of a single random Ginibre matrix.[36] In terms of free probability theory, it is the free multiplicative convolution product of s copies of the Marchenko-Pastur distribution,[11,37] which is written as $\pi^{(s)} = [\pi^{(1)}]^{\boxtimes s}$.

An explicit expression of the spectral density for $s = 2$ is

$$\pi^{(2)}(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi}\ \frac{\left[\sqrt[3]{2}\left(27 + 3\sqrt{81 - 12x}\right)^{\frac{2}{3}} - 6\sqrt[3]{x}\right]}{x^{\frac{2}{3}}\left(27 + 3\sqrt{81 - 12x}\right)^{\frac{1}{3}}}, \tag{36}$$

where $x \in [0, 27/4]$, was derived first in Ref. 38 in context of construction of generalized coherent states from combinatorial sequences. More recently, it was applied in Ref. 5 to describe random quantum states associated with certain graphs.

The spectral distribution of a product of an arbitrary number of s random Ginibre matrices was recently analyzed by Burda *et al.*[39] also in the general case of rectangular matrices. The distribution was expressed as a result of a polynomial equation and it was conjectured that the finite size effects can be described by a simple multiplicative correction. Another recent work of Liu *et al.*[40] provides an integral representation of the distribution $\pi^{(s)}$ derived in the case of s square matrices of size N, which is assumed to be large.

Making use of the inverse Mellin transform and the Meijer G–function one may find a more explicit form of this distribution. It can be represented[41] as a superposition of s hypergeometric functions of the type ${}_sF_{s-1}$,

$$\pi^{(s)}(x) = \sum_{n=1}^{s} \Lambda_{n,s}\ x^{\frac{n}{s+1}-1}\ {}_sF_{s-1}\Big(\Big[\Big\{1 - \frac{1+j}{s} + \frac{n}{s+1}\Big\}_{j=1}^{s}\Big];$$
$$\Big[\Big\{1 + \frac{n-j}{s+1}\Big\}_{j=1}^{n-1}, \Big\{1 + \frac{n-j}{s+1}\Big\}_{j=n+1}^{s}\Big]; \quad \frac{s^s}{(s+1)^{s+1}}x\Big), \tag{37}$$

where the coefficients $\Lambda_{n,s}$ read for $n = 1, 2, \ldots, s$,

$$\Lambda_{n,s} := s^{-3/2}\sqrt{\frac{s+1}{2\pi}}\left(\frac{s^{s/(s+1)}}{s+1}\right)^{n}\ \frac{\left[\prod_{j=1}^{n-1}\Gamma\left(\frac{j-n}{s+1}\right)\right]\left[\prod_{j=n+1}^{s}\Gamma\left(\frac{j-n}{s+1}\right)\right]}{\prod_{j=1}^{s}\Gamma\left(\frac{j+1}{s} - \frac{n}{s+1}\right)}\ . \tag{38}$$

Here ${}_pF_q([\{a_j\}_{j=1}^{p}]; [\{b_j\}_{j=1}^{q}]; x)$ stands for the hypergeometric function[42] of the type ${}_pF_q$ with p "upper" parameters a_j and q "lower" parameters b_j, and of the argument x. The symbol $\{a_i\}_{i=1}^{r}$ represents the list of r elements, $a_1, \ldots, a_r$. The above distribution is exact and it describes the density of squared singular values of s square Ginibre matrices in the limit of large matrix size N.

Observe that in the simplest case $s = 1$ the above form reduces to the Marchenko–Pastur distribution,

$$\pi^{(1)}(x) = \frac{1}{\pi\sqrt{x}}\ {}_1F_0\Big([-\frac{1}{2}]; [\ ]; \frac{1}{4}x\Big) = \frac{\sqrt{1 - x/4}}{\pi\sqrt{x}}\ , \tag{39}$$

while the case $s = 2$

$$\pi^{(2)}(x) = \frac{\sqrt{3}}{2\pi x^{2/3}}\ {}_2F_1\Big([-\frac{1}{6}, \frac{1}{3}]; [\frac{2}{3}]; \frac{4x}{27}\Big) - \frac{\sqrt{3}}{6\pi x^{1/3}}\ {}_2F_1\Big([\frac{1}{6}, \frac{2}{3}]; [\frac{4}{3}]; \frac{4x}{27}\Big) \tag{40}$$

is equivalent to the form (36) obtained in Ref. 38.

The distributions (37), shown in Fig. 8, are thus directly applicable to describe the level density of random mixed states obtained from a $2s$–partite pure states by projection onto maximally entangled states and partial trace as described in Sec. V. This result becomes exact in the asymptotic limit, it which the dimension N of a single subsystem tends to infinity. However, based on recent results of Burda *et al.*[39] one can conjecture that the finite N effects can be described by a multiplicative correction.

Note that the upper edge $b(s) = (s+1)^{s+1}/s^s$ of the FC distribution $\pi^{(s)}(x)$ for large matrices determines the size of the largest eigenvalue λ_{max} of the reduced density matrix ρ of size N. In the case of the structureless ensemble of random pure states on $N \times N$ system, corresponding to $s = 1$, one has $b(1) = 4$ so that $\lambda_{max} \approx 4/N$. For states obtained by the projection of a $2s$ partite system on maximally entangled states, as described in Sec. V, the largest component behaves as $\lambda_{max} \approx b(s)/N$.

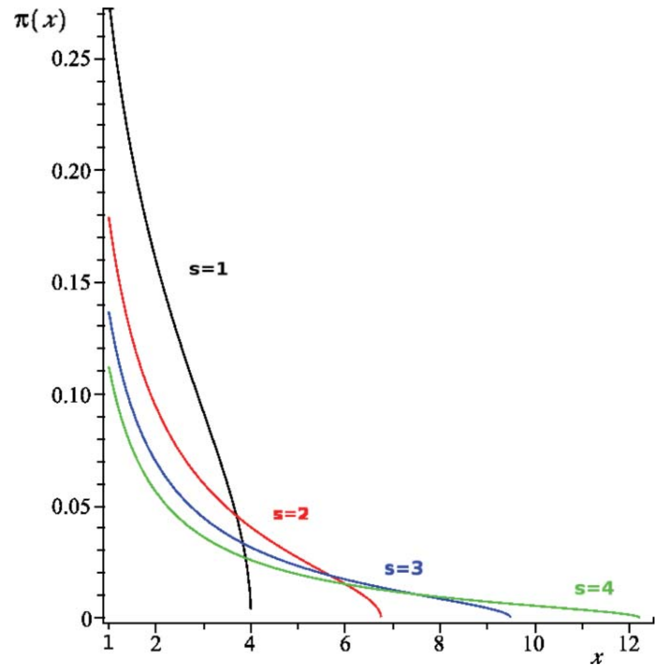

FIG. 8. (Color online) Fuss–Catalan distributions $\pi^{(s)}(x)$ plotted for $s = 1, 2, 3, 4$ supported on the interval $[0, (s+1)^{s+1}/s^s]$. To demonstrate the behavior at the right edge of the support the figure is depicted for $x \geq 1$.

The number λ_{max}, equal to the largest component of the Schmidt vector of a random pure state on the bi-partite system, can be used to measure the degree of quantum entanglement. For instance, for a bi-partite system, the "geometric measure" of entanglement, related to the distance to the closest separable state (in the sense of the natural Fubini-Study distance) reads,[43] $E_g(|\phi\rangle) = -\ln \lambda_{\max}$. This quantity can also be considered as the Chebyshev entropy S_∞ – the generalized Renyi entropy $S_q = \frac{1}{1-q} \ln \mathrm{Tr}\rho^q$ in the limit $q \to \infty$.[9]

Thus, the right edge $b(s)$ of the support of the spectral density for the reduced state $\rho = \mathrm{Tr}_B|\phi\rangle\langle\phi|$ determines the geometric measure of entanglement of the corresponding random pure states. In the case of structureless random pure states, related to the Marchenko–Pastur distribution one becomes an asymptotic expression for the average $\langle E_r(|\phi>)\rangle = -\ln(4/N)$. In the general case of random state corresponding the FC distribution $\pi^{(s)}(x)$ the typical value of the entropy reads

$$\langle S_\infty \rangle_s = \ln N - \ln b(s) = \ln N + s \ln s - (s+1)\ln(s+1). \tag{41}$$

The larger the value of s, the smaller the Chebyshev entropy S_∞, and the less entangled a typical random state obtained by the projection of the $2s$ partite system.

The average von Neumann entropy, $S_1 = -\mathrm{Tr}\rho \ln \rho$, of mixed states of size N generated according to the FC distribution reads $\langle S(\rho)\rangle_s = \ln N - \sum_{j=2}^{s+1} \frac{1}{j}$.[5] The second moment of the FC distribution given in (35) implies the asymptotic average purity $\langle \mathrm{Tr}\rho^2 \rangle_s \approx (s+1)/N$ – the larger the number s, the more pure the typical mixed state generated by a projection onto $s-1$ maximally entangled states.

VII. CONCLUDING REMARKS

In this work we analyzed structured ensembles of random pure states on composite systems. They are defined with respect to a given decomposition of the entire system into its subsystems, what induces a concrete tensor product structure in the Hilbert space. The structured ensembles are thus invariant with respect to the group of local unitary transformations.

TABLE I. Reduction of pure states from structured ensembles leads to random mixed states of the form $\rho = WW^\dagger/\mathrm{Tr}WW^\dagger$. Random matrix W is constructed out of random unitary matrices U_i distributed according to the Haar measure and/or (independent) random Ginibre matrices G_j of a given size N. Asymptotic distribution $P(x)$ of the density of a rescaled eigenvalue $x = N\lambda$ of ρ for $N \to \infty$ is characterized by the singularity at 0, its support $[a, b]$, the second moment M_2 determining the average purity $\langle \mathrm{Tr}\rho^2 \rangle = M_2/N$, and the mean entropy, $\int_a^b [-x \ln x P(x) dx]$, according to which the table is ordered.

k	s	Matrix W	Distribution $P(x)$	Singularity at $x \to 0$	Support $[a, b]$	M_2	Mean entropy
1	0	U_1	$\delta(1) = \pi^{(0)}$	–	$\{1\}$	1	0
2	0	$U_1 + U_2$	arcsine	$x^{-1/2}$	$[0, 2]$	$3/2$	$\ln 2 - 1 \approx -0.307$
3	0	$U_1 + U_2 + U_3$	3 entangled states	$x^{-1/2}$	$[0, 2\frac{2}{3}]$	$5/3$	≈ -0.378
4	0	$U_1 + U_2 + U_3 + U_4$	4 entangled states	$x^{-1/2}$	$[0, 3]$	$7/8$	≈ -0.411
1	1	$G \sim UG$	Marchenko–Pastur $\pi^{(1)}$	$x^{-1/2}$	$[0, 4]$	2	$-1/2 = -0.5$
2	1	$(U_1 + U_2)G$	Bures	$x^{-2/3}$	$[0, 3\sqrt{3}]$	$5/2$	$-\ln 2 \approx -0.693$
1	2	$G_1 G_2$	Fuss–Catalan $\pi^{(2)}$	$x^{-2/3}$	$[0, 6\frac{3}{4}]$	3	$-5/6 \approx -0.833$
1	..	...	...	...	...	...	...
1	s	$G_1 \cdots G_s$	Fuss–Catalan $\pi^{(s)}$	$x^{-s/(s+1)}$	$[0, (s+1)^{s+1}/s^s]$	$s + 1$	$-\sum_{j=2}^{s+1} \frac{1}{j}$

Performing a partial trace over selected subsystems one obtains an ensemble of random mixed states defined on the remaining subsystems. The particular ensemble depends thus on the number of systems traced out and on the way the initial random pure states are prepared.

Quantum states obtained by the partial trace of a superposition of k maximally entangled pure states of the bi-partite system involve the sum of k random unitary matrices. To generate states which involve a product of an arbitrary number of s matrices one needs to consider a system consisting of $2s$ subsystems, in which an orthogonal measurement is performed in the product of $s - 1$ maximally entangled bases. Selected ensembles of random states, recipe to generate numerically the corresponding density matrices, and some properties of the distribution of the Schmidt coefficients are collected in Table I.

We are going to conclude this work by writing down a more general class of structured random states, which contains all particular cases discussed in the paper and listed in Table I. Consider the following ensemble of non-hermitian random matrices parametrized by an arbitrary k–dimensional probability vector $p = \{p_1, \ldots, p_k\}$ and a non-negative integer s:

$$W_{k,s} := \left[p_1 U_1 + p_2 U_2 + \cdots + p_k U_k\right] G_1 \cdots G_s. \tag{42}$$

Here $U_1, \ldots, U_k$ denote k independent random unitary matrices distributed according to the Haar measure on $U(N)$, while $G_1, \ldots, G_s$ are independent square random matrices of size N from the complex Ginibre ensemble. Random density matrix is obtained as a normalized Wishart–like matrix:

$$\rho_{k,s} := \frac{W_{k,s} W_{k,s}^\dagger}{\mathrm{Tr}(W_{k,s} W_{k,s}^\dagger)}. \tag{43}$$

Note that any particular ensemble from the above class can be physically realized by taking a superposition of k random pure states weighted by the vector p. Each pure state is defined on the system containing $2s$ subsystems. Performing a measurement in the product of $(s - 1)$ maximally entangled bases one gets a random pure state of the desired structured ensemble. Eventually, averaging over the last subsystem one arrives at the mixed state (43).

Consider first the case of a uniform probability vector, $p_i = 1/k$ for $i = 1, \ldots, k$. For $k = 1$ one obtains ensembles leading to the Fuss–Catalan distributions $\pi^{(s)}$, which in the case $s = 1$ reduces to the Marchenko–Pastur distribution. Taking $k = 2$ and $s = 0$ one obtains the arcsine ensemble (7), while for larger k one obtains the distribution (10), which converges to $\pi^{(1)}$ in the limit $k \to \infty$. Moreover, the case $k = 2$ and $s = 1$ corresponds to the Bures ensemble (23). Thus, the case $k = 2$ and arbitrary s can be called *higher order Bures ensemble*.

In a more general case, taking an arbitrary probability vector p and varying the weights in a continuous manner one can study the transition between given structured ensembles. For instance, by fixing the parameter s, setting $k = 2$, and varying the weight $p_2 = 1 - p_1$, one defines a continuous interpolation between the higher order Bures ensemble and the Fuss–Catalan ensemble. We have shown therefore that having at our disposal simple algorithms to generate random unitary and random Ginibre matrices we can construct a wide class of ensembles of random quantum states. Furthermore, we provide constructive physical recipe to generate such states by means of generic two-particle interaction, superposition of states, selective measurements in maximally entangled basis, and performing averages over certain subsystems.

As discussed in Appendix B it is also possible to introduce analogous ensembles of real random density matrices. Physically this corresponds to imposing restrictions on the class of the interactions used to generate random pure states. In contrast with the complex case, the ensemble based on square real Ginibre matrices does not lead to the Bures measure in the space of real states. To achieve such a measure one needs to generalize the ensemble even further to allow also rectangular Ginibre matrices.[15] In physical terms this implies that the dimension of the principal system and the auxiliary system have to be different in this case.

The notion of random quantum states is closely related with the concept of random quantum maps. Due to the Jamiołkowski isomorphism any quantum operation Φ acting on density matrices of size N can be represented by a state on the extended Hilbert space,[9]

$$\sigma = \big(\Phi \otimes \mathbb{1}\big)|\psi^+\rangle\langle\phi^+|. \tag{44}$$

Here $|\psi^+\rangle = \frac{1}{\sqrt{N}}\sum_{j=1}^{N}|j, j\rangle$ denotes the maximally entangled state from the bi-partite Hilbert space $\mathcal{H} = \mathcal{H}_N \otimes \mathcal{H}_N$. Any state σ on the composed Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ defines a completely positive, trace preserving map provided the following partial trace condition is satisfied, $\mathrm{Tr}_A\sigma(\Phi) = \mathbb{1}/N$.

It is possible to impose this partial trace condition on an arbitrary state ω acting on $\mathcal{H}$. To this end one finds the reduced state $Y := \mathrm{Tr}_{\mathrm{A}}\omega$ which is positive and allows one to take its square root $\sqrt{Y}$, and write the normalized cognate state:[44]

$$\sigma = \frac{1}{N}\Big(1 \otimes \frac{1}{\sqrt{Y}}\Big)\omega\Big(1 \otimes \frac{1}{\sqrt{Y}}\Big)\,. \tag{45}$$

The required property, $\mathrm{Tr}_A\sigma = \mathbb{1}/N$, is satisfied by construction, so the state σ represents a quantum operation. As the matrix elements of the corresponding superoperator Φ can readily be obtained by reshuffling[9] the entries of the density matrix σ, any random state ω determines by (45) and (44) a quantum operation. Therefore, any ensemble of random states introduced in this paper, applied for bi-partite, $N \times N$ systems determines the corresponding ensemble of random operations. For instance, the induced measure with $K = N^2$ corresponds to the flat measure in the space of quantum operations,[44,45] but other ensembles of random states can be also applied to generate random quantum operations.[9]

The present study on ensembles of random states should be concluded with a remark that apart of the methods developed in this paper several other approaches are advocated in the literature. In very recent papers[46,47] the authors follow a statistical approach introducing a partition function which leads to a generalization of the Hilbert-Schmidt measure. Varying the parameter of the model, which corresponds to the inverse temperature, they demonstrate a phase transition during an interpolation between Marchenko-Pastur and semicircle distribution of spectral density. In another recent approach Garnerone *et al.* studied statistical properties of random matrix product states,[48] which are obtained out of products of truncated random unitary matrices. Although these models of random states do differ from the one presented in this work, possible links and relations between results obtained in these approaches are currently under investigation.

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APPENDIX A: SUM OF k RANDOM UNITARIES AND THE DISTRIBUTION ν_k

We compute, in the limit of large matrix size $N \to \infty$, the asymptotic eigenvalue distribution of the random matrix,

$$V_k = \frac{1}{k}(U_1 + U_2 + \cdots + U_k)(U_1^* + U_2^* + \cdots + U_k^*), \tag{A1}$$

where $U_1, \dots, U_k$ are $N \times N$ random independent Haar unitary matrices. Obviously, this is equivalent to computing the singular value distribution of $k^{-1/2}\sum_{i=1}^k U_i$.

For now, we forget about the normalization pre-factor and we put $W_k = kV_k$. It is a well known result in free probability theory that independent large unitary matrices are free from each other (and, very importantly, but not of interest here, they are also free from deterministic diagonal matrices).

Theorem A.1 (Refs. 49 and 50): *Let $U_{1,N}, \dots, U_{k,N} \in \mathcal{U}(N)$ be k independent Haar unitary random matrices. Then, as $N \to \infty$,*

$$U_{1,N}, \dots, U_{k,N} \xrightarrow{*-distr} u_1, \dots, u_k, \tag{A2}$$

where $u_1, \dots, u_k$ are free Haar unitary elements in a non-commutative W^*-probability space $(\mathcal{M}, \tau)$.

Hence, computing the limit distribution of W_k amounts to understanding the distribution of a sum of k free Haar unitary elements in a von Neumann algebra $w_k = (u_1 + \cdots + u_k)(u_1^* + \cdots + u_k^*)$. This problem has been related to random walks on k-regular trees by Kesten.[51] Indeed, the number of alternating words of length $2p$ in the letters u_i, u_i^* which reduce to the unit is bijectively the same as the number of walks of length $2p$ on the k-regular tree beginning and ending at some fixed vertex. Using standard formulas for the number of such walks, we can deduce moment information for $k = 2$:

$$\lim_{N\to\infty} \mathbb{E}\left[\frac{1}{N}\operatorname{Tr} W_2^p\right] = \tau(w_2^p) = \binom{2p}{p}^2, \tag{A3}$$

and a moment generating function in the general case:

$$F_k(z) = \sum_{p=0}^{\infty} \tau(w_k^p) z^p = \frac{2(k-1)}{k - 2 + k\sqrt{1 - 4(k-1)z}}. \tag{A4}$$

From the last formula, using Cauchy transform techniques, one can easily deduce the probability density function of the distribution of w:

$$d\mu_k(x) = \frac{k}{2\pi}\frac{\sqrt{4(k-1)x - x^2}}{k^2 x - x^2}\mathbf{1}_{[0,4(k-1)]}(x)dx. \tag{A5}$$

This result has been obtained by Haagerup and Larsen in Example 5.3 of Ref. 52. The authors were interested in the Brown measure of the non-normal element $\tilde{w} = u_1 + \cdots + u_k$. They found that the distribution of $|\tilde{w}|$ is given by

$$d\tilde{\mu}_k(x) = \frac{k}{\pi}\frac{\sqrt{4(k-1) - x^2}}{k^2 - x^2}\mathbf{1}_{[0,2\sqrt{k-1}]}(x)dx. \tag{A6}$$

One can easily recover Eq. (A5) by using $w = \tilde{w}\tilde{w}^*$ and by noticing that $\mu_k = \tilde{\mu}_k \circ \mathrm{sq}$, where sq is the square function $\mathrm{sq}(x) = x^2$.

If ν_k is the distribution of the rescaled element $v = w/k$, then one arrives at the desired distribution function,

$$d\nu_k(x) = \frac{1}{2\pi} \frac{\sqrt{4k(k-1)x - k^2x^2}}{kx - x^2} \mathbf{1}_{[0,4\frac{k-1}{k}]}(x)dx. \tag{A7}$$

In Ref. 52 it is shown that the Brown measure of $\tilde{w}$ is a rotationally invariant measure, supported on the centered disk of radius $\sqrt{k}$ with radial density

$$f_{\tilde{w}}(r) = \frac{k^2(k-1)}{\pi(k^2 - r^2)^2}, \qquad 0 < r < \sqrt{k}. \tag{A8}$$

With the proper $k^{-1/2}$ rescaling, it is easy to see that the above Brown measure converges to the uniform measure on the unit disk. Hence, we recover the Ginibre behavior in the limit $k \to \infty$ (one has to take first the limit $N \to \infty$). Let us add that a more general study on statistical properties of a sum of random unitary matrices was recently presented by Jarosz.[53]

APPENDIX B: REAL RANDOM STATES, REAL GINIBRE, AND RANDOM ORTHOGONAL MATRICES

Although most often one considers complex density matrices, it is also interesting to study quantum states described by real density matrices. The dimensionality of the set of real states on $\mathcal{H}_N$ is $N(N+1)/2 - 1$, so its geometry is easier to study than that of the $N^2 - 1$ dimensional set of complex states.[9] For instance, the set of real states of a qubit forms a two-dimensional disk, which can be considered as a cross-section of the three-dimensional Bloch ball of complex states. Euclidean volume of the set of real density matrices of size N was derived in Ref. 54, while the corresponding measure can be derived from the real Ginibre ensemble.

In this appendix we define ensembles of real states based on random orthogonal matrices and real Ginibre ensemble, and show that the level density does not differ from the complex case. To this end we formulate two lemmas.

Lemma B.1: Consider k independent orthogonal matrices $O_1, \ldots, O_k$, distributed according to the Haar measure on $O(N)$ and define a normalized real density matrix as

$$\rho_{\mathrm{ort}} = \frac{(O_1 + \cdots + O_k)(O_1^T + \cdots + O_k^T)}{\mathrm{Tr}(O_1 + \cdots + O_k)(O_1^T + \cdots + O_k^T)}. \tag{B1}$$

Then for large N its spectral density is described by the distribution ν_k given in (10).

This lemma follows directly from the fact that the moments of orthogonal and unitary random matrices have the same behavior for large matrix size N, since random orthogonal matrices are asymptotically free.[55]

Lemma B.2: Consider s independent random matrices $R_1, \ldots, R_s$ taken from the real Ginibre ensemble of square matrices of size N. Define a normalized real density matrix as

$$\rho_R = \frac{R_1R_2\cdots R_s(R_1R_2\cdots R_s)^T}{\mathrm{Tr}[R_1R_2\cdots R_s(R_1R_2\cdots R_s)^T]}. \tag{B2}$$

Then for large N its spectral density is described by the Fuss-Catalan distribution $\pi^{(s)}$ given in (37).

To prove this, one needs to show that in the case of large matrix size N the moments of this distribution are indeed given by the Fuss–Catalan numbers (35), exactly as for the product of complex Ginibre matrices. This follows from the interpretation of the Wick formula for random matrices in terms of maps gluing and from the fact that leading terms have to be non-crossing, therefore orientable as for complex Gaussian matrices – see, e.g., Refs. 56 and 57.

It is natural to combine both definitions and propose a more general ensemble of real density matrices, each obtained out of k random orthogonal matrices and s square real Ginibre matrices in a direct analogy to Eq. (42).

Let us close this work with a remark that the differences between the real and complex case can be significant in some cases. For instance, to get a real state distributed according to the Bures measure one needs to use a symmetric random unitary matrix and a rectangular, $N \times (N+1)$, real Ginibre matrix, while the complex Bures state is obtained from a random unitary and a square matrix of the complex Ginibre ensemble.[15]

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