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Eigenvector statistics of random band matrices

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The statistics of eigenvector elements is studied for random band matrices as a function of band and matrix sizes. It is shown that the statistics obeys a scaling law similar to that found for the mean localization length of eigenvectors. To describe the statistics we propose an expression for the probability distribution of elements based on the assumption of an exponential form of eigenvectors. We demonstrate a fundamental role of fluctuations of the localization length.

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I. INTRODUCTION

The properties of random band matrices have attracted great attention in recent years [1–7], mainly due to their relevance for the studies of quantum chaos [8, 10]. In particular Casati, Molinari, and Izrailev [1] have shown, on the basis of numerical data, that the average localization length of eigenvectors of random band matrices follows a scaling law. The scaling parameter is b^2/N, where b measures the bandwidth and N is the size of the matrix. The subsequent investigations revealed that a similar scaling is obeyed by the distribution of eigenvalues spacings [10, 11].

Various numerical [1, 3] and theoretical [5, 7] studies have proved that the eigenvalue density of random band matrices follows Wigner semicircular law [12, 13] in the limit when b behaves as N^{β} with some $\beta > 0$ and $N \to \infty$. On the other hand, preliminary calculations of density-density correlation functions indicate [14] that fluctuations of level spacings, contrary to the case of full random matrices, depend on a position in the spectrum.

The aim of our paper is to study statistics of eigenvector components of real symmetric random band matrices. The motivations stem from the theory of quantum chaotic systems, where it is known that eigenvector statistics provide very useful quantum signatures of chaos [9, 10, 15–17]. First, on the basis of numerical experiments we show that the probability distribution of eigenvector elements obeys similar scaling as the one discovered in Ref. [1] for the average localization length. Then, we try to find a simple model of such distribution based on two observations stemming from numerical experiments: We postulate the truncated exponential form of the localization of eigenvectors of finite random band matrices and take into account strong fluctuations of the localization length. On the other hand, we assume independent statistical fluctuations of elements, as in the standard Gaussian orthogonal ensemble of matrices (GOE).

The paper is organized as follows. In Sec. II we present a model of random band matrices and discuss results concerning scaling properties of the distribution of eigenvectors elements. Section III is devoted to the study of interplay between localization of eigenvectors and the form of distribution of their elements. First, we show that the simple model of such a distribution, which does not take into account fluctuations of localization length, gives rather poor agreement with numerical data. Second, we present numerical evidence that a remarkable improvement is achieved by assuming the special type of distribution of localization lengths.

II. SCALING OF EIGENVECTOR ELEMENT DISTRIBUTION

We consider the ensemble of random band matrices recently studied by Casati, Molinari, and Izrailev [1]. The matrix elements A_{ij} are real, symmetric and defined by

$$A_{ij} = \xi_{ij}\Theta(b - |i-j|), \quad i,j = 1,\dots,N, \tag{2.1}$$

where b is the bandwidth and $\Theta(\,)$ denotes the unit step function. The coefficients ξ_{ij} are statistically independent Gaussian random variables distributed according to

$$P(\xi_{ij}) = \frac{1}{\sqrt{2\pi}\sigma_{ij}} \exp\left(-\frac{\xi_{ij}^2}{2\sigma_{ij}^2}\right). \tag{2.2}$$

The variance σ_{ij}^2 equals to

$$\sigma_{ij}^2 = \sigma^2(1+\delta_{ij}) , \quad \text{where} \quad \sigma^2 = \frac{N}{b(2N-b+1)}. \tag{2.3}$$

The last condition assures the normalization $\mathrm{Tr}\,(A^2) = N$.

In the limit $b = 1$ the random matrix A is diagonal and its level spacing distribution is Poissonian. On the other hand for $b = N$ it is a member of GOE and displays linear level repulsion. The level spacings statistics are in that case well described by the Wigner surmise [12]. In an intermediate case $1 \ll b < N/2$ level spacing distribution depends only on the scaling parameter $x = b^2/N$ [11] and is surprisingly well approximated by Brody *et al.* [13] or Izrailev [10] distributions. It is tempting to find analogous simple models describing statistical properties of eigenvectors.

Let c_k^l denote the lth component of the kth normalized eigenvector of A. The distribution $P(y)$ of squared moduli of components $y = |c_k^l|^2$ is called eigenvector statistics. In the limit $b \rightarrow 1$ we have obviously

$$P(y) \rightarrow \frac{N-1}{N}\delta(0) + \frac{1}{N}\delta(1), \tag{2.4}$$

whereas for $b = N$ and $N \rightarrow \infty$, $P(y)$ becomes the Porter-Thomas distribution [12],

$$P(y) = \frac{1}{\sqrt{2\pi y \langle y \rangle}} \exp\left(-\frac{y}{2\langle y \rangle}\right), \tag{2.5}$$

where $\langle y \rangle$ stands for the mean value of y.

Our main purpose is to study intermediate cases $1 \ll b < N/2$. To this end we have performed numerical experiments verifying the hypothesis that the distribution $P(y)$ scales with the parameter x for sufficiently large N and b. To check this conjecture, we have compared the eigenvector statistics $P(y)$ for matrices of different sizes N and bandwidths b, keeping the same value of x. The resulting data clearly display scaling of $P(y)$. As an example, we present in Fig. 1(a) two histograms of $P(\log_{10} y)$ with the mean $\langle y \rangle$ normalized to unity. The results were obtained from an ensemble of seven independently generated matrices with the same $N = 588$ and $b = 14$ (i.e., $x = 1/3$). The total number of $\simeq 2 \times 10^6$ statistical data was divided into 300 bins. The result is compared in the same figure with a similar one in which the histogram was obtained from an ensemble of matrices of different sizes and bandwidths, but with the same value of the parameter x . In particular b took the values of $6, 7, \ldots, 13, 14$ with N ranging from 108 to 588. The number of generated matrices of each size was chosen to keep the number of statistical data for $P(\log_{10} y)$ of the order of 10^6 for a given N. Both histograms in Fig. 1(a) can hardly be distinguished. This result indicates that the eigenvector statistics indeed obeys the same scaling law as the localization length [1].

Similar results are presented in Fig. 1(b) for a larger value of x, i.e., $x = 1.0$. The first histogram was obtained for $b = 16$ and $N = 256$. The second one displays results for the set of matrices with $N = 100, 144, \ldots, 484, 576$ and $b = 10, 12, \ldots, 22, 24$, respectively. The coincidence of two histograms is as remarkable as previously. As reference distributions, the Porter-Thomas distribution (2.5) is plotted (dashed lines) in Fig. 1. As expected the distribution $P(y)$ approaches the Porter-Thomas one as the parameter x increases.

In all cases the distributions in Fig. 1 were calculated using all eigenvectors of the matrices from statistical samples. Since the localization length averaged over fluctuations (ensemble averaging) is dependent on the energy and goes to zero when energy approaches the edges of the spectrum [18], we have compared the above results with those pertaining to the eigenvectors that correspond to eigenvalues from the center of the spectrum. This was achieved by neglecting contributions from eigenvectors belonging to the $N/4$ smallest and the $N/4$ largest eigenvalues. In this case the remaining eigenvectors from

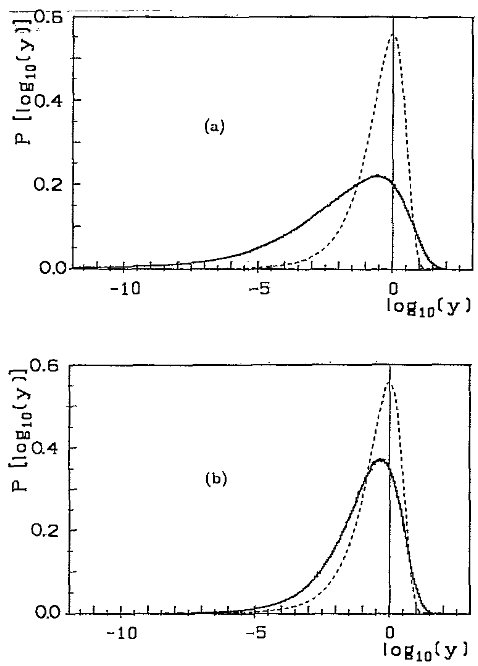

FIG. 1. Histograms of eigenvectors statistics in the logarithmic scale for two values of the scaling parameter $x = b^2/N$: (a) $x = 1/3$ and (b) $x = 1.0$ and different values of N: $100 < N < 600$. The dashed line represents the Porter-Thomas distribution.

the center of the spectrum turn to have the average localization length practically independent on the energy and, as a result, the distribution $P(y)$ of the components of eigenvectors is entirely caused by the fluctuations of localization length for individual eigenvectors. These eigenvectors are more delocalized than the average ones, and the resulting histograms $P(y)$ are slightly narrower than those obtained from all eigenvectors, but the scaling dependence on the parameter x remains. In what follows all presented results are obtained taking into account all eigenvectors.

III. EIGENVECTOR STATISTICS AND FLUCTUATIONS OF LOCALIZATION LENGTH

The Porter-Thomas distribution (2.5) is obtained for GOE from the observation that components c_k^l are independent Gaussian random variables of a common variance and a vanishing mean. The eigenvectors are thus fully delocalized. In other limits of finite b and $N \rightarrow \infty$ all eigenstates appear to be exponentially localized. It is thus rather tempting to give a description of an intermediate situation when the distribution $P(y)$ changes from one limit of completely localized states (2.4) to delocalized chaotic states (2.5). To describe this transition one can conjecture that, apart from fluctuations, an exponential form for the absolute value of eigenvectors components is also valid for finite matrices. Fluctuations, it could then be assumed, have influence only on amplitudes, i.e.,

$$|c_k^l|^2 = \xi_{kl}^2 e^{-|l-l_k|a}. \qquad (3.1)$$

The above expression represents an eigenvector whose center of localization is at $l = l_k$. The parameter a is the inverse of localization length, assumed constant for all eigenvectors, whereas ξ_{kl} are, as in the case of GOE, taken to be independent Gaussian random variables with zero mean and common, N- and a-dependent variance. We choose the variance such that the average norm of a vector $\langle\sum_{l=1}^N |c_k^l|^2\rangle$ equal to one. We shall use the notion of the exponential localization of the eigenvectors for the situation described by Eq. (3.1), notwithstanding the fact that for finite matrices the exponential decrease is cut by the finite size of a matrix and hence can lose its physical meaning (e.g., when the localization length $1/a$ is comparable with the matrix dimension N). We assume further a uniform distribution of l_k in the range from 1 to N. Averaging over the (independent) distributions of ξ_{kl} and l_k we obtain, after a straightforward integration, the following form of the eigenvector statistics:

$$P_a(y) = \frac{1}{ay}\left\{ \operatorname{erf}\left[\left(\frac{y}{2a}(e^a - 1)\right)^{1/2}\right] - \operatorname{erf}\left[\left(\frac{y}{2a}(1 - e^{-a})\right)^{1/2}\right]\right\}, \qquad (3.2)$$

where erf( ) denotes the error function $\operatorname{erf}(x) = 2\pi^{-1/2}\int_0^x \exp(-t^2)dt$. Note that, in the limit of infinite localization length ($a \to 0$), the distribution $P_a(y)$ tends to the Porter-Thomas one.

It turns out, however, that the distribution (3.2) does not provide a satisfactory fit to the numerical results presented in Fig. 1. From our numerical analysis, however, follows that localization length, or alternatively speaking its inverse a, fluctuates very strongly. Therefore, it cannot be assumed constant for all eigenvectors, as was done in Eq. (3.1). This is definitely true at the edges of the spectrum, where the average localization length depends strongly on energy (i.e., on the eigenvalue connected to a given eigenvector). From the numerical experiments it is also clear that even in a narrow energy range, where the dependence of the localization length on energy should be negligible, strong fluctuations of a for individual eigenvectors do persist. Hence, the main point of our conjecture consists of assuming that the parameter a (the inverse localization length) vary from eigenvector to eigenvector and finding an appropriate distribution of a. The resulting eigenvector distribution will thus have the form

$$P(y) = \int_0^\infty P_a(y)P(a)da, \qquad (3.3)$$

where $P(a)$ stands for the probability distribution of a. Equation (3.3) holds, provided that fluctuations of a are statistically independent of those of ξ's.

Looking for the simplest model of the distribution $P(a)$ it is very attractive to assume that it takes a Gaussian form truncated to $a > 0$ (see also [19, 20]):

$$P(a) \propto \exp\left(-\frac{(a - \langle a\rangle)^2}{2\delta^2}\right)\Theta(a), \qquad (3.4)$$

The above approach leads to a better agreement with numerical results, especially in cases of strong ($x < 1/3$) and weak ($x > 1$) localization. The agreement is achieved by the least-squares fitting of the two free parameters $\langle a\rangle$ and δ characterizing the distribution (3.4). For the intermediate case, however, the results call for further improvements. This is illustrated in Fig. 2, which presents the histogram $P(y)$ for $N = 600$, $x = 2/3$ and the best fit of (3.3) with $P(a)$ as in Eq. (3.4) with $\langle a\rangle = 6.25$ and $\delta = 4.40$. In order to understand still persisting discrepancies we turned to the more thorough study of the fluctuations of the localization length.

In principle the inverse localization length a can be found for each eigenvector separately by fitting the formula (3.1) to numerical results, but practically it is very difficult (or next to impossible) due to enormously large fluctuations. It is more convenient, however, to follow the ideas of Ref. [1] introducing the entropic localization length. To this end we define for each eigenvector its entropy,

$$H_k = -\sum_{l=1}^N |c_k^l|^2 \ln |c_k^l|^2, \qquad (3.5)$$

and its average,

$$\langle H\rangle = \frac{1}{N}\sum_{k=1}^N H_k. \qquad (3.6)$$

For the Gaussian orthogonal ensemble one easily finds [21] that the average entropy

$$\langle H\rangle = H_{\rm GOE} = \Psi(N/2 + 1) - \Psi(3/2), \qquad (3.7)$$

where $\Psi(x)$ denotes the Digamma function [22].

In the case of strong localization (small x) typical values of H_k are much smaller than $H_{\rm GOE}$ and $\langle H\rangle$ tends to zero. For each eigenvector we introduce now the so called entropic localization length normalized to the size of a matrix N:

$$\beta_k = e^{H_k - H_{\rm GOE}}. \qquad (3.8)$$

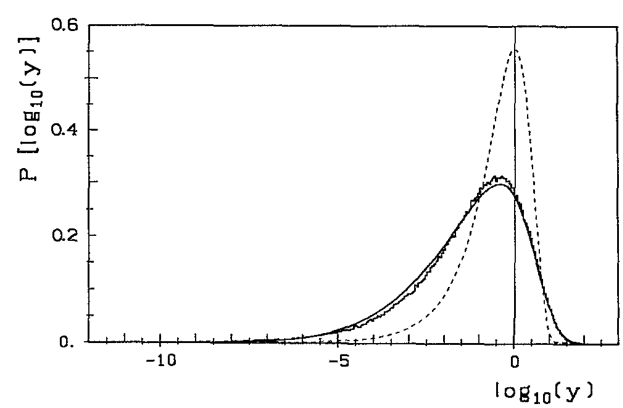

FIG. 2. Eigenvector statistics for $x = 2/3$, $N = 600$. The solid line refers to the best fit of the numerical data with the help of Eqs. (3.3) and (3.4); the dashed line stands for Porter-Thomas distribution.

From the above definitions and Eq. (3.1) one finds the following relation between β_k and the inverse localization length a_k of the kth eigenvector:

$$\beta_k = \frac{1-e^{-a_k}}{a_k} \exp\left(1 - \frac{a_k e^{-a_k}}{1-e^{-a_k}}\right). \tag{3.9}$$

In deriving (3.9) we employ the fact that β_k self-averages over fluctuations of ξ's and is therefore a unique function of a_k. The relation (3.9) can be inverted, and a_k can be expressed in terms of β_k provided $\beta_k \le 1$. This is typically fulfilled except in the case of very strong delocalization.

In numerical simulations we have calculated β_k for each eigenvector and then determined a_k using the inverse of the relation (3.9). The dependence of the average inverse localization length $\langle a\rangle = \sum_{k=1}^{N} a_k/N$ on the scaling parameter $x = b^2/N$ is presented in Fig. 3. Different symbols correspond to various sizes of matrices. Evidently all the results fall onto a universal scaling curve. A very good approximation to this curve may be obtained on the basis of the following reasoning. First we introduce the so-called entropic localization length [1] (strictly speaking its ensemble average):

$$\beta = e^{\langle H\rangle - H_{\rm GOE}}. \tag{3.10}$$

It was found numerically [1] that with a very good accuracy

$$\beta = \frac{\gamma x}{1+\gamma x}, \quad \gamma \simeq 1.4. \tag{3.11}$$

This allows us to check our approach in the following way. Combining the above expression with the relation between a and β (3.9) we obtain the approximate formula for average inverse localization length $\langle a\rangle$ as a function of x. The result is indicated in Fig. 3 as the solid line and agrees very well with the numerical data providing a clear confirmation of scaling.

Another interesting evidence of scaling is presented in Fig. 4. Here we study the relation between the variance δ and the mean $\langle a\rangle$ of the (unknown) distribution of a_k. Both quantities depend on x only and amazingly are proportional to each other independently on the matrix dimension N. The best fit for the slope $\kappa = \delta/\langle a\rangle$ gives $\kappa \simeq 0.64$. This result is still not understood, it should be pointed out that the similar problem of the relation between the mean localization length and its variance is of great interest in the theory of conductance fluctuations (see, e.g., [23]).

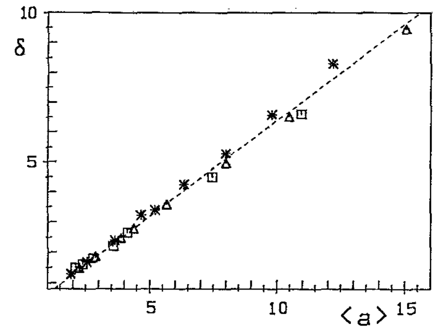

FIG. 4. Parameters of the distribution $P(a)$: width δ vs mean $\langle a\rangle$ for three different sizes of matrices: $N = 600$ (asterisks), $N = 300$ (triangles), and $N = 200$ (squares). The dashed line represents the best linear fit with the slope 0.64.

The numerical data concerning the relation between the mean of a and the variance δ presented in Fig. 4 allow us to check more carefully the conjecture about Gaussian character of the fluctuation of the inverse localization length. In Fig. 5 we show another plot of the relation δ versus $\langle a\rangle$ for $N = 600$. Each asterisk represents numerical results obtained in the same way as discussed above. Circles, on the other hand, were obtained by finding the best fit of the parameters δ and $\langle a\rangle$ in the formula (3.4) for each set of eigenvectors. One observes quite a good agreement (in particular the circles lie on a straight line), but the Gaussian fit seems to overestimate the width and underestimate the mean systematically.

This systematic discrepancy can be also noticed by looking directly at the distribution $P(a)$. In Fig. 6 we have plotted the histogram of $P(a)$ using a sample of ten

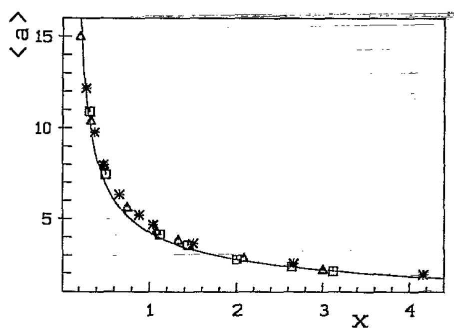

FIG. 3. Average inverse localization length $\langle a\rangle$ as a function of the scaling parameter x for $N = 600$ (asterisks), $N = 300$ (triangles), and $N = 200$ (squares). The solid line represents the theoretical dependence obtained from Eqs. (3.9) and (3.10).

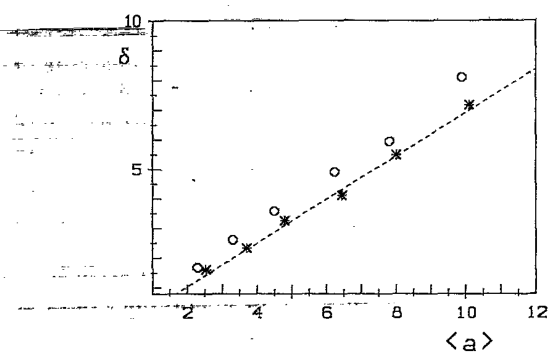

FIG. 5. Parameters of the distribution $P(a)$ for $N = 600$ obtained directly from the numerical data (asterisks) and from the best fit of the numerical data with the help of Eqs. (3.3) and (3.4) (circles). The dashed line represents the best linear fit to the asterisks data.

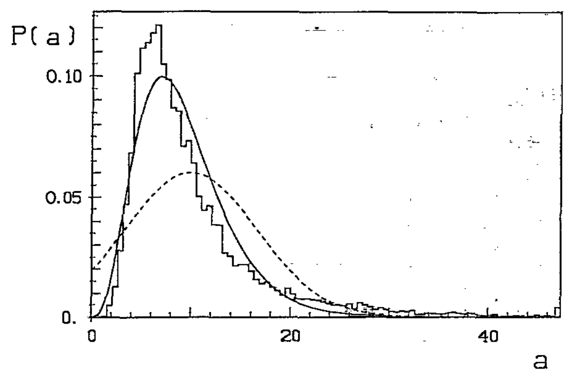

FIG. 6. Distribution of $P(a)$ for $x = 0.375$, $N = 600$ and the best fit of Eq. (3.4) ($\langle a \rangle = 9.9$, $\delta = 6.6$) - dashed line and Eq. (3.12) ($\langle a \rangle = 9.25$, $p = 3.23$) - solid line.

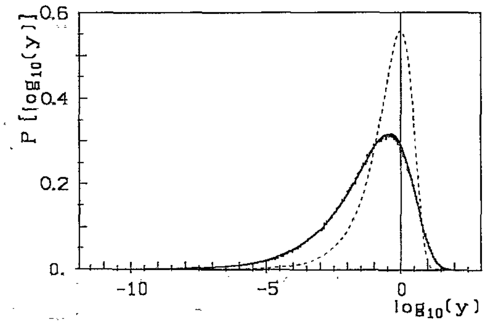

FIG. 7. Histogram $P(\log_{10} y)$ as in Fig. 2. The solid line represents the best fit ($\langle a \rangle = 6.85$ and $p = 1.23$) of the numerical data with the help of Eqs. (3.3) and (3.4); the dashed line represents the Porter-Thomas distribution.

matrices with $N = 600$ and $b = 15$. The dashed line represents a truncated Gaussian (3.4) with the same mean and width as those found from numerical calculations. Evidently for such relatively small values of x real distribution departs dramatically from the Gaussian shape. All these results show that the truncated Gaussian form of the distribution of a is rather poor approximation.

We made a preliminary study to improve the agreement with numerical data by fitting other forms of the distribution. One of the possible expressions, which gives much better correspondence with numerical data, has a form of a two-parameter family of curves:

$$P(a) \propto a^p \exp\left(-\frac{a}{\langle a \rangle}(p+1)\right) \qquad (3.12)$$

The solid curve in Fig. 6 corresponds to the best fit of Eq. (3.12) obtained for $\langle a \rangle = 9.25$ and $p = 3.23$.

The above two-parameter distribution $P(a)$ leads via Eq. (3.3) to a very satisfactory fit for eigenvector statistics valid for all values of the scaling parameter. Figure 7 presents the same data as Fig. 2 with solid line representing the distribution $P(y)$ obtained from Eqs. (3.3) and (3.12) with $\langle a \rangle = 6.85$ and $p = 1.23$.

Concluding, we have shown that eigenvector statistics of random band matrices depends only on the scaling parameter $x = b^2/N$, and is dominantly determined by fluctuations of localization length.

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