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Eigenvector statistics for the transitions from the orthogonal to the unitary ensemble

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A description for the eigenvector statistics of classical chaotic systems with a time reversal destroying part of the Hamilton operator is proposed in closed form. We have checked our form by a numerical simulation of a model system capable to show all intermediate stages between orthogonal and unitary behaviour, and proved the superiority to a chi-square distribution.

1. Introduction

In the last years a wide class of quantum systems has been investigated, which have a classical counterpart with global chaotic motion in their phase space. It has become a folkway to look at the eigenvalue statistics of those systems. In most cases it is found that, the probability to find the next eigenvalue as s, called the nearest neighbor distribution $P(s)$, is well approximated by the Wigner surmise obtained for one of the three gaussian random matrix ensembles, the Orthogonal Ensemble (OE), the Unitary Ensemble (UE) and the Symplectic Ensemble (SE). They are commonly called the three universality classes [1]. The symmetry properties of the system determine which of the three ensembles could be used to fit the level distribution.

A similar generic behaviour is shown, under the condition of classical chaos, by the eigenvector statistics of the systems [2]. The eigenvectors behave as if they were chosen out of one of the universality classes. For these ensembles even the exact results are known [3]. In the limit of large matrices one gets the well known Porter Thomas distribution for orthogonal ensemble and χ_ν^2 for the UE and the SE. Recently there has been a growing interest to study systems which could undergo a transition between two of these universal ensembles [4]. For the eigenvalue correlation functions of intermediate systems a great deal of work has been done [5], but it is difficult to confirm the predictions numerically or in experiments because a great number of eigenvalues is needed to get a statistically significant comparison. Our knowledge about intermediate eigenvector statistics is not so well founded. Despite the fact that the statistics of the eigenvectors consists of more elements, until now there is no explicit known family of interpolating functions between the χ_ν^2 distributions with $\nu=1, 2$ characteristic to OE, UE respectively. It had been proposed that a χ_ν^2 distribution of degree ν with ν continuously varying between 1 and 2 could be used for this purpose [6]. Apart from the fact that this family contains for $\nu=1, 2$ the exact results for the universality classes, there are no rational arguments behind this conjecture. In this work we present another family of interpolating distribution functions and supply theoretical arguments. We checked our prediction by a numerical study of a periodically kicked top [7].

2. Intermediate eigenvector statistics

Most considerations about eigenvector statistics and transition strength fluctuations can be reduced to the determination of the distribution function $P(\tilde{y})$ of the modulus of an eigenvector component.

$$\tilde{y}=|\langle\phi_i|c_k\rangle|^2, \quad i=1,\dots,N;\ k=1,\dots,N. \tag{1}$$

$|c_k\rangle$ is element of an "generic" basis [8], and $|\phi_i\rangle$ is a eigenvector of the system under consideration. In the following we will skip the explicit dependence of $\tilde{y}$ on i and k. The different distribution functions for $\tilde{y}$ in the three universality classes could be written in a unified manner if we use chi-square distributions

$$\chi^2_{\nu,\langle\tilde{y}\rangle}(\tilde{y})=\frac{\left(\dfrac{\nu}{2\langle\tilde{y}\rangle}\right)^{\nu/2}}{\Gamma\left(\dfrac{\nu}{2}\right)}\tilde{y}^{(\nu/2)-1}\exp\left(-\frac{\nu\tilde{y}}{2\langle\tilde{y}\rangle}\right) \quad \nu=1,2,4 \tag{2}$$

★ On leave from Instytut Fizyki, Uniwersytet Jagielloński, ul. Reymonta 4, PL-30-059 Kraków, Poland with $v=1, 2, 4$ for OE, UE and SE and $\langle\tilde{y}\rangle=1/N$ since $\sum_{i=1}^{N} y_i=1$. It is now convenient to rescale $\tilde{y}$ to the new variable $y=N\tilde{y}$, so that its mean value is one $\langle y\rangle=1$. This rescaling is of great importance for the comparison of experimental data with the theoretical predictions, because similar to the behaviour of eigenvalue spectra only the fluctuations about the mean value could be expected to be generic.

The following observation is important. For the orthogonal ensemble y is simple the square of a real number, while for the unitary ensemble $y=c_r^2+c_i^2$ is the squared modulus of a complex number i.e. the sum of two squares. Due to Kramers degeneracy in the case of symplectic ensemble y is the sum of four squares. In the limit of large matrix dimensions these components are practically independent random variables distributed according to a $\chi^2_{1,v^{-1}}$ (with variance v^{-1}) distribution for the three cases. For the distribution function of y then easily the form (2) with $\langle y\rangle=1$ follows. Let us now focus on the transition from the orthogonal ensemble to the unitary ensemble. It is very intuitive to guess that one must assign different weights to the different variables. The imaginary part of a component y grows from a distribution with variance zero to a distribution of variance 1/2. In other words we assume that the variable y is a squeezed state. Its shape varies from a line along the real axis $-1\leqq y\leqq 1$ (OE), through an ellipse with the sum of the axis equal to one (intermediate case), to a circle with radius 1/2 (UE). Let us now denote weights for c_r^2 and c_i^2 respectively by $\sigma_r^2=1/b$ and $\sigma_i^2=1-1/b$, $b\in\{1, 2\}$. The distribution of $y=c_r^2+c_i^2$ is given by the convolution of the distributions of c_r^2 and c_i^2

$$P_b(y)=\chi^2_{1,\sigma_r^2}*\chi^2_{1,\sigma_i^2}(y) = \frac{b\,e^{-\left(y\frac{b^2}{4(b-1)}\right)}}{2\sqrt{b-1}}\, I_o\left(\frac{yb(2-b)}{4(b-1)}\right), \quad (3)$$

where I_o is the Bessel function of order zero. More precise arguments supporting the above formula are provided in the appendix A. It can be easily checked that in the limiting cases $P_b(y)$ tends to $\chi^2_{1,1}$ (for $b\to 1$) and $\chi^2_{2,1}$ (for $b\to 2$). Figure 1a displays a set of distributions $\chi^2_{v,1}$ for some values of the parameter $v\in\{1, 2\}$, Fig. 1b shows a family of $P_b(y)$ distributions with $b\in\{1, 2\}$. Note the shift of the maximum of the $P_b(y)$ distribution for the intermediate values of $b\in\{1, 2\}$.

An analogous method can be also applied to get a family of distributions interpolating between the orthogonal and symplectic or unitary and symplectic ensembles. Formulae for all cases are derived in the appendix B. It should be noted, however, that by investigating the transition from or to the symplectic ensemble special case has to be paid to provide appropriate treatment of the Kramers degeneracy in the system.

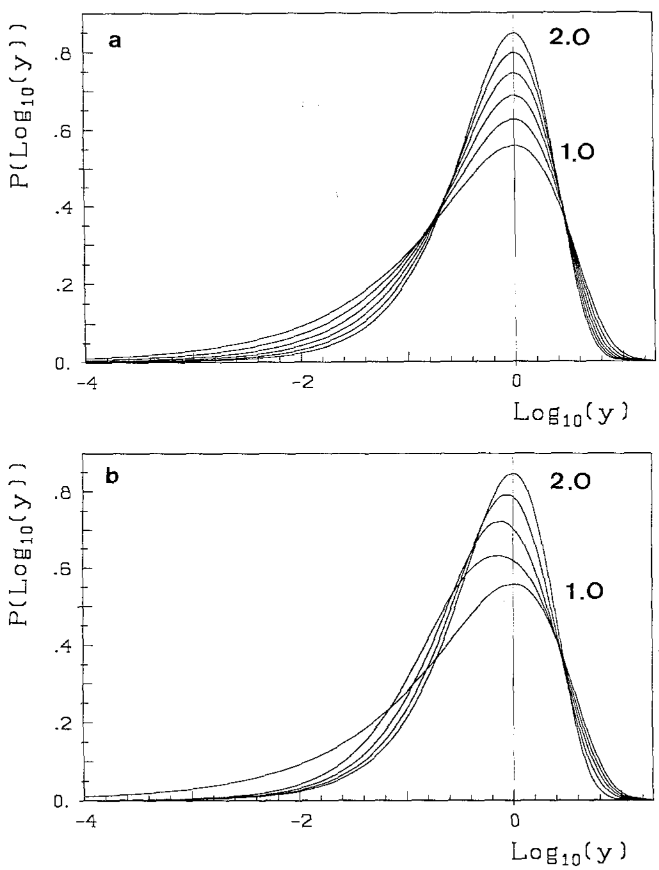

Fig. 1a, b. Distributions of eigenvector statistics interpolating between orthogonal and unitary ensembles for several values of control parameter denoted in the picture. **a** The $\chi^2_{v,1}(y)$ distribution; **b** The $P_b(y)$ distribution

3. Numerical results

We analysed the dynamical system of a kicked top. The dynamical variables are the three components of the angular momentum operator $\hat{J}$ obeying standard commutation relations $[\hat{J}_i, \hat{J}_k]=i\varepsilon_{ikl}\hat{J}_l$, $\hat{J}|j\rangle=j(j+1)|j\rangle$. The system is defined by the following Floquet operator:

$$\hat{F}=\exp(-ip\hat{J}_y)\exp\left(-iK_1\frac{\hat{J}_z^2}{2j}\right)\exp\left(-iK_2\frac{\hat{J}_x^2}{2j}\right). \quad (4)$$

For generic values of the parameters p, K_1 and K_2 this system pertains to the unitary universality class. If the parameter K_2 vanishes, a generalized antiunitary symmetry appears [7], and the orthogonal ensemble should be used. Thus changing values of the parameter K_2 one can study the transition from orthogonal to unitary ensemble.

Numerical results obtained from $j=400$ are presented in Fig. 2. Dashed lines denote the limiting cases $\chi^2_{1,1}$ and $\chi^2_{2,1}$ for OE and UE. Two curves have been fit to the numerical results: $P_b(y)$ of (3) (bold line) and $P_v(y)$ of (2) (narrow line). Both drawings for different coupling constants K_2 (Fig. 2a $K_2=0.160$, $b=1.108$, $v=1.34$ and Fig. 2b $K_2=0.183$, $b=1.183$, $v=1.50$) clearly show advantage of the suggested Bessel function distribution given by (3). In each case the square deviation $S=\sum_{i=1}^{ibin=200}[f(y_i)-\text{hist}(i)]^2$ is six times smaller for the dis-

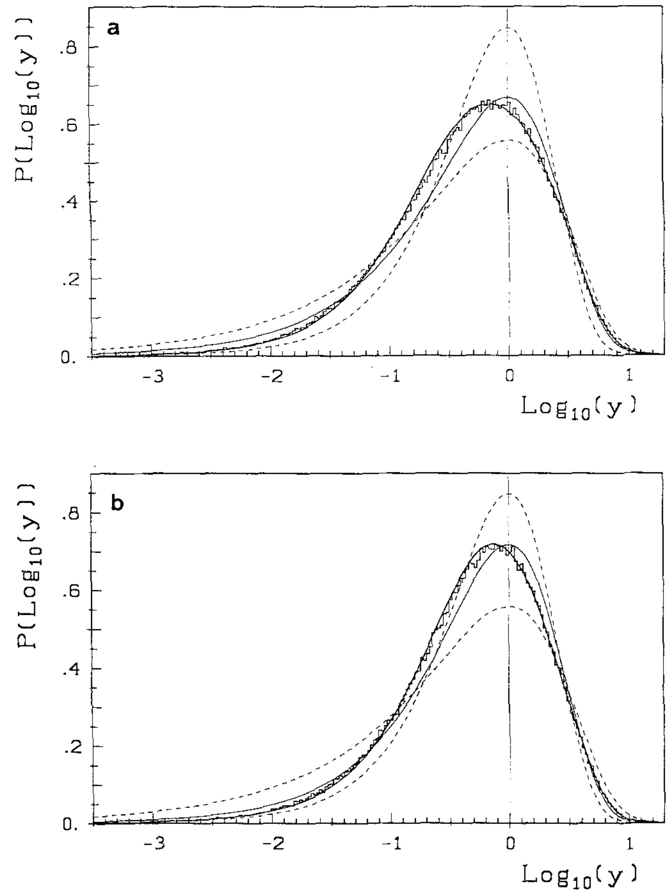

Fig. 2a, b. Histogram of eigenvector components of kicked top with $j=400$, $K_1=14.59$, $p=1.7$. Bold lines – best fit of $P_b(y)$, narrow line – best fit of $\chi^2_{v,1}(y)$, dashed lines – limiting cases of $\chi^2_{1,1}$ (OE) and $\chi^2_{2,1}$ (UE), **a** $K_2=0.160$, $b=1.108$. $v=1.34$; **b** $K_2=0.184$, $b=1.183$, $v=1.50$

tribution $P_b(y)$ than for the distribution $P_v(y)$ defined in (2). The applicability of the proposed family of distributions seems therefore to be justified. The behaviour of b, when K_2 varies, will be discussed at an other place. There we will investigate numerically all aspects of the transition OE$\leftrightarrow$UE, including eigenvalue statistics.

4. Conclusions

We have proposed an eigenvector distribution for classical chaotic systems, which cannot be described by one universality class. Such eigenvector statistics are of interest for the study of systems the Hamiltonians of which have a small part responsible for generalized time reversal symmetry breaking [4, 9]. For example, eigenvector statistics are used to analyse the fluctuations in the neutron widths of complex nuclei to set a upper bound on a possible time reversal breaking part in their Hamiltonian operators. For the quantum description of classical chaotic systems, which do not belong in one of the three universality classes the eigenvector statistics are useful as they give a new generic quantum property of such systems. Despite the fact that there is a long history of considerations of those statistics [10], the explicit form is still lacking. Our approach was verified in a numerically reliable way and supported by theoretical arguments, so we hope that we could close this gap.

On the same line of considerations as our problem is the question, how the eigenvector statistics of a system behaves when it undergoes a transition from regular to chaotic motion [11, 12]. The hope is that one can borrow some of the ideas we introduced here to come closer to the solution of this problem.

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Appendix A

We provide an additional argument supporting the assumption that the family of interpolating distributions for eigenvectors is given by (3). Concentrating on the distribution $P(y_1)$ of the first component of a given eigenvector, we integrate out all other variables $y_2, \ldots, y_N$ in the joint distribution function [3]. For this purpose the invariance of the interpolating ensembles [5] under orthogonal transformations will be exploited. The joint distribution function of the eigenvector has the form:

$$P(u_k, v_k) = \delta(1-\sqrt{(r^2+R^2)})\, X_N(r^2, R^2, \lambda)\, \Omega_{N-1}^{-2}, \tag{A.1}$$

where u_k and v_k with $k=1, 2, \ldots, N$ are the real and imaginary parts of the eigenvector components, r and R are the norm of the real and imaginary parts of the eigenvector under consideration, and Ω_{N-1} is the volume of the N-dimensional hypersphere. The freedom in the phase of the eigenvector has been used to fix the value of the scalar product $\langle u|v\rangle = \sum_{k=1}^{N} u_k v_k$ to zero. The delta factor guarantees the normalization of the eigenvectors and X_N is an unknown function of r^2 and R^2, which depends on a transition parameter $\lambda \in \{0, \infty\}$ [5, 11]. For $\lambda=0$ we start with the orthogonal case and for $\lambda=\infty$ we have the unitary case. The desired distribution function for $y=|c_1|^2$ is:

$$P(y) = \int_0^1 r^{N-1} R^{N-1}\, \delta(y - N(u_1^2 + v_1^2))\, \delta(1-\sqrt{(r^2+R^2)}) \cdot X_N(r^2, R^2, \lambda)\, \Omega_{N-1}^{-2}\, \mathrm{d}r\, \mathrm{d}R\, \mathrm{d}\Omega_u\, \mathrm{d}\Omega_v. \tag{A.2}$$

For the vectors u, v we introduce N-dimensional polar coordinates i.e.:

$$\begin{aligned}
u_1 &= r\cos(\phi_1)\\
u_2 &= r\sin(\phi_1)\cos(\phi_2)\\
&\vdots\\
u_{N-1} &= r\sin(\phi_1)\ldots\cos(\phi_{N-1})\\
u_N &= r\sin(\phi_1)\ldots\sin(\phi_{N-1}).
\end{aligned}$$

With this transformation the integration measure reads:

$$\mathrm{d}\Omega_u^{N-1}=\prod_{i=1}^{N-1}\sin(\phi_i)^{N-1-i}\mathrm{d}\phi_i$$
$$=\sin(\phi_1)^{N-2}\mathrm{d}\phi_1\,\mathrm{d}\Omega_u^{N-2}. \tag{A.4}$$

After integrating over the delta functions we gain:

$$P(y)=\int_0^1 \alpha \bar{X}_N(r^2,\lambda)\,G_N(y,r^2)\,2r\,\mathrm{d}r \tag{A.5}$$

where

$$\alpha=\frac{2\pi}{N}\left(\frac{\Omega^{N-2}}{\Omega^{N-1}}\right)^2,$$

$$\bar{X}_N(r^2,\lambda)=r^{N-2}(1-r^2)^{\frac{N-2}{2}}X_N(r^2,1-r^2,\lambda),$$

$$G_N(y,r^2)=\frac{1}{\pi r\sqrt{1-r^2}}$$
$$\cdot\int_0^1\frac{\left(1-\frac{y\xi^2}{Nr^2}\right)^{\frac{N-3}{2}}\left(1-\frac{y(1-\xi^2)}{N(1-r^2)}\right)^{\frac{N-3}{2}}}{\sqrt{1-\xi^2}}\mathrm{d}\xi. \tag{A.6}$$

For $N\gg 1$ the constant α and the function G_N have the following behaviour:

$$\alpha\simeq 1, \tag{A.7}$$

and

$$G_N(y,r^2)\simeq\frac{1}{\pi r\sqrt{1-r^2}}\int_0^1\frac{\mathrm{e}^{-(y\xi^2/2r^2)}\,\mathrm{e}^{-(y(1-\xi^2)/2(1-r^2))}}{\sqrt{1-\xi^2}}\mathrm{d}\xi$$
$$=\frac{\mathrm{e}^{-(y/4r^2(1-r^2))}}{2r\sqrt{1-r^2}}I_o\left(\frac{y(1-2r^2)}{4r^2(1-r^2)}\right), \tag{A.8}$$

where I_o is the Bessel function of order zero [13]. The function $\bar{X}_N(r^2,\lambda)$ obeys for all N the normalisation:

$$\int_0^1 \bar{X}_N(r^2,\lambda)r\,\mathrm{d}r=1. \tag{A.9}$$

The following symmetry relations can be found [10]:

$$G_N(y,r^2)=G_N(y,1-r^2), \tag{A.10}$$

$$\bar{X}_N(r^2,\lambda)=\bar{X}_N(1-r^2,\lambda). \tag{A.11}$$

Introducing notation:

$$r^2=1-\frac{1}{b},\quad b\in\{1,2\}, \tag{A.12}$$

we get:

$$P(y)=\int_1^2 \bar{X}_N\left(\frac{\bar{b}-1}{\bar{b}},\lambda\right)G_N\left(y,\frac{\bar{b}-1}{\bar{b}}\right)\frac{\mathrm{d}\bar{b}}{\bar{b}^2}. \tag{A.13}$$

In the limiting cases of OE ($\lambda=0$) the distribution $\bar{X}_N\left(\frac{b-1}{b},\lambda\right)\Big/b^2$ is equal to $\delta(b-1)$ and for UE ($\lambda=\infty$) this distribution is equal to $\delta(b-2)$. Assuming for the whole range of $0\leqq\lambda\leqq\infty$

$$\frac{\bar{X}_N\left(\frac{b-1}{b},\lambda\right)}{b^2}=\delta(b-b(\lambda)) \tag{A.14}$$

one obtains from (A.8) and (A.13) the proposed form of (3) of the interpolating distribution function. It might be worthy to find an additional argument for this assumption connected with explicit form (A.14) for $\bar{X}_N\left(\frac{b-1}{b},\lambda\right)\Big/b^2$. The rigorous prove of (A.14) might be tedious, since one has to integrate out all eigenvalues e_i in the joint probability distribution of eigenvalues and eigenvectors $P(e_i,\varphi_j)$, $i=1,\dots,N$; $j=1,\dots,N^2-N$, [9] (φ_j parameterize the manifold of the eigenvectors). In the resulting distribution parameters of all but one eigenvector should integrated out in order to obtain the explicit form of $\bar{X}_N$.

Appendix B

We will derive formulae for distributions interpolating between a) OE and SE, b) UE and SE. In general one has to consider convolution of two $\chi^2_{v,\langle y\rangle}$ distributions. Resulting distribution:

$$P_{v_1,\langle y_1\rangle;v_2,\langle y_2\rangle}:=\chi^2_{v_1,\langle y_1\rangle}*\chi^2_{v_2,\langle y_2\rangle}, \tag{B.1}$$

can be expressed in terms of the Kummer function $M(a,b,z)$ [13]. Performing the integration one gets:

$$P_{v_1,\langle y_1\rangle;v_2,\langle y_2\rangle}$$
$$=\left(\frac{v_1}{2\langle y_1\rangle}\right)^{v_1/2}\left(\frac{v_2}{2\langle y_2\rangle}\right)^{v_2/2}\frac{y^{\frac{(v_1+v_2)}{2}-1}}{\Gamma\left(\frac{v_1}{2}+\frac{v_2}{2}\right)}\mathrm{e}^{-y(v_1/2\langle y_1\rangle)}$$
$$\cdot M\left(\frac{v_1}{2},\frac{v_1+v_2}{2},\frac{y}{2}\left(\frac{v_1}{\langle y_1\rangle}-\frac{v_2}{\langle y_2\rangle}\right)\right). \tag{B.2}$$

Note that due to the Kummer Transformation $M(a,b,z)=\mathrm{e}^z M(a-b,b,-z)$ [13], the above formula posses the necessary symmetry:

$$P_{v_1,\langle y_1\rangle;v_2,\langle y_2\rangle}=P_{v_2,\langle y_2\rangle;v_1,\langle y_1\rangle}. \tag{B.3}$$

Equation (B.2) provides families of distributions interpolating between all universality classes. The analysed case of transition OE↔UE is received by putting $v_1=v_2=1$, $\langle y_1\rangle=\frac{1}{b}$, $\langle y_2\rangle=1-\frac{1}{b}$, $b\in\{1,2\}$ and (B.2) reduces to (3). To study transition OE↔SE one has to put $v_1=1$, $v_2=3$, $\langle y_1\rangle=\frac{1}{b}$, $\langle y_2\rangle=1-\frac{1}{b}$, $b\in\{1,4\}$. The last case of transition UE↔SE will be achieved by setting $v_1=2$, $v_2=2$, $\langle y_1\rangle=\frac{2}{b}$, $\langle y_2\rangle=1-\frac{2}{b}$, $b\in\{2,4\}$.

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