Volume of the set of separable states

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The question of how many entangled or, respectively, separable states there are in the set of all quantum states is considered. We propose a natural measure in the space of density matrices ϱ describing N-dimensional quantum systems. We prove that, under this measure, the set of separable states possesses a nonzero volume. Analytical lower and upper bounds of this volume are also derived for $N=2\times2$ and $N=2\times3$ cases. Finally, numerical Monte Carlo calculations allow us to estimate the volume of separable states, providing numerical evidence that it decreases exponentially with the dimension of the composite system. We have also analyzed a conditional measure of separability under the condition of fixed purity. Our results display a clear dualism between purity and separability: entanglement is typical of pure states, while separability is connected with quantum mixtures. In particular, states of sufficiently low purity are necessarily separable. [S1050-2947(98)02808-X]

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I. INTRODUCTION

The question of quantum inseparability and entanglement of mixed states has attracted much attention recently. This problem is, by far, more complicated than the analogous one for pure states [1], and involves subtle effects like ''hidden nonlocality'' [2] or ''distillation of entanglement'' [3,4]. Generally speaking, one is interested in inseparable states as the states containing Einstein-Podolsky-Rosen correlations. In fact, all inseparable mixed states have a nonzero ''entanglement of formation'' [5] which means that to build them a nonzero amount of pure entangled states is needed. In particular if a source emits pairs of particles in *unknown* pure states, so that they form a quantum ensemble described by an inseparable density matrix, then it follows that the source *must* emit some entangled pairs with a nonzero probability. In this sense the inseparable mixed states can be viewed as entangled, in correspondence to the entangled pure states.

One of the fundamental questions concerning these subjects is to estimate how many entangled (disentangled) states exist among all quantum states. More precisely, one can consider the problem of quantum separability or inseparability from a measurement theoretical point of view, and ask about relative volumes of both sets. There are three main reasons of importance in this problem. The first reason, of some philosophical implication, may be contained in the questions ''Is the world *more classical* or *more quantum*? Does it contain more quantum-correlated (entangled) states than classically correlated ones?'' The second reason has a more practical origin. Analyzing some features of entanglement, one often has to rely on numerical simulations. It is then important to know to what extent entangled quantum states may be considered as typical. Finally, the third reason has a physical origin. The physical meaning of separability has recently been associated with the possibility of partial time reversal [6] (see also Ref. [7]). Separable states of composite systems allow time reversal in one subsystem, without losing their physical relevance. However, for a system of a dimension $N\geqslant8$, the fact that a state admits partial time reversal is not sufficient to assure separability, and counterexamples have been found [8]. Moreover, it has recently been shown that none of those counterexamples can be distilled to a singlet form [9]. Therefore, it seems pertinent to investigate how frequently such peculiar states appear. At first glance it seems quite likely that such states form a set of measure zero, and that from a measurement theoretical point of view the set of separable states and states that admit partial time reversal have equal volumes.

In this paper we make an attempt to answer at least the two first of the above-formulated questions. We also give a qualitative argument of why the last conjecture fails. To this aim we propose a simple and natural measure on the set $\mathcal{S}$ of density matrices acting on a finite-dimensional Hilbert space $\mathcal{H}$. Using this measure we estimate the relative volume of the set of separable states $\mathcal{S}_{\rm sep}$. The upper (lower) bound on this volume is obviously the lower (upper) bound on the relative volume of the set of inseparable (entangled) states $S_{\rm insep}=S\setminus S_{\rm sep}$.

The paper is organized as follows. Sec. II contains our definition of the natural measure in $\mathcal{S}$. In Sec. III, we recall basic definitions of separable states, and prove that, for any compound system $\mathcal{S}$, the volume of $\mathcal{S}_{\rm sep}$ is nonzero regardless of the number of subsystems it contains and its (finite) dimension. This is achieved by proving the existence of a

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topological lower bound of this volume. Better lower bounds are also calculated analytically by analyzing the relation between the purity of the state and its separability. In Sec. IV, analytic upper bounds on the volume of $\mathcal{S}_{\rm sep}$ are found. The study of inseparable states with positive partial transposition is presented in Sec. V. In Sec. VI, we present estimates on the volume of separable states obtained by the Monte Carlo numerical simulations. This section is self-contained, and also includes a simplified corollary of the results of Secs. II–V, and a discussion of the dualism between purity and separability. We conjecture that the volume of separable states decreases exponentially with the dimension of the Hilbert space. Finally, Sec. VII contains our conclusions and open questions.

The reader should note that Secs. II–V have a rather formal mathematical character. The results of these sections provide a rigorous base for the numerical calculations of Sec. VI, but detailed knowledge of the proofs is by no means necessary to understand the main message of the paper. The reader who is not interested in such rigorous proofs of the presented results may well skip Secs. III–V, and go straight to Secs. VI and VII.

II. NATURAL MEASURE OF THE SET OF QUANTUM STATES

Let us consider a set of states in an N-dimensional Hilbert space $\mathcal{H}$. In particular, $\mathcal{H}$ may describe a composite system with m component subsystems: $\mathcal{H}=\otimes_{i=1}^{m}\mathcal{H}_i$, where $\Pi_{i=1}^{m}N_i=N$.

An operator ϱ acting on $\mathcal{H}$ describes a *state* if $\mathrm{Tr}\varrho=1$ and if ϱ is a positive operator, i.e.,

$$\mathrm{Tr}(\varrho P)\geqslant 0, \tag{1}$$

for any projector P. Any state represented by a density matrix ϱ can in turn be represented by its spectral decomposition:

$$\varrho=\sum_{n=1}^{N}\Lambda_n P_n, \quad \sum_{n=1}^{N}\Lambda_n=1, \quad \Lambda_n\geqslant 0, \tag{2}$$

where P_n form a complete set of orthogonal projectors. Thus the set of states can be viewed as a Cartesian product of sets:

$$\mathcal{S}=\mathcal{P}\times\triangle. \tag{3}$$

The set $\mathcal{P}$ denotes the family of complete sets of orthonormal projectors $\{P_i\}_{i=1}^{N}$, $\Sigma_{n=1}^{N}P_n=I$, where I is the identity matrix. There exist the unique, uniform measure ν on $\mathcal{P}$ induced by the Haar measure on the group of unitary matrices $U(N)$. Integration over the set $\mathcal{P}$ thus amounts to an integration of the corresponding angles and phases in N-dimensional complex space that determine the families of orthonormal projectors (or, alternatively speaking, the unitary matrix that diagonalizes ϱ).

The symbol $\triangle$ in Eq. (3) represents there the set of all Λ_n's, which is a subset of the $(N-1)$-dimensional linear submanifold of real space $\mathbb{R}^N$, defined by the trace condition, $\Sigma_{n=1}^{N}\Lambda_n=1$. Geometrically, $\triangle$ is defined as a convex hull (i.e., a set of all convex combinations of the edge points) $\triangle=\mathrm{conv}\{\mathbf{x_i}\in\mathbb{R}^{\mathbb{N}}:\mathbf{x_i}=(0,\ldots,1_i,\ldots,0),\ i=1,\ldots,N\}$. Since the simplex $\triangle$ is a subset of the $(N-1)$-dimensional hyperplane, there exist a natural measure on $\triangle$ which is defined as a usual normalized Lebesgue measure $\mathcal{L}_{N-1}$ on $\mathbb{R}^{N-1}$. More specifically, any measurable function $f(.)$ of $\Lambda_1,\ldots,\Lambda_N$ can be integrated with the measure

$$\frac{1}{V_s}\int_0^1 d\Lambda_1\cdots\int_0^1 d\Lambda_N f(\Lambda_1,\ldots,\Lambda_N)\,\delta\left(\sum_{1}^{N}\Lambda_n-1\right)=\frac{1}{V_s}\int_0^1 d\Lambda_1\cdots\int_0^1 d\Lambda_{N-1} f\left(\Lambda_1,\ldots,\Lambda_{N-1},1-\sum_{0}^{N-1}\Lambda_n\right), \tag{4}$$

where the normalization constant V_s equals the volume of the set $\triangle$ in $\mathcal{R}^{N-1}$, whereas $\delta(\ )$ denotes Dirac's delta distribution. The two above-discussed measures induce a natural measure on $\mathcal{S}$:

$$\mu=\nu\times\mathcal{L}_{N-1}. \tag{5}$$

III. VOLUME OF THE SET OF SEPARABLE STATES

A. Preliminaries—separable states

Throughout this paper we shall assume that the Hilbert space of the considered quantum system has an arbitrary but finite dimension. To make further considerations more clear, we start from the following notation and definitions. Recall first that the space $\mathcal{A}$ of operators acting on $\mathcal{H}$ constitute a new Hilbert space (a so-called Hilbert-Schmidt space) with the scalar product

$$\langle A,B\rangle=\mathrm{Tr}(B^{\dagger}A). \tag{6}$$

It induces a natural norm (a trace norm)

$$\|A\|=\sqrt{\mathrm{Tr}(A^{\dagger}A)} \tag{7}$$

which, according to the condition $\dim\mathcal{H}<\infty$, is topologically equivalent to all other norms on $\mathcal{A}$, in particular to the norm $\|A\|'=\mathrm{Tr}|A|$. Furthermore, let us recall the following.

Definition 1. The state ϱ acting on the Hilbert space $\mathcal{H}=\mathcal{H}_1\otimes\mathcal{H}_2$ is called separable[1] if it can be approximated in the trace norm by the states of the form

[1]The presented definition of separable states is due to Werner [1], who called them classically correlated states.

$$\varrho=\sum_{i=1}^{k} p_i \varrho_i \otimes \tilde{\varrho}_i , \quad (8)$$

where ϱ_i and $\tilde{\varrho}_i$ are states on $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Usually one deals with a finite-dimensional Hilbert space $\dim\mathcal{H}=N$. For this case it has been shown [8] that any separable state can be written as a convex combination of *finite* product pure states, i.e., in those cases the ''approximation'' part of the definition is redundant.

It has also been shown [11] that the necessary condition for separability of the state ϱ is positivity of its partial transposition ϱ^{T_2}. The latter is defined in an arbitrary orthonormal product basis $|f_i\rangle\otimes|f_j\rangle$ as a matrix with elements:

$$\varrho^{T_2}_{mm',nn'} \equiv \langle f_m|\otimes\langle f_{m'}|\varrho^{T_2}|f_n\rangle\otimes|f_{n'}\rangle=\varrho_{mn',nm'} . \quad (9)$$

Although the matrix ϱ^{T_2} depends on the used basis, its eigenvalues do not. Consequently, for any state the above condition can be checked using *an arbitrary* product orthonormal basis.[2] For systems of dimensions 2×2 and 2×3 the partial transposition condition is also a sufficient one [10], and thus the set of separable states is completely characterized by this condition. The definition of separable states can be easily generalized to systems composed of more than two subsystems.

Definition 2. The state ϱ acting on the Hilbert space $\mathcal{H}=\otimes_{l=1}^{m}\mathcal{H}_l$ is called separable if it can be approximated in the trace norm by the states of the form

$$\varrho=\sum_{i=1}^{k} p_i \otimes_{l=1}^{m} \varrho_i^l , \quad (10)$$

where ϱ_i^l are states on $\mathcal{H}_l$. Straightforward generalization of the proof about decomposition from Ref. [8] gives us the possibility of omitting the approximation part in the definition.

Lemma 1. Any separable state ϱ of a system composed by m subsystems can be written as

$$\varrho=\sum_{i=1}^{k} p_i P^i_{\mathrm{prod}}, \quad k\leqslant N^2 \quad (11)$$

where P^i_{prod} are pure product states having the m-decomposable form $\otimes_{l=1}^{m}P_l$, where P_l are projectors acting on $\mathcal{H}_l$. It is worth mentioning that minimal decompositions with $k=N$ can be always found for $N=4$ [6,12].

B. Existence of nonzero lower bound for the volume of separable states

We shall prove now that the volume of the set of separable states is nonzero independently of the dimension of the Hilbert space and the number of subsystems m composing it. For our purposes we first prove the following simple lemma.

[2] As the full transposition of a positive operator is also positive, positivity of the partial transposition ϱ^{T_2} is equivalent to positivity of the partial transposition ϱ^{T_1} (defined in an analogous way).

Lemma 2. If the Hermitian operator $A\in\mathcal{A}$ satisfies $\langle A,\otimes_{i=1}^{m}P_i\rangle=0$ for any product projectors $\otimes_{i=1}^{m}P_i$, then it is a trivial zero one.

Proof. Let us consider an arbitrary orthogonal [in the sense of scalar product (6)] product Hermitian basis in the space of operators $\mathcal{A}$, i.e., a basis such that any of its elements is a product of Hermitian matrices (for instance, in the 2×2 case the basis could consist of products of Pauli matrices $\sigma_n\otimes\sigma_m$, with $n,m=0,1,2,3$ and $\sigma_0=I$). Hermicity assures that any element of the basis can be written as a real combination of product projectors. Any coefficient of the expansion of A in this basis is given by the scalar product (6) of A and the corresponding basis element. From the general assumption of formulas of type $\langle A,\otimes_{i=1}^{m}P_i\rangle=0$, we obtain immediately that all expansion coefficients must vanish. Hence A must be equal to the zero operator. We can now propose the following theorem.

Theorem 1. Let $\triangle_\epsilon$ be a simplex defined as $\triangle_\epsilon=\mathrm{conv}\{\mathbf{y_i}\in\mathbb{R}^N:\mathbf{y}_i=\epsilon\mathbf{x}_i+(1-\epsilon)\mathbf{z}_I;i=1,\ldots,N;\mathbf{z}_I=(1/N,\ldots,1/N)\}$. Let us define a set $\mathcal{Q}_\epsilon=\mathcal{P}\times\triangle_\epsilon$. Then there exists some positive ϵ such that $\mathcal{Q}_\epsilon\subset\mathcal{S}_{\mathrm{sep}}$, where $\mathcal{S}_{\mathrm{sep}}$ represents the set of separable states.

The meaning of the above theorem is straightforward. It proposes that all states in the sufficiently small neighborhood of the maximally mixed state $\varrho_I=I/N$ [which is represented in $\triangle$ as a point $\mathbf{z}_I=(1/N,\ldots,1/N)$ for any chosen spectral decomposition of unity] are necessarily separable. Note that by definition, the simplex $\triangle_\epsilon$ has edges ϵ times smaller than $\triangle$, so that its volume $\mu(\triangle_\epsilon)=\epsilon^{N-1}\mu(\triangle)=\epsilon^{N-1}$, since according to our normalization $\mu(\triangle)=1$.

Proof. Suppose, conversely, that for any positive ϵ the set $\mathcal{Q}_\epsilon$ contains some inseparable state ϱ_{insep}. It is easy to see that then there must exist a sequence of inseparable states $\varrho^n_{\mathrm{insep}}$ convergent to the maximally mixed state ϱ_I. According to lemma 1 and theorem 1 from Ref. [10], there exist a sequence of operators A_n separating the states $\varrho^n_{\mathrm{insep}}$ from the state ϱ_I in the sense that for any n it holds that $\langle A_n,\varrho^n_{\mathrm{insep}}\rangle<0$ and $\langle A_n,\varrho_I\rangle\geqslant0$. Moreover, from the quoted results it follows that $\langle A_n,\sigma\rangle\geqslant0$ for any $\sigma\in\mathcal{S}_{\mathrm{sep}}$. Let us normalize the operators A_n by introducing $\tilde{A}_n=A_n/\|A_n\|(\|A_n\|=\sqrt{\langle A^\dagger A\rangle})$. These operators satisfy

$$\langle\tilde{A}_n,\varrho^n_{\mathrm{insep}}\rangle<0, \quad \langle\tilde{A}_n,\sigma\rangle\geqslant0 \quad \text{for any } \sigma\in\mathcal{S}_{\mathrm{sep}} . \quad (12)$$

In particular it holds that $\langle\tilde{A}_n,\varrho_I\rangle\geqslant0$. From construction, the sequence $\tilde{A}_n$ belongs to the sphere in the finite-dimensional space $\mathcal{A}$. As the latter is a compact set, the sequence includes some subsequence $\tilde{A}_{n(k)}$ which is convergent to some nonzero operator $\tilde{A}_*(\|\tilde{A}_*\|=1)$. From Eq. (12) and continuity of the scalar product, it follows that the limit operator also satisfies

$$\langle\tilde{A}_*,\sigma\rangle\geqslant0 \quad \text{for any } \sigma\in\mathcal{S}_{\mathrm{sep}} . \quad (13)$$

Now using Eq. (12) and the Schwarz inequality, we obtain

$$0\leqslant\langle\tilde{A}_{n(k)},\varrho_I\rangle=\langle\tilde{A}_{n(k)},\varrho_I-\varrho_{\rm insep}^{n(k)}\rangle+\langle\tilde{A}_{n(k)},\varrho_{\rm insep}^{n(k)}\rangle$$
$$\leqslant\langle\tilde{A}_{n(k)},\varrho_I-\varrho_{\rm insep}^{n(k)}\rangle\leqslant\|\tilde{A}_{n(k)}\|\|\varrho_I-\varrho_{\rm insep}^{n(k)}\|=\|\varrho_I-\varrho_{\rm insep}^{n(k)}\|. \tag{14}$$

Taking the limit with respect to k, we obtain

$$\mathrm{Tr}\tilde{A}_*=\langle\tilde{A}_*,\varrho_I\rangle=\lim_{k\to\infty}\langle\tilde{A}_{n(k)},\varrho_I\rangle=0. \tag{15}$$

Hence $\tilde{A}_*$ is traceless, which is in contradiction with Eq. (13). Indeed, if the operator $\tilde{A}_*$ is to be nontrivial (the construction implies its unit norm) then there must exist some product state $P_{\rm prod}\equiv\otimes_{i=1}^m P^i$ such that $\langle\tilde{A}_*,P_{\rm prod}\rangle\neq 0$ (see lemma 2). Since, on the other hand, one requires the trace of $\tilde{A}_*$ to vanish, one obtains that $\langle\tilde{A}_*,P_{\rm prod}\rangle=-\langle\tilde{A}_*,I-P_{\rm prod}\rangle$. Hence, one of the separable states $\sigma'=P_{\rm prod}$, $\sigma''=[1/(N-1)](I-P_{\rm prod})$ violates condition (13), which gives the expected contradiction. The above theorem leads immediately to the following one.

Theorem 2. The measure $\mu(\mathcal{S}_{\rm sep})$ of separable states is a nonzero one. In particular there exists always some $\epsilon>0$ such that the following inequality holds:

$$\mu(\mathcal{S}_{\rm sep})\geqslant\mu(\triangle_\epsilon)=\epsilon^{N-1}>0. \tag{16}$$

As an illustration of the above theorem, let us consider the 2×2 or 2×3 cases ($N=4$ and 6) for which separability is equivalent to the positivity of the partial transposition. It is easy to see that the spectrum of the partially transposed density matrix must belong to the interval $[-\frac{1}{2},1]$. Hence any state of the form $\varrho=(1-p)(I/N)+p\tilde{\varrho}$, for an arbitrary $\tilde{\varrho}$ and $p\leqslant 2/(2+N)$, has a positively defined partial transposition, and thus is separable for the considered cases. As the maximal value of p is $\frac{1}{3}$ or $\frac{1}{4}$, this means that the value of ϵ in the above theorem can be estimated just by $\frac{1}{3}$ or $\frac{1}{4}$ for $N=4$ or 6, respectively. In Sec. III C we shall show that those bounds can be significantly improved.

C. Purity and separability

As we have shown, all states in the small enough neighborhood of the totally mixed state $\varrho_I=I/N$ are separable. On the other hand, we know that in the subspace of all pure states, the measure of separable states is equal to zero [2]. It is, therefore, interesting to investigate the relationship between entanglement and mixture of quantum states. A qualitative characterization of the degree of mixture is provided by the von Neumann entropy $H_1(\varrho)=-\mathrm{Tr}(\varrho\ln\varrho)$. Another quantity, called the participation ratio,

$$R(\varrho)=\frac{1}{\mathrm{Tr}(\varrho^2)}, \tag{17}$$

is often more convenient for calculations. It varies from the unity (for pure states) to N (the totally mixed state ϱ_I), and may be interpreted as an effective number of states in the mixture. This quantity, applied in solid-state physics a long time ago [15], is related to the von Neumann–Renyi entropy of order 2, $H_2(\varrho)=\ln R(\varrho)$. The latter, called also the purity of the state, together with other quantum Renyi entropies $H_q(\varrho)=(\ln[\mathrm{Tr}\varrho^q])/(1-q)$ is used, for $q\neq1$, as a measure of how much a given state is mixed. It has also been applied for the derivation of some necessary conditions of separability in Ref. [16]. In Sec. VI, we shall demonstrate, using numerical simulations, that the participation ratio (and other von Neumann–Renyi entropies) allows one to establish a dualism between purity and separability of the states of composite systems. In this subsection we use it to calculate a natural lower bound on the volume of separable states for dimensions $N=4$ and 6. For this purpose consider the following lemma.

Lemma 3. If the state ϱ satisfies

$$R(\varrho)\geqslant N-1, \tag{18}$$

where N is the dimension of $\mathcal{H}$, then ϱ^{T_2} is positive defined, i.e., its spectrum $s(\varrho^{T_2})$ belongs to the simplex $\triangle$.

Proof. Let us denote by $B_N(r,P)$ the ball in the space $\mathbb{R}^N$ with radius r and center P, and by $S_N(r,P)$ its surface. Condition (18) is invariant with respect to the partial transposition, because $\mathrm{Tr}(\varrho^2)=\mathrm{Tr}((\varrho^{T_2})^2)$. That implies that $s(\varrho^{T_2})\in B_N(r,\mathbf{z}_I)$ with $r=1/\sqrt{N-1}$ and $\mathbf{z}_I=(1/N,\ldots,1/N)$. Let us define the $(N-1)$-dimensional linear manifold $\mathcal{M}_{N-1}=\{\mathbf{x}=(x_1,\ldots,x_N),\Sigma_{i=1}^N x_i=1\}$. We only need to show that its intersection with the ball is included in the simplex $\triangle$, i.e., that the new $(N-1)$-dimensional ball $B'_{N-1}\equiv B_N(r,\mathbf{z}_I)\cap\mathcal{M}_{N-1}\subset\triangle$. This can be seen in the following way. It follows, from the high symmetry of the sphere and the invariance of the simplex under cyclic permutations of coordinates, that the center of this intersection is again $\mathbf{z}_I$. Hence the radius r' of B_{N-1} can be calculated immediately by taking the distance from an arbitrary point from the surface $S_N(r,\mathbf{z}_I)\cap\mathcal{M}_{N-1}$ [say, for example from the point $(0,1/N-1,\ldots,1/N-1)$ to the point $\mathbf{z}_I$]. It is elementary to show that $r'=1/\sqrt{N(N-1)}$. On the other hand, one can calculate the maximal radius r'' of the ball of the type $B''_{N-1}(r'',\mathbf{z}_I)$ included in $\triangle$ by calculating the minimal distance of $\mathbf{z}_I$ to the boundary of $\triangle$. To this aim we have to minimize $(r'')^2=\Sigma_{i=1}^N(x_i-1/N)^2$ with the constraints $\Sigma_{i=1}^N x_i=0$, and $x_0=0$. Using Lagrange multipliers we immediately obtain $r''=r'$, and hence B_{N-1} belongs to $\triangle$, which ends the proof.

Now using the explicit expressions for the volume of a $(N-1)$-dimensional ball $(V_N(r)=\pi^{(N-1)/2}r^{N-1}/\Gamma[(N-1)/2])$, and for the volume of the simplex $\triangle$ belonging to the manifold $\mathcal{M}_{N-1}[V_\triangle=\sqrt{N}/(N-1)!]$, one can obtain the lower bound of the volume of states with positive partial transposition,

$$\tau_N=\frac{(N-1)!\,\pi^{(N-1)/2}}{N^{N/2}(N-1)^{(N-1)/2}\Gamma\left(\dfrac{N+1}{2}\right)}. \tag{19}$$

Recalling that for Hilbert spaces of dimensions $N=2\times2$ and $N=2\times3$ the states with positive partial transposition are the separable states, Eq. (19) leads directly to the following theorem.

Theorem 3. If the participation ratio satisfies $R(\varrho)\geqslant3$ $(R(\varrho)\geqslant5)$ for $N=4$ $(N=6)$ then the state ϱ is separable. Therefore, the measure $\mu(\mathcal{S}_{\rm sep})$ of separable states is restricted from below by the inequality $\mu(\mathcal{S}_{\rm sep}) \geqslant \pi/6\sqrt{3} \simeq 0.302$ for $N=4$, and $8\pi^2/625\sqrt{5} \simeq 0.056$ for $N=6$.

IV. UPPER BOUNDS ON THE VOLUME OF SEPARABLE STATES

In this section we seek for upper bounds on the volume of $V_{\rm sep}$, or, equivalently, lower bounds on the set of inseparable states $V_{\rm insep}$. Several necessary conditions for separability have recently been established with the aid of positive maps. We should use them to determine an upper bound on $V_{\rm sep}$. As we shall see, these conditions are in some way complementary, and can be combined to obtain a better estimate of the upper bound of $V_{\rm sep}$. Our first estimate relies on the positivity of the partial transposition. It is valid for any dimension, but we shall apply it to composite systems of dimension 2×2. Note that if a state has a partial transposition which is not positively defined, then the state is necessarily inseparable. Before proceeding further we should first recall the Schmidt decomposition of a pure state $|\Psi\rangle \in \mathcal{H}=\mathcal{H}_1 \otimes \mathcal{H}_2$, $\dim\mathcal{H}_1 = N_1$, $\dim\mathcal{H}_2 = N_2$, $N_1 \times N_2 = N$,

$$|\Psi\rangle = \sum_{i=1}^{\min(N_1,N_2)} a_i |e_i\rangle \otimes |f_i\rangle, \tag{20}$$

where $|e_i\rangle\otimes|f_i\rangle$ form a biorthogonal basis $\langle e_i|e_j\rangle = \langle f_i|f_j\rangle = \delta_{ij}$, and $0 \leqslant a_i \leqslant 1$ denote the coefficients of the Schmidt decomposition with the condition $\Sigma_i a_i^2 = 1$. It is straightforward to see that $P_\Psi^{T_2} = (|\Psi\rangle\langle\Psi|)^{T_2}$ has eigenvalues a_i^2 for $i=1,\dots,\min(N_1,N_2)$ and $\pm a_i a_j$ for $i\neq j$. We can now state our first lemma.

Lemma 4. If in the range of a state ϱ there exists $|\Psi\rangle$ such that

$$\Lambda = \langle \Psi|\varrho^{-1}|\Psi\rangle^{-1}\rangle \frac{1}{1+\max_{i\neq j}(a_i a_j)}, \tag{21}$$

then ϱ is inseparable.

Proof. According to Ref. [17], any state ϱ can be expressed as

$$\varrho = \Lambda P_\Psi + (1-\Lambda)\tilde{\varrho}, \tag{22}$$

where P_Ψ is a projector onto $|\Psi\rangle$ and $\tilde{\varrho}$ is a (positively defined) state. Thus

$$\varrho^{T_2} = \Lambda P_\Psi^{T_2} + (1-\Lambda)\tilde{\varrho}^{T_2}. \tag{23}$$

Recall that for any N_1 or N_2, the eigenvalues of $\tilde{\varrho}^{T_2}$ belong to the interval $[-\frac{1}{2},1]$. Let $|\Psi_{\rm neg}\rangle$ denote the eigenvector corresponding to the minimal eigenvalue of $P_\Psi^{T_2}$: $-\max_{i\neq j}(a_i a_j)$. We thus have

$$\langle\Psi_{\rm neg}|\varrho^{T_2}|\Psi_{\rm neg}\rangle \leqslant -\Lambda(\max_{i\neq j}(a_i a_j)) + 1 - \Lambda < 0, \tag{24}$$

because $\langle\Psi_{\rm neg}|\tilde{\varrho}^{T_2}|\Psi_{\rm neg}\rangle \leqslant 1$. The above inequality implies that ϱ^{T_2} is not positively defined when condition (21) holds, and therefore ϱ is not separable. Note, that the lemma 4 can be applied in particular to the eigenvectors of ϱ.

Lemma 5. If ϱ has an eigenvector $|\Psi\rangle$ corresponding to the eigenvalue Λ such that the condition (21) holds, then ϱ is inseparable.

The eigenvalue Λ can fulfill the above condition if and only if it is the largest eigenvalue, because it must be larger than $\frac{2}{3}$. The corresponding normalized eigenvector, however, is absolutely arbitrary, and, according to invariant measure on the group, it can be generated simply by a uniform probability on the N-dimensional unit sphere. This implies, as we shall see below, that the Schmidt coefficients a_i are also absolutely arbitrary and distributed uniformly on the octant of the $\sqrt{N}$-dimensional sphere.

Consider the $N_1 = N_2 = K$ case. Any vector in $(N=K^2)$-dimensional space from the unit sphere can be represented by a row of complex numbers x_i with the condition $\Sigma_i^N |x_i^2| = 1$. In any product basis, we can view it as a $K\times K$ matrix C_{ij} with $i,j=1,\dots,K$, and with the condition $\mathrm{Tr}(C^\dagger C)=1$. We seek the uniform distribution on the set of such matrices. But, from the polar decomposition theorem, any matrix of such a type can be represented in the form

$$C = U'DU \tag{25}$$

where U' and U are some unitary matrices, while D is a diagonal matrix with non-negative elements (eigenvalues). These eigenvalues are nothing else but a_i. The reason is that the above form, which is the analog of the spectral decomposition of the Hermitian matrix, is at the same time the Schmidt decomposition written in the matrix notation. In our case (taking into account the above-mentioned trace condition), the spectrum of D is represented by the point belonging to the octant area of the sphere. This leads to the measure

$$\mu'(\Psi) = \nu(U'(K))\nu(U(K))\mu(D), \tag{26}$$

where the first two measures are Haar measures on the unitary group $U(K)$, and the last one is the uniform (Lebesgue) measure on the octant of the ball in K-dimensional space. Similar results can be straightforwardly generalized for the cases $N_1 \neq N_2$.

If one calculates now measure (26) for $|\Psi\rangle$, and combines it with the uniform measure on the simplex $\mu_\triangle$, one could estimate an upper bound of separable states:

$$\mu(S_{\rm sep}) \leqslant 1 - \int \Theta(\max_i \Lambda_i - [1+\max_{i\neq j}(a_i a_j)]^{-1}) \times d\mu'(\Psi)\, d\mu_\triangle, \tag{27}$$

where Θ denotes the Heaviside function. The double integration over the unitary groups that contains $\mu'(\Psi)$ can be easily performed, since neither Λ_i nor a_i depend on the direction of $|\Psi\rangle$. This is the first qualitative argument that the measure of inseparable states does not vanish.

Moreover, recently [18] a new separability condition has been introduced with the aid of positive maps condition [10]: if the state ϱ is separable, then $I\otimes\varrho_1 - \varrho$ must be positive, where ϱ_1 is the reduced density matrix. It implies for any $|\Psi\rangle$ that $\mathrm{Tr}[(I\otimes\varrho_1)P_\Psi] \geqslant \mathrm{Tr}(\varrho P_\Psi)$. Straightforward estimation tells us that for any separable state it must hold that

$\langle\Psi|\varrho|\Psi\rangle \leq \max_i a_i^2$, where a_i are again the Schmidt decomposition coefficients of Ψ. That implies a lemma analogous to lemma 4.

Lemma 6. If in the range of a state ϱ there exists $|\Psi\rangle$ such that

$$\Lambda = \langle\Psi|\varrho|\Psi\rangle > \max_i a_i^2, \tag{28}$$

then ϱ is inseparable.

This lemma is neither stronger nor weaker than lemma 4. If we apply it to eigenvectors of ϱ, however, the relevant eigenvalue need not be the maximal. In the case 2×2 we can combine both conditions (lemmas 4-6) to obtain a better estimate on the upper bound of $\mu(S_{\rm sep})$

$$\begin{aligned} 1-\mu(S_{\rm sep}) \geq \frac{4}{V_\Delta V_{\rm oct}} &\times \int_0^1 d\Lambda_1 \int_0^{1-\Lambda_1} d\Lambda_2 \int_0^{1-\Lambda_1-\Lambda_2} d\Lambda_3 \\ &\times \int_{a_1\geq 0} da_1 \int_{a_2 \geq 0} da_2 \\ &\times \Theta[\Lambda_1 - (1+a_1 a_2)^{-1}]\Theta(\Lambda_1 - \max(a_1^2, a_2^2)) \\ &\times \delta(a_1^2 + a_2^2 - 1). \end{aligned} \tag{29}$$

Notice that in the above expression the first three integrals are over the eigenvalues of ϱ that are located in the simplex Δ, whereas the remaining two integrals are on the eigenvalues of D from the octant area of the sphere. The integrals can be calculated analytically, but the resulting expressions are very complex. After a tedious, but straightforward, calculation we obtain

$$\mu(S_{\rm sep}) \leq 0.863. \tag{30}$$

In general for an arbitrary dimension $N = N_1 \times N_2$ and $K = \min(N_1, N_2)$; $1-\mu(S_{\rm sep})$ can be estimated from below by a bound $b(N_1, N_2)$ using the above method. This bound, on the other hand, can be estimated from above by the volume of the ''corners'' of the simplex Δ of the sides $1-(1/K)$ regardless of the uniform measure on pure states, by setting it equal to unity there. This follows from the fact that condition (28) can only then be fulfilled, when an eigenvalue of ϱ is larger than $\max_i a_i^2 \geq 1/K$. The relative volume of such corners equals $N[1-(1/K)]^{N-1}$. Keeping formula (20) in mind the above simple estimation leads to the following corollary:

Corollary. Consider a quantum system $\in \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\dim\mathcal{H}_1 = N_1$, $\dim\mathcal{H}_2 = N_2$, $N = N_1 \times N_2$, and $K = \min(N_1, N_2)$. Then using lemmas 4–6, the volume of separable states is restricted from above by

$$\mu(S_{\rm sep}) \leq 1 - b(N_1, N_2), \tag{31}$$

where

$$b(N_1, N_2) \leq \left(1 - \frac{1}{K}\right)^{N-1}. \tag{32}$$

The volume [Eq. (32)], however, converges asymptotically to the value 1 as N_1 and N_2 grow, so that in the limit of large N we obtain a trivial result $\mu(S_{\rm sep}) < 1$. At the same time, the numerical results which we shall present subsequently strongly suggest that there should exist an upper bound for $\mu(S_{\rm sep})$ converging to zero. So far, the rigorous proof that in the limit of higher dimensions $\mu(S_{\rm sep}) \to 0$ remains an open problem.

V. INSEPARABLE STATES WITH POSITIVE PARTIAL TRANSPOSITION

As it was mentioned in Sec. I, for $N \geq 8$, there are states which are inseparable but have positive partial transposition [8,10]. Moreover, it was recently shown that all states of such type represent ''bound'' entanglement in the sense that they cannot be distilled to the singlet form [9]. The immediate question that arises is how frequently such peculiar states appear in the set of all the states of a given composite system. This question is related to the role of time reversal in the context of entanglement of mixed states [6,7]. Below we provide a qualitative argument that the volume of the set of those states is also nonzero.

Lemma 7. For $N \geq 8$, the set of inseparable states with positive partial transposition includes a nonempty ball.

Proof. Consider the two sets of quantum states for some composite system: the set of separable states $S_{\rm sep}$ and the set of states with positive partial transposition T. The first of them is convex and compact. The second one is a convex set.[3] Since positivity of partial transposition is necessary for separability, we have obviously $S_{\rm sep} \in T$. Consider any state σ belonging to T but not to $S_{\rm sep}$ (we know that for $N \geq 8$ such states exist). Let us take the ball $B(r, \varrho_I)$ around the maximally chaotic state ϱ_I such that the whole $B(r, \varrho_I)$ belongs to $S_{\rm sep}$ (the ball can be in principle defined in an arbitrary norm, as all norms are topologically equivalent; cf. Sec. III). Obviously such a ball exists; otherwise ϱ_I would belong to the boundary of $S_{\rm sep}$. It can be shown that the latter would contradict the fact of a nonzero volume (see Sec. III). Consider the sequence of balls obtained from $B(r, \varrho_I)$ by a translation of the center r, and a rescaling $B_n = B(r/n, [1-(1/n)]\sigma + (1/n)\varrho_I)$. Some B_n's have to include no separable states. Otherwise, if any B_n included some separable states, e.g., $\varrho_n^{\rm sep}$, by virtue of compactivity of $S_{\rm sep}$, the state σ would be separable as the limit of a sequence of separable states $\varrho_n^{\rm sep}$. Hence some B_{n_0}'s do not belong to set of separable states. But on the other hand, any state from B_{n_0} is a convex combination of elements of T. Thus the whole ball B_{n_0} belongs to T, as the latter is a convex set. In this way we have shown that there is some ball $B_{n_0} \subset T$ that does not intersect with $S_{\rm sep}$. But the inseparable states with positive partial transposition are just the ones belonging to T and not to $S_{\rm sep}$. This ends the proof of the lemma.

VI. NUMERICAL RESULTS

In this section we provide rather precise estimates of the measure $\mu(S_{\rm sep})$ for $N = 4$ and 6, as well as upper bounds of

[3] In fact it is also compact, but we shall not need this property here.

such measure for $N>6$. Our results are obtained numerically. This section is self-contained, in the sense that it can be read directly without passing though the more technical sections III–V. It also includes the results obtained in these previous sections.

A. Estimation of volume of separable states

Our goal is to estimate the volume of the set of separable states $\mu(S_{\rm sep})$. For simplicity we discuss states consisting of two subsystems. Any state describing a mixture of N_1- and N_2-dimensional subspaces may be represented by a positive defined $(N_1N_2)\times(N_1N_2)$ Hermitian matrix ϱ with trace equal to unity. ϱ^{T_2} denotes, as before, the matrix partially transposed with respect to the second subsystem. As mentioned previously, if ϱ is separable, then necessarily ϱ^{T_2} is positive [11]. Moreover, for the simplest 2×2 and 2×3 problems this condition is also a sufficient one [10], which is not true in the general case $N\geqslant 8$ [8,10]. Therefore, the set of separable states is *a subset* of states with positive partial transpositions. Thus, in order to estimate the volume of separable states from above, it is sufficient to find the volume of the set of states with positive partial transpositions.

Let us remind the reader that, according to Sec. II, any state (any density matrix) can be represented in a family $\mathcal{P}$ of complete sets of orthogonal projectors, and the simplex $\triangle$ representing all possible spectra,

$$S=\mathcal{P}\times\triangle. \tag{33}$$

On the other hand, any element of $\mathcal{P}$ can be represented by a unitary transformation, and any element of $\triangle$ as a diagonal matrix D with the matrix elements $\Lambda_{ij}=\delta_{ij}\Lambda_i$, such that they fulfill $\Sigma_{i=1}^{N}\Lambda_i=1$. Such a representation corresponds to the form

$$\varrho=UDU^{\dagger}. \tag{34}$$

Thus a uniform distribution on the set of all density matrices represented by Eq. (33) is constructed naturally by postulating an uniform distribution on unitary transformations $U(N)$ (the Haar measure), and an uniform distribution on diagonal matrices D.

We have numerically calculated the volume of the set of matrices with positive partial transpose, and in such a way have estimated the volume of the separable states. An algorithm to generate random $U(N)$ matrices was recently given in Refs. [13,14]. The random diagonal matrix D fulfills that $\Sigma_{i=1}^{N}\Lambda_i=1$, so the vector $\tilde{\Lambda}=(\Lambda_1,\ldots,\Lambda_N)$ is localized on the $(N-1)$-dimensional simplex $\triangle$ (see Sec. II). Physically speaking, no component of this vector is distinguished in any sense. Random vectors $\tilde{\Lambda}$ are thus generated *uniformly* on this subspace according to the simple method described in Appendix A.

The numeric algorithm is then straightforward: first, we generate random density matrices of any size $N=N_1\times N_2$; second, we construct their partial transpositions; and finally, we diagonalize them and we check whether their eigenvalues $\lambda_i;i=1,\ldots,N$ are all positive. This procedure has been repeated several million times in order to obtain an accuracy on the order of $\frac{1}{1000}$.

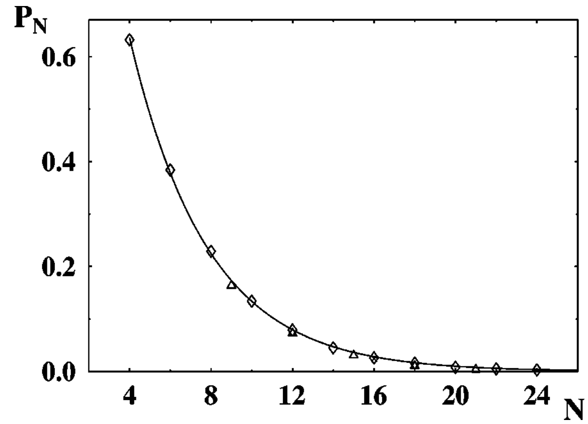

FIG. 1. Probability of finding a state with a positive partial transpose as a function of the dimension of the problem N. For $N>6$, this gives an upper bound only of the relative volume of the separable states. Different symbols distinguish different sizes of one subsystem [$k=2(\diamond)$, $3(\triangle)$, and $4(\times)$], while the solid line represents the exponential fit.

We should recall that in previous sections we have obtained rigorous analytical lower and upper bounds of $\mu(S_{\rm sep})$, i.e., we have proven that $0<\mu(S_{\rm sep})<1$. The lower bounds follow from the fact that the states sufficiently close to the totally mixed state are separable for all N. We have also shown that states which have sufficiently large participation ratio (i.e. which are sufficiently mixed) have a positive partial transpose, and are thus separable for $N=4$ and 6. On the other hand, the upper bounds come from the fact that matrices with large eigenvalues corresponding to an entangled eigenvector, are necessarily inseparable. Let us denote the measure $\mu(S_{\rm sep})$ in the $N_1\times N_2$ case by $P_{N_1\times N_2}$. Our analytical bounds for the cases 2×2 and 2×3 are summarized below:

$$0.302<P_{2\times2}<0.863,\quad 0.056<P_{2\times3}. \tag{35}$$

Our numerical results agree with these bounds, but to our surprise the probability that a mixed state $\varrho\in\mathcal{H}_2\times\mathcal{H}_2$ is separable exceeds 50%. The results are

$$P_{2\times2}\approx0.632\pm0.002\quad\text{and}\quad P_{2\times3}\approx0.384\pm0.002. \tag{36}$$

For higher dimensions, our results are summarized in Fig. 1. This figure displays the probability P_N that the partially transposed matrix ϱ^{T_2} is positive as a function of $N=N_1\times N_2$. For $N=4$ and 6, this is just the required probability of encountering a separable state, while for $N>6$ it gives an upper bound for this quantity.

Due to symmetry of the problem $P_{N_1\times N_2}$ must be equal to $P_{N_2\times N_1}$. Numerical results strongly suggest that this quantity depends only on the product $N=N_1\times N_2$, e.g., $P_{2\times6}=P_{3\times4}$. Moreover, the dependence can be well reproduced by an exponential decay. The best fit gives $P_N\sim1.8e^{-0.26N}$. We conjecture, therefore, that the measure of separable states decreases exponentially with the size of the system under consideration.

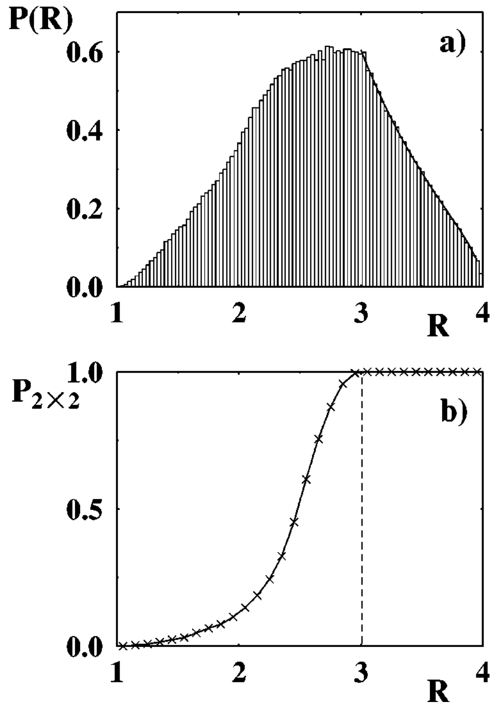

FIG. 2. Purity and separability in $(N=4)$-dimensional Hilbert space: (a) probability of finding a quantum state with a participation ratio R; (b) probability of finding a separable state $P_{2\times 2}$ as a function of the participation ratio R (crosses). All states beyond the dashed vertical line placed at $R=N-1$ are separable. Circles show the mean entanglement $\langle t\rangle$, as defined in Appendix B.

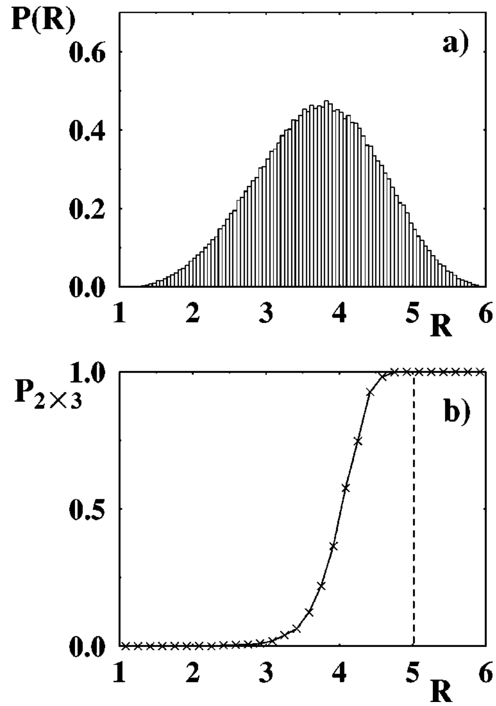

FIG. 3. Same as in Fig. 2 for $N=6$.

B. Purity versus separability

Here we would like to illustrate the physical connection between the participation ratio and entanglement, which was already discussed in Sec. III C. We recall that the participation ratio $R(\varrho)=1/\mathrm{Tr}(\varrho^2)$ gives a characterization of the degree of mixture, and can be interpreted as the effective number of states on the mixture. We have demonstrated that if the state ϱ has a sufficiently large participation ratio, or equivalently a sufficiently low von Neuman–Renyi entropy $H_2(\varrho)=\ln R(\varrho)$, then its partially tranposed density matrix is always positive. This holds for any arbitrary N, so in particular it means it is separable for $N=4$ and 6. A more precise estimate can be performed numerically.

For example consider the $(N=4)$-dimensional Hilbert space. A manifold of the constant participation ratio R is given by the ellipsoid in the space of eigenvalues:

$$\Lambda_1^2+\Lambda_2^2+\Lambda_3^2+(1-\Lambda_1-\Lambda_2-\Lambda_3)^2=1/R.$$

The probability distribution $P(R)$ obtained numerically using the natural uniform measure on the three-dimensional (3D) simplex is plotted in Fig. 2(a). This corresponds to the relative volume of the cross section of a 3D hypersphere of radius $R^{-1/2}$ centered at (0,0,0,0), with the simplex defined by a condition $\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4=1$. For $N=4$ and $R>3$, we obtain $P(R)=6\pi R^{-2}\sqrt{1/R-1/4}$.

We have randomly generated a million points in the 3D simplex, computed the corresponding participation ratio, rotated the corresponding state by a random unitary matrix U, and checked whether the generated state is separable. This procedure allows us to investigate the dependence of the probability of the separable states on the participation ratio. Our numerical results are summarized in Fig. 2(b), and, again, are fully compatible with the theorems and lemmas obtained in the previous sections.

Similar results and compatibility are obtained for the case

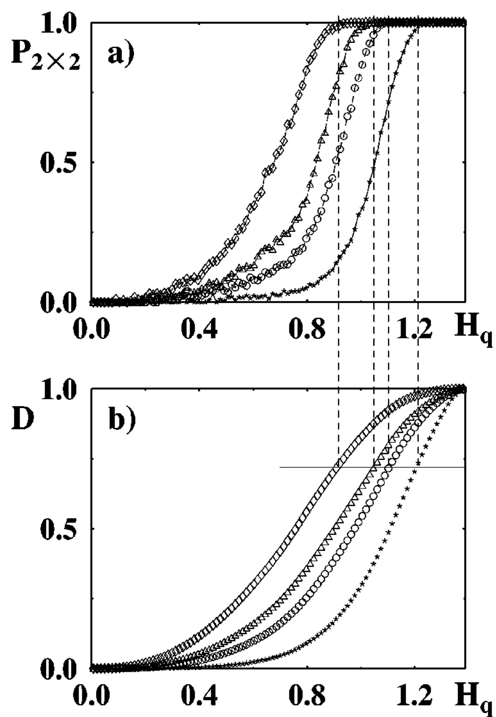

FIG. 4. (a) Probability of separable states $P_{2\times 2}$ for $N=4$ vs von Neumann–Renyi entropies $H_q\in[0,\ln(4)]$ for $q=1(\star)$, $2(\bigcirc)$, $3(\triangle)$, and $10(\diamond)$. (b) Integrated distribution function $D(H_q)$. The vertical line drawn at $D\sim 0.7$ corresponds to these values of H_q for which $P_{\rm sep}=1$.

$N=6$. In this case we deal with a 5D simplex. Numerical data, displayed in Fig. 3, support the general fact that the quantum states with $R(\varrho)\geqslant N-1$ have positive partial transpose (i.e., are separable for $N=4$ and 6) in the N-dimensional Hilbert space. It is interesting to note that the relative amount of separable states unambiguously increases as the participation ratio grows. Moreover, the mean ''degree of entanglement'' $\langle t\rangle$, defined in Appendix B, decreases monotonically with R. This illustrates that for composite systems there exist a *dualism* between two quantities: the purity and separability of the state. The purer a quantum state is, the smaller its probability of being separable. This conclusion is supported by Fig. 4, where we plot the dependence of the probability of finding a separable state $P_{\rm sep}$ on the values of several quantum Renyi entropies $H_q(\varrho)$. For $q\neq 1$ the Renyi entropy is defined as $H_q(\varrho):=(\ln[{\rm Tr}\varrho^q])/(1-q)$, while in the limit $q\rightarrow 1$ it tends to the standard von Neumann entropy $H_1(\varrho)=-{\rm Tr}\varrho\ln\varrho$. We have not been able to generalize rigorously the results for $q=2$, for which large entropy implies necessarily the positivity of the partial transpose. Nevertheless, the numerical results suggest that a similar result holds for arbitrary values of the Renyi parameter q. All states with sufficiently large $H_q(\varrho)$ are separable, as shown in Fig. 4(a). Can this fact be used to obtain a better lower bound for $P_{N_1\times N_2}$? Figure 4(b) suggests that this is not the case: the cumulative probability P of $H_q(\varrho)$ attains more or less the same value for all q at the point at which $P_{\rm sep}$ becomes $\simeq 1$.

VII. CONCLUSIONS

Summarizing, we have developed a measurement theoretical approach to the separability-inseparability problem. To this aim, we have proposed a natural measure in the space of density matrices ϱ on the N-dimensional space. We have proven that, under this measure, the set of separable states has a nonzero volume, although this volume is not maximal in the set of all states. Analytical lower and upper bounds of this volume have been found for $N=2\times 2$ and $N=2\times 3$ cases. We have also provided qualitative evidence that for $N\geqslant 8$ the peculiar set of inseparable states with positive partial transposition has, under this measure, a nonzero volume.

We have used Monte Carlo simulations to estimate with much higher precision the volume of separable states. Our numerical simulations give strong evidence that this volume decreases exponentially with the dimension of the composite quantum system. Finally, we have also discussed the dualism between purity and separability, and have shown that while entanglement is typical of pure states, separability is connected with quantum mixtures.

Several questions concerning this subject remain still as open problems, and, so far, we have not been able to prove them rigorously. Particularly challenging are the two following related questions. (i) Does the volume of the set of separable states really go to zero as the dimension of the composite system N grows, and how fast? (ii) Does the set of separable states really have a volume strictly smaller than the volume of the set of states with a positive partial transpose?

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APPENDIX A: GENERATION OF UNIFORM DISTRIBUTION ON THE SIMPLEX

Our aim is to construct the uniform distribution of the points *on* the manifold given by $\Sigma_{i=1}^{N}\Lambda_i=1$, where each component Λ_i is non-negative. Let $\xi_i\,;i=1,\ldots,N-1$ be independent random numbers generated uniformly in the interval (0,1). We start with uniform distributions *inside* the $(N-1)$-dimensional simplex Δ_{n-1} defined by $\Sigma_{i=1}^{N-1}\Lambda_i<1$. Its volume is proportional to the product $\Pi_{k=1}^{N-1}x_k^{N-1-k}dx_k=\Pi_{k=1}^{N-1}d[x_k^{N-k}]$, which enables us to find the required densities for each component. Since the vertex of the simplex Δ_{N-1} is situated at $\{0,\ldots,0\}$, the largest weight corresponds to the small values of x_1. Therefore,

$$\Lambda_1=1-\xi_1^{1/(N-1)},$$

$$\Lambda_2=[1-\xi_2^{1/(N-2)}](1-\Lambda_1),$$

$$\Lambda_k=[1-\xi_k^{1/(N-k)}]\left(1-\sum_{i=1}^{k-1}\Lambda_i\right),$$

$$\ldots$$

$$\Lambda_{N-1}=[1-\xi_{N-1}]\left(1-\sum_{i=1}^{N-2}\Lambda_i\right).$$

Eventually the last component Λ_N is already determined as

$$\Lambda_N=1-\sum_{i=1}^{N-1}\Lambda_i\,.$$

The vector $\tilde{\Lambda}=\{\Lambda_1,\ldots,\Lambda_N\}$ constructed in this way is distributed uniformly in the requested subspace. An alternative procedure, albeit more time consuming, is to take any vector of an $N\times N$ auxiliary random unitary matrix V and obtain the random vector as $\Lambda_k=|V_{kj}|^2$ with arbitrary j.

APPENDIX B: AVERAGED ''DEGREE OF ENTANGLEMENT''

The problem of defining a quantity capable of measuring a ''degree of entanglement'' is a subject of several recent studies [19–22]. Let us define, for a given density matrix ϱ, the quantity

$$t := \sum_{i=1}^{N} |\lambda_i'| - 1,$$

where λ_i', $i=1,\ldots,N$ denotes the eigenvalues of the partially transposed matrix ϱ^{T_2}. For any separable matrix all eigenvalues are positive; its trace is equal to unity and t is equal to zero. On the other hand, for the maximally entangled states belonging to a 2×2 system, the spectrum of eigenvalues λ' consists of $\{-\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\}$, so that $t=1$. Moreover, for the often studied 2×2 Werner states [1] depending on the parameter x, the quantity t vanishes for $x<\frac{2}{3}$ (separable states) and equals $t=(3x-2)/(4-3x)$ for entangled states ($\frac{2}{3}\leqslant x\leqslant 1$).

We could not resist the temptation to investigate the mean value of t averaged over random density matrices generated as described above. For the 2×2 problem the mean value $\langle t\rangle$ equals 0.057 and increases to 0.076 for the 2×3 problem. For large systems this quantity seems to saturate at $t\sim 0.10$, as the ratio of the matrices with positive values of t (some eigenvalues of ϱ^{T_2} are negative) tends to unity. Moreover, as shown in Figs. 2(b) and 3(b), the average degree of entanglement $\langle t\rangle$ decreases monotonically with the participation ratio R, which provides a quantitative characterization of the relation between the entanglement and purity of mixed quantum states.

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