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LETTER TO THE EDITOR

Linewidth of a periodically pumped laser with sub-Poissonian photon statistics

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Abstract. We study the dynamical properties of a periodically pumped laser which displays sub-Poissonian photon statistics. Using a stroboscopic quantum map, we establish the linewidth for the ideal case where pump fluctuation and spontaneous emission are ineffective. Corrections due to spontaneous emission are also discussed.

A lot of work has recently been carried out on noise reduction schemes for lasers. One such scheme, originally due to Golubev and Sokolov [1], aims at reducing the fluctuation of the photon number towards sub-Poissonian statistics [2–4]. The question naturally arises as to whether and how the photon noise suppression influences the linewidth of the laser [5]. In this letter we shall focus our attention on the dynamics of such a laser and determine its linewidth. In contrast to a related study by Benkert *et al* [3], our work will be based on a quantum map arising from the stroboscopic description of a periodically pumped laser. The latter approach was introduced in [2] and further developed in [6].

Consider N active atoms in a resonator with mode frequency ω. The working levels of each atom are assumed, for simplicity, in resonance with the mode frequency, $E_2-E_1=\hbar\omega$. A train of periodic pulses (with period T) repeatedly brings all atoms from the ground state (with energy $E_0<E_1$) to the upper working level.

In order to fully suppress pump fluctuations each such pump step must be effectively instantaneous, i.e. much faster than the subsequent relaxation process. Each excited atom now faces two possibilities. Either, it may leave level 2 incoherently due to spontaneous emission with a characteristic time τ_{sp}. Alternatively, it may undergo a coherent Rabi oscillation with a Rabi frequency $g\sqrt{\bar{m}}$; here g denotes the elementary Rabi frequency referring to a single photon in the laser mode and $\bar{m}$ is a typical photon number in that mode. Ideal noise suppression requires $\tau_{sp}\gg 1/(g\sqrt{\bar{m}})$. For the same reason one forces the atoms to perform precisely one-half of a Rabi cycle. To that end, each atom must be incoherently sucked into the ground level immediately upon arrival in the lower working level. Denoting the rate constant of the latter incoherent process by γ_1 and requiring $\gamma_1\gg g\sqrt{\bar{m}}$, we encounter the rate constant $\gamma=4g^2/\gamma_1$ for the generation of photons in the laser mode. Roughly a time $\tau_{em}=1/\gamma\bar{m}$ after their preparation in the upper working level the atoms will then have deposited precisely N photons in the laser mode.

In practice the resonator will be non-ideal, with a photon lifetime $1/\kappa$. If a stationary regime is to be approached after many cycles, with the stationary number of photons much larger than N, we must further require $\tau_{em} \ll 1/\kappa$.

We shall describe the resonator field mode by the density operator ρ in the Fock representation. The Fock states are defined as usual by $b|n\rangle = \sqrt{n}|n-1\rangle$, $n = 0, 1, 2 \ldots$, where b and b^+ are anihilation and creation operators fulfilling the standard commutation relation $[b, b^+] = 1$. The stroboscopic cycle-to-cycle description mentioned above involves the map

$$\rho(t+1) = M\rho(t) \tag{1}$$

where M is an operator governing the dynamics of the system over one period and the time is measured in units of the period T. Since we have assumed the coherent emission to be much faster than the damping these two processes take place effectively successively during each period. The operator M thus takes the form of a product

$$M = DE \tag{2}$$

where E represents emission and D the damping.

In principle, the operators M, D and E are represented by tetrads. However, as was shown in [2], these operators assign separate evolutions to density matrix elements $\rho_{m,m+\nu}$ with different degrees of diagonality ν. We thus confront the dyadic maps

$$\rho_{m,m+\nu}(t+1) = \sum_n M^{(\nu)}_{mn}\rho_{n,n+\nu}(t) \qquad M^{(\nu)}_{mn} = \sum_k D^{(\nu)}_{mk}E^{(\nu)}_{kn}. \tag{3}$$

For $\nu = 0$, (3) describes the stroboscopic evolution of the probabilities ρ_{mm} which was treated in detail in [2].

In order to determine the laser linewidth we calculate the correlation function

$$C(t) = \langle b^+(t)b(0)\rangle. \tag{4}$$

For large times t this will decay exponentially

$$C(t) \sim e^{-\lambda t} \tag{5}$$

and the decay rate λ can be identified with the laser linewidth λ. According to map (1) we have

$$C(t) = \mathrm{Tr}\{b^+ M^t(b\rho)\} = \sum_{mn} [(M^{(1)})^t]_{mn}[(m+1)(n+1)]^{1/2}\rho_{m+1,m+1}(0). \tag{6}$$

To proceed with calculation of $C(t)$ we need the two matrices $D^{(1)}_{m,n}$ and $E^{(1)}_{m,n}$. Since we are interested in the linewidth for the stationary regime we shall eventually identify $\rho_{m,m}(0)$ with the steady-state probabilities $\bar{\rho}_{mm}$.

The various damping dyads $D^{(\nu)}_{mk}$ for $\nu = 0, 1, 2, \ldots$ can simply be read off the well known master equation of the damped harmonic oscillator [7] as

$$D^{(\nu)}_{mk} = \begin{cases} \left(\binom{k}{m}\binom{k+\nu}{m+\nu}\right)^{1/2} d^{m+\nu/2}(1-d)^{k-m} & k \leq m \\ 0 & k > m \end{cases} \tag{7}$$

where d is the fractional photon survival per cycle,

$$d = e^{-2\kappa T}. \tag{8}$$

Incidentally, for $\nu = 0$ one here encounters the well known Bernoulli distribution.

If the limit $\tau_{sp} \gg \tau_{em}$ is taken to the extreme, such that spontaneous emission is completely negligible, the emission dyads simply express the addition of precisely N photons per cycle,

$$E_{mn}^{(\nu)} = \delta_{m,\,n+N}. \tag{9}$$

In that case the full dyad (3) takes the form

$$M_{mn}^{(\nu)} = D_{m,\,n+N}^{(\nu)}. \tag{10}$$

As was shown in [2], for a large number of atoms, $N \gg 1$ and/or weak damping, $1 - d \ll 1$, the stationary probabilities $\bar{\rho}_{mm}$ are well approximated by the Gaussian

$$G_{x_0,\sigma}(x) := \frac{1}{\sqrt{2\pi\sigma^2}} \exp[-(x - x_0)^2/2\sigma^2] \tag{11}$$

as

$$\bar{\rho}_{mm} = G_{\bar{m},\bar{\sigma}}(m). \tag{12}$$

The mean and the variance of the photon number come out as

$$\bar{m} = \frac{Nd}{1-d} \qquad \bar{\sigma}^2 = \frac{\bar{m}}{1+d} \tag{13}$$

the ratio $\bar{\sigma}^2/\bar{m} = 1/(1+d)$ signalling sub-Poissonian statistics. In the same spirit the dyads $D_{mk}^{(\nu)}$ can also be represented by Gaussians. The case $\nu = 1$ which we need for the linewidth reads

$$D_{mk}^{(1)} = \left(d\,\frac{k+1}{m+1}\right)^{1/2} G_{x_0(k),\,\sigma(k)}(m)$$

$$x_0(k) = (k+N)d \qquad \sigma^2(k) = x_0(k)(1-d). \tag{14}$$

By inserting the stationary probabilities (12) and the damping dyad (14) into (10) and (6) we obtain the correlation function as

$$C(1) = \sqrt{d} \sum_m \bar{\rho}_{mm} m \left(1 + \frac{N}{m+1}\right)^{1/2} \sim C(0)\sqrt{d} \left\langle\!\left\langle \left(1 + \frac{N}{m+1}\right)^{1/2} \right\rangle\!\right\rangle_{\bar{m},\,\bar{\sigma}} \tag{15}$$

where brackets $\langle\langle \ldots \rangle\rangle_{x_0,\sigma}$ denote an average over m with the Guassian $G_{x_0,\sigma}(m)$. Since we have $\bar{m} \gg 1$ the variation of $[1 + N/(m+1)]^{1/2}$ across the Gaussian is quite negligible, we may simply replace the variable m under the square root by the mean $\bar{m}$. We thus have

$$C(1) \approx C(0)\sqrt{d}\left(1 + \frac{N}{\bar{m}+1}\right)^{1/2} = C(0)\left(1 - \frac{1-d}{2\bar{m}}\right) \tag{16}$$

up to corrections of relative weight $(1-d)$. Analogously, we find $C(t+1)/C(t) = C(1)/C(0)$. Comparing this with the asymptotic exponential (5) we read off the linewidth

$$\lambda = \frac{1-d}{2\bar{m}}. \tag{17}$$

This resembles the classic Schawlow–Townes result [8] in its inverse proportionally to the laser intensity. We would like to note that $\lambda \to 0$ for $d \to 1$ since both pumping and spontaneous emission are eliminated as noise sources.

It is appropriate to discuss the perturbative influence of spontaneous emission, i.e. to allow for a non-zero value of the parameter

$$\varepsilon = \gamma_2/\gamma \tag{18}$$

where γ_2 is the rate of spontaneous emission from the upper working level. For spontaneous emission to be a small perturbation we have to assume $\varepsilon/\bar{m} \ll 1$. In that situation the result (7) for the damping dyad remains valid, while the emission dyad (9) must be corrected. In calculating $E^{(\nu)}_{mn}$ to first order in ε one must exploit the assumed smallness of the time scale ratios $g\sqrt{\bar{m}}/\gamma_1$ and $\gamma_2/\gamma\bar{m} = \varepsilon/\bar{m}$. We have employed the method of adiabatic elimination described in [2] for the special dyad $E^{(0)}_{mn}$. Repeating that procedure for $E^{(\nu)}_{mn}$ we arrive at

$$E^{(\nu)}_{m,n} = \left(\frac{\Gamma(m+1)\Gamma(m+1+\nu)}{\Gamma(n+1)\Gamma(n+1+\nu)}\right)^{1/2} \frac{\Gamma(n+1+\varepsilon+\nu/2)}{\Gamma(m+1+\varepsilon+\nu/2)\Gamma(n+N-m+1)} \times \left[\varepsilon \ln\left(\frac{m+2+\varepsilon+\nu/2}{n+1+\varepsilon+\nu/2}\right)\right]^{n+N-m} \tag{19}$$

where $\Gamma(x)$ stands for the Euler Γ function.

We need this result for $\nu = 1$ and in the limit $\bar{m} \gg 1$ for which the Gaussian approximation holds

$$\begin{aligned} E^{(1)}_{mk} &= G_{x_0(m),\sigma(m)}(k) \\ x_0(m) &= k + N - \varepsilon N/k \qquad \sigma^2(m) = \varepsilon N/k. \end{aligned} \tag{20}$$

In particular, this $E^{(1)}_{mk}$ reduces to the Kronecker form (9) for $\varepsilon \to 0$.

As the final ingredient for the correlation function we need the stationary distribution $\bar{\rho}_{mm}$ of the photon number to first order in ε which was already given in [2] as

$$\begin{aligned} \bar{\rho}_{mm} &= G_{\bar{m},\bar{\sigma}}(m) \\ \bar{m} &= \frac{Nd}{1-d} + \varepsilon\frac{d \ln d}{1-d} \qquad \bar{\sigma}^2 = \frac{\bar{m}}{1-d} - \varepsilon d^2 \frac{\ln d}{1-d^2}. \end{aligned} \tag{21}$$

In analogy with (16) we at last obtain the recursion relation for the correlation function

$$C(t+1) = C(t)\sqrt{d}\left[1 + \frac{N}{\bar{m}+1}\left(1 - \frac{\varepsilon}{m}\right)\right]^{1/2} \tag{22}$$

The linewidth thus takes the form

$$\lambda = \frac{1-d}{2\bar{m}}[1 + \varepsilon(1-d)]. \tag{23}$$

Spontaneous emission is seen to deteriorate the phase uncertainty by the factor $1 + \varepsilon(1-d)$.

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