Quantum chaotic system in the generalized Husimi representation

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The generalized Husimi distribution based on coherence-preserving coherent states is defined. We analyze the eigenfunctions of the evolution operator of the quantum kicked rotator in this representation.

The correspondence between quantum and classical dynamics is often studied in problems dealing with quantum chaos. In the analysis of quantum stochastic systems the Wigner distribution function W is frequently employed as an analogue of the classical probability distribution.[1] However, since W takes also negative values, it cannot be directly compared with the probability distribution function. It was suggested recently[2–4] that the Husimi distribution may be more useful for that purpose. The Husimi distribution H_s can be defined for an arbitrary state $|\Psi\rangle$ as a coarse-grained Wigner distribution,[5] or with the help of the standard coherent states,[3]

$$H_s(p,q)=(2\pi\hbar)^{-1}\,|\langle\Phi_{p,q}\,|\,\Psi\rangle|^2, \tag{1}$$

where $|\Phi_{p,q}\rangle$ is the minimum dispersion Gaussian wave packet:

$$\Phi_{p,q}(x)=[2\pi(\Delta q)^2]^{-1/4}\times\exp\left[\frac{i}{\hbar}\langle p\rangle-\left[\frac{(x-\langle q\rangle)}{2(\Delta q)}\right]^2\right]. \tag{2}$$

There is, however, a wide range of systems which do not preserve ordinary coherent states (CS).[6] Much work has been done to find generalized CS which remain coherent during time evolution for an arbitrary Hamiltonian.[7,8] One of the most important generalizations is due to Perelomov, who found CS associated with Lie group.[9]

We define the generalized Husimi distribution H_g by replacing in Eq. (1) the gaussian wave packets with generalized CS, linked with the symmetry of the problem. Such a distribution function preserves all basic features of the standard Husimi function H_s. As a non-negative, coarse-grained function it may also be regarded as a probability distribution. In this paper the H_g distribution is applied in the analysis of the intensively exploited model of periodically kicked rotator.[10–14] This chaotic quantum system can be defined by the following Hamiltonian:

$$H=p^2/2-[K/(2\pi)^2]\cos(2\pi q)\sum_{n=-\infty}^{n=\infty}\delta(t-n)\,, \tag{3}$$

where q,p are the scaled angle and angular momentum, respectively, K is a kicking strength parameter, and the rotator mass and kicking period are equal to 1. Expanding the wave function Ψ in terms of the eigenstates of $H_0=p^2/2$ as

$$\Psi(q,t)=(1/2\pi)\sum_{n=-\infty}^{n=\infty}a_n(t)e^{2\pi niq}\,, \tag{4}$$

we obtain a quantum map $-a_n^{(l+1)}=\sum_j a_j^{(l)}U_n^j$. The infinite matrix of the evolution operator $\hat{U}$ is given by[10]

$$U_n^j=e^{-ij^2 2\pi\tau}i^{(n-j)}J_{n-j}(k)\,, \tag{5}$$

where $\tau=\hbar/4\pi$, $k=K/4\pi\tau$, and $J_r(x)$ is the ordinary Bessel function of the first kind and order r. The kicking strength K parameter governs the motion of the analogous classical model. For $K\approx 1$ the last KAM orbit breaks down, and the unlimited diffusion in energy takes place (see Ref. 15 for a review of the classical standard map). In the quantum model, dynamics of the system is much more complicated and depends on two relevant parameters. For any rational value of τ the energy growth caused by the quantum resonance is unbounded in time,[13] while for a dense set of irrational τ values, the energy remains limited for any initial conditions and values of K.[11]

Diagonalization of the matrix U enables us to look at the properties of eigenvalues and eigenfunctions from the point of view of quantum chaos. A character of the quasienergy spectrum is strongly connected with the time evolution of the system.[13] Analyzing the generalized Husimi distribution of an eigenstate, we will confirm the conclusion,[2,16] that in the quasiclassical regime the eigenstates follow the classical orbits.

The generalized CS for the rotator problem are defined with help of the raising and lowering operators $\hat{E}_\pm$.[17] Let $|n\rangle$ be an eigenstate of H_0, so $\hat{E}_\pm|n\rangle=|n\pm1\rangle$. The CS $|\alpha\pm\rangle$ are then

$$|\alpha\pm\rangle=N^{-1/2}e^{\alpha\hat{E}_\pm}|0\rangle\,, \tag{6}$$

where $\alpha=re^{-i\chi}$ is an arbitrary complex number and the normalization constant N is equal to the Bessel function of the zero order of an imaginary argument $N=I_0(2r)$. In particular the CS in the angular representation has the following form $\psi_{\alpha_s}(\varphi)=(2\pi N)^{-1/2}\exp(\alpha e^{Si\varphi})$, where s denotes a sign. It is easy to show that the coherent states defined above have most properties of the standard CS. They are nonorthogonal and overcomplete. The average angle is equal $\langle\varphi\rangle=\langle\alpha_s\,|\,\varphi\,|\,\alpha_s\rangle=\chi$, and the angular momentum $\langle\alpha_s\,|\,-i\hbar\partial/\partial\varphi\,|\,\alpha_s\rangle=s\,|\alpha|$. Thus each point in the phase space can be connected with an appropriate CS $|\alpha_s\rangle=|\alpha_{p,q}\rangle$. A coherent state can also be expanded in terms of free Hamiltonian H_0 eigenstates $|n\rangle$ as

$$|\alpha+\rangle=N^{-1/2}\sum_{n=0}^{n=\infty}\frac{\alpha^n}{n!}|n\rangle\,. \tag{7}$$

Our generalized CS does not remain coherent during a time evolution. However, up to a certain time $t_c \sim r^{-1}$ the wave packet follows the classical trajectory and the dispersion growth is limited.[18] These CS are inherently associated with the underlying dynamics of the rotator problem. In fact, operators $\hat{E}_+$, $\hat{E}_-$, and $\hat{p}$ generate the dynamical group.[19] Thus our states are coherent in sense of Perelomov[9] on the dynamical group of the system. On the contrary, the Gaussian wave packets used in Ref. 2 were chosen arbitrarily, without any link with the dynamics of the system. Therefore we will use states (6) in the definition of the generalized Husimi distribution function H_g:

$$H_g(p,q)=(2\hbar\pi)^{-1} \, | \langle \alpha_{p,q} | \Psi \rangle |^2 \, . \tag{8}$$

We apply the above-defined H_g distribution in the analysis of quantum and semiclassical features of the quantum rotator. As a state $|\Psi\rangle$ in Eq. (8) we take the eigenfunction of the evolution operator U. Since the off-diagonal elements of the infinite matrix (5) rapidly decrease with the order of the Bessel function,[11] we can consider, as a useful approximation, the M dimensional matrix only. The matrix size M should be at least ten times larger than the value of k parameter. We could diagonalize this matrix numerically up to $M=120$, for any value of the relevant τ parameter.[20] Using the expansions (4) and (7), the generalized Husimi distribution can be easily evaluated for different points (p,q) of the phase space.

As an example we have chosen the eigenstate $|\Psi_r\rangle$, which corresponds to the orbits around the secondary resonance ($p=0.5$) in the classical standard map.[15] Figures 1(a)–3(a) present the H_g distribution as a function of phase space for different values of system parameters. To make the comparison with the classical dynamics easier, the second part of each picture [Figs. 1(b)–3(b)] presents the standard map generated by a selected set of initial points for the appropriate K value.

Figure 1 is depicted for such a small value of kicking strength $K=0.25$ that the classical orbits remain, in general, horizontal lines. The relatively large value of τ causes the corresponding wall in H_g distribution to be rather thick. The secondary resonance is already visible in the classical map [Fig. 1(b)], but the H_g distribution has, due to coarse graining, only two mild peaks. For smaller values of τ the wall becomes thinner, and in the limit of $\tau \sim \hbar \to 0$ might be associated with a single orbit. Increasing the K parameter to 0.51, the classical secondary resonances occupy enough place in the phase space [Fig. 2(b)] to generate well-distinguishable peaks in Husimi distribution [see Fig. 2(a)]. Figure 3(a) shows a "more classical" (smaller τ) case of the model for $K=1.02$. The resonant peaks are better localized in the phase space and due to smaller dispersion more details appear. For such a large value of K parameter the connected stochasticity occurs in the classical system. During larger evolution times the chaotic trajectories in Fig. 3(b) would diffuse in angular momentum. The chaotic motion exhibits itself in the quantum model as the irregular, rippled shape between two resonant peaks in H_g distribution. This case is, therefore, an example of a quantum system, which exposes regular and chaotic features simultaneously.

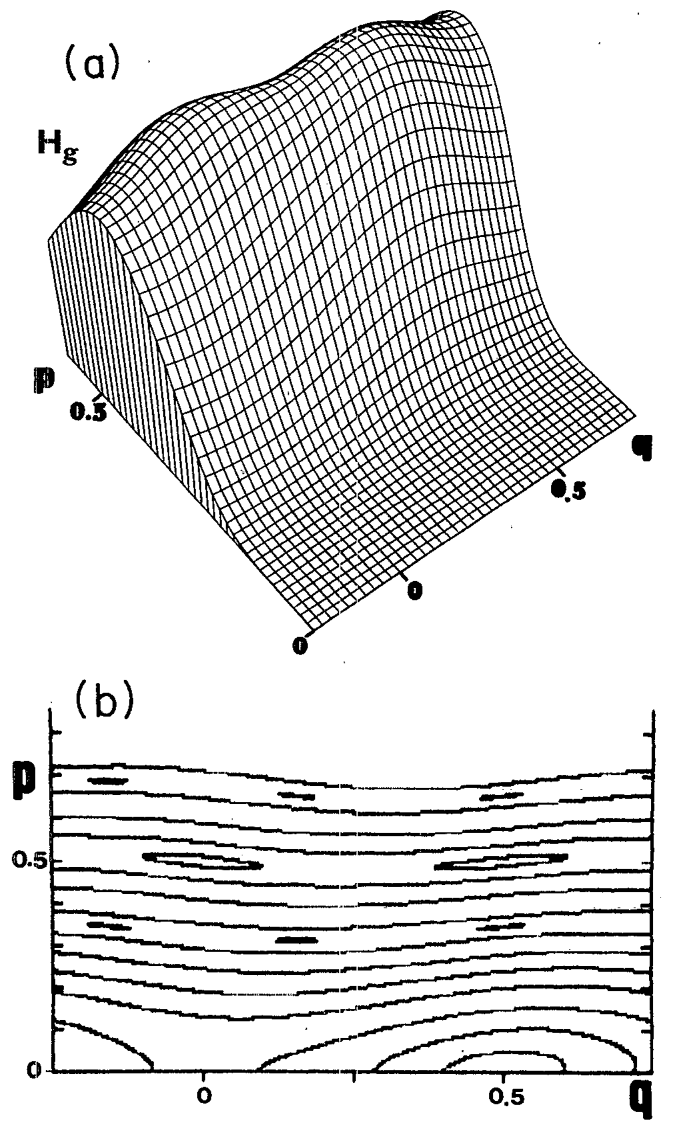

FIG. 1. (a) The generalized Husimi distribution H_g for the eigenfunction $|\Psi_r\rangle$ of the evolution operator depicted as a function of phase space (p,q) for $K=0.25$ and $\tau=0.04084$. The chosen eigenfunction corresponds to the orbits in the vicinity of the secondary resonance ($p=0.5$) in the classical standard map. (b) Classical standard map at $K=0.25$ in appropriate units of p and q. The phase-space cylinder is cut at $q=-0.25=0.75$ (mod 1) to match with (a).

It is worthwhile to note the averaging over a finite volume $\Delta p \Delta q \sim \hbar$ is involved in the definition of H_g, so precise details of the classical phase space are smeared out. However, the similarity between the H_g function and orbits in the classical phase space is evident. Our model enables a continuous change of the τ parameter, which determines the size of quantum effects, as opposed to Ref. 2, where τ is restricted to discrete values only. In particular, in the pure quantum regime of large τ values, the averaging is done over a large volume of the phase space, which causes a rather flat shape of the distribution function H_g. Since the ratio $\hbar/K \sim k$ is limited by the size of matrix M, the quasiclassical regime is difficult to achieve

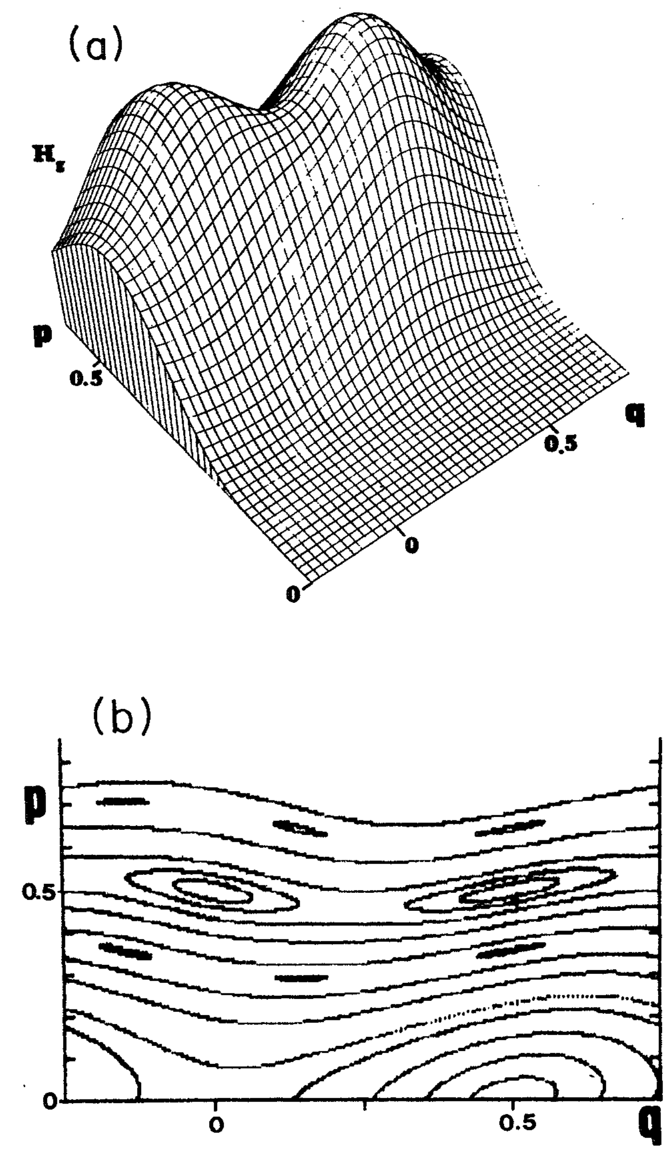

FIG. 2. (a) Husimi distribution for the same eigenfunction $|\Psi_r\rangle$ as in Fig. 1(a), but $K=0.51$, $\tau=0.040\,84$; (b) corresponding classical map.

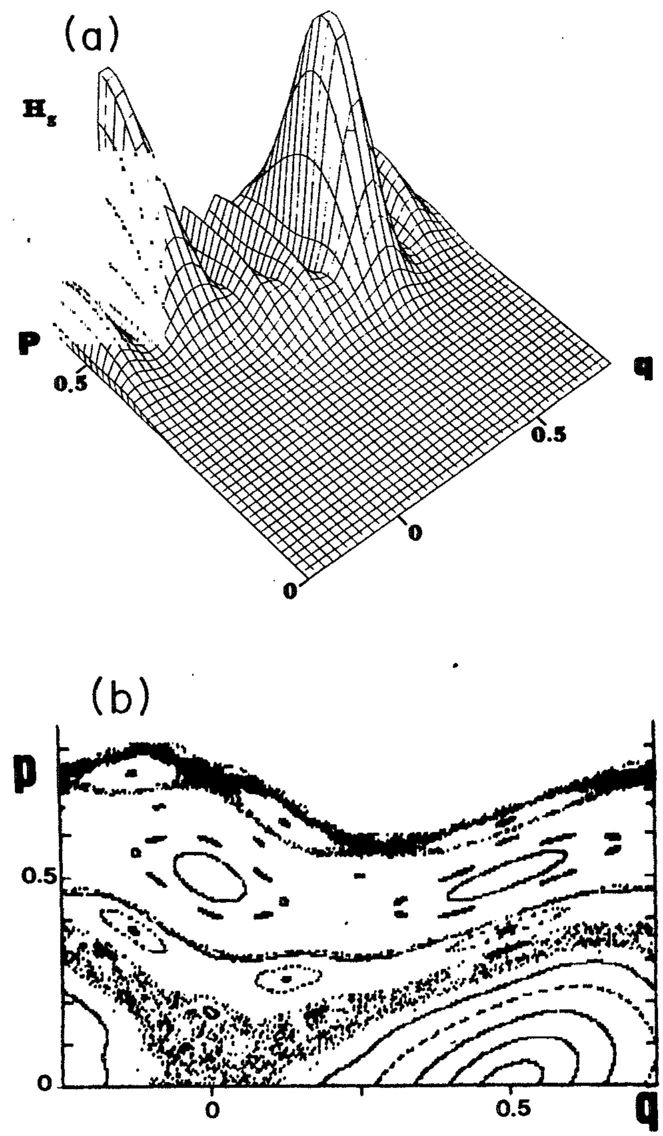

FIG. 3. As in Figs. 1 and 2; $K=1.02$, $\tau=0.010\,21$.

in this model for a strongly chaotic system. Nevertheless, a rippled, irregular shape of the generalized Husimi distribution of the eigenfunction may be considered as one of the qualitative criteria for a quantum chaotic system.

These conclusions are not in contradiction with the results of Chang and Shi.[2] It is due to the fact that for large angular momentum $p \gg \Delta p$ any wave packet, well localized in phase space, might be used to mimic a classical point. The Gaussian CS applied in Ref. 2 could, therefore, lead to the proper interpretation.

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