

Positive maps, positive polynomials and entanglement witnesses

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Abstract

We link the study of positive quantum maps, block positive operators and entanglement witnesses with problems related to multivariate polynomials. For instance, we show how indecomposable block positive operators relate to biquadratic forms that are not sums of squares. Although the general problem of describing the set of positive maps remains open, in some particular cases we solve the corresponding polynomial inequalities and obtain explicit conditions for positivity.

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1. Introduction

The set of positive maps acting on a finite-dimensional Hilbert space is a long-standing subject of mathematical interest. In spite of many efforts (see [1–5] and references therein), the structure of this set in spaces of arbitrary dimension is still not well understood. Of particular interest are positive maps, which are not completely positive [6–8]. The theorem of Jamiołkowski implies [2] that any such map can be represented by an operator, acting on a bi-partite Hilbert space, which is not positive, but is block-positive.

Non-completely positive maps recently attracted considerable attention amongst the physics community [9–11]. Positive maps have mainly been studied in view of their possible application to characterize quantum entanglement [12] and in connection with entanglement witnesses [13–16]. An entanglement witness is a Hermitian operator W such that $\text{Tr}(W\sigma) \geq 0$ for any separable state σ , while the negativity of $\text{Tr}(W\rho)$ implies that the state ρ is entangled. Note that a Hermitian operator W may be considered as an observable, so the expectation value $\text{Tr}(W\rho)$ can be measured in an experiment [17]. From a mathematical perspective any entanglement witness is a block positive operator which is not positive.

In the present paper we aim to clarify the relation between positive maps and positive polynomials. Definitions and basic information can be found in section 2. In section 3, we

explore the link between positive maps and positive polynomials and we address problems related to early contributions on the subject. In particular, we analyze implications of the work of Jamiołkowski [18, 19] and show why the results of these papers do not allow one to formulate a conclusive test for the positivity of a given map.

On the other hand, in some particular cases such results can be obtained. In sections 4 and 5, we investigate two families of maps and working with the corresponding polynomials we find explicit conditions for positivity. Furthermore, we demonstrate how positive maps relate to the existence of positive polynomials which are not sums of squares and we formulate an open problem concerning entanglement witnesses in $(2 \times m)$ -dimensional spaces.

2. Block positivity—motivation and definitions

Let $\mathcal{H}_1, \mathcal{H}_2$ be finite-dimensional spaces over \mathbb{C} , both equipped with Hermitian inner products ($\dim \mathcal{H}_1 = N_1, \dim \mathcal{H}_2 = N_2$). Let $\mathcal{L}(\mathcal{H}_1)$ denote the algebra of linear operators on \mathcal{H}_1 . We denote with $\mathcal{L}(\mathcal{H}_1)^+$ the set of positive elements of $\mathcal{L}(\mathcal{H}_1)$. A linear map $\Phi : \mathcal{L}(\mathcal{H}_1) \rightarrow \mathcal{L}(\mathcal{H}_2)$ is called *positive* if and only if it maps elements of $\mathcal{L}(\mathcal{H}_1)^+$ to elements of $\mathcal{L}(\mathcal{H}_2)^+$. It is well known [2] that the set of positive maps is isomorphic to the set of *block positive operators* (block positive over \mathbb{C}). Therefore, instead of asking whether a given map is positive, in this work we will be concerned with the equivalent question of whether the corresponding operator is block positive, so that it can serve as an entanglement witness.

A Hermitian operator A on $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ is called *block positive over \mathbb{C}* if it satisfies the following condition

$$\langle u \otimes v | A(u \otimes v) \rangle \geq 0 \quad \forall_{u \in \mathcal{H}_1, v \in \mathcal{H}_2}. \tag{1}$$

Note that condition (1) is not invariant with respect to global unitary transformations on \mathcal{H} , so this definition depends on the particular form of the decomposition of \mathcal{H} .

It will also be useful to introduce the concept of block positivity for real linear spaces. Let X and Y be finite-dimensional vector spaces over \mathbb{R} ($\dim X = M_1, \dim Y = M_2$). Let A be a linear operator on $X \otimes Y$. In analogy to (1), we say that A is *block positive over \mathbb{R}* if it satisfies

$$(x \otimes y) \cdot A(x \otimes y) \geq 0 \quad \forall_{x \in X, y \in Y}. \tag{2}$$

Condition (2) does not imply symmetry of A , but we may always assume that A is symmetric because the antisymmetric part of A in (2) vanishes. Thence $(X \otimes Y)^2 \ni (w_1, w_2) \mapsto w_1 \cdot A(w_2) \in \mathbb{R}$ is a symmetric bilinear form on $X \otimes Y$.

In index notation, condition (2) reads

$$A_{ab,cd} x^a y^b x^c y^d \geq 0 \quad \forall_{\{x^a\}_{a=1}^{M_1}, \{y^b\}_{b=1}^{M_2} \subset \mathbb{R}}, \tag{3}$$

where x^a and y^b are the coordinates of x and y with respect to the orthonormal bases $\{e_i\}_{i=1}^{M_1}, \{f_j\}_{j=1}^{M_2}$ of X, Y (resp.) which we use.

Obviously, (3) is a positivity condition for a real multivariate polynomial of degree 4. If the polynomial $A_{ab,cd} x^a y^b x^c y^d$ is a *sum of squares* (SOS) of some other polynomials P_i , then we must have

$$A_{ab,cd} x^a y^b x^c y^d = \sum_i P_i^2 = \sum_i (B_{ab}^i x^a y^b)^2, \tag{4}$$

where the real coefficients B_{ab}^i ($a = 1, \dots, M_1, b = 1, \dots, M_2$) are arbitrary and the range of the index i is finite.

Indeed, the polynomials P_i must be homogeneous and of degree 2. They cannot have terms of the form $x^a x^b$, neither of the form $y^a y^b$, since there are no terms $(x^a x^b)^2$ nor $(y^a y^b)^2$

in the sum $A_{ab,cd}x^ay^bx^cy^d$. Thus we conclude that if $A_{ab,cd}x^ay^bx^cy^d = \sum_i P_i^2$ for some polynomials P_i , then $P_i = B_{ab}^i x^a y^b$. But (4) looks just like a quadratic form on $X \otimes Y$, written in the product basis $\{e_i \otimes f_j\}_{i=1, j=1}^{M_1, M_2}$. It is tempting to say that (4) implies positive semidefiniteness of A , but *this is not true*. Nevertheless, a similar result can be proved if we assume that A is *symmetric with respect to partial transpose*, $A^\tau := (\mathbb{1} \otimes T)A = A$, where T denotes the transposition. Putting this in a different way, A should satisfy

$$(x_1 \otimes y_1) \cdot A(x_2 \otimes y_2) = (x_1 \otimes y_2) \cdot A(x_2 \otimes y_1) \quad \forall_{x_1, x_2 \in X, y_1, y_2 \in Y} \quad (5)$$

For any operator A being a SOS and expressed by equation (4), we may define the following operator \tilde{A} :

$$\tilde{A}_{ab,cd} = \frac{1}{2} \left(\sum_i B_{ab}^i B_{cd}^i + B_{ad}^i B_{cb}^i \right). \quad (6)$$

It is easy to see that $(x \otimes y) \cdot \tilde{A}(x \otimes y) = (x \otimes y) \cdot A(x \otimes y)$ for all $x \in X, y \in Y$. In Appendix A we show that this property together with (5) and (6) implies $\tilde{A} = A$. But \tilde{A} is of the special form (6), which we did not assume about A . More precisely, \tilde{A} is proportional to a sum of a semipositive definite operator B with matrix elements $\sum_i B_{ab}^i B_{cd}^i$ and its partial transposition B^τ with matrix elements $\sum_i B_{ad}^i B_{cb}^i$. We conclude that $A_{ab,cd}x^ay^bx^cy^d = \sum_i P_i^2$ implies

$$A = \frac{1}{2}(B + B^\tau), \quad B \geq 0 \quad (7)$$

for the operators A with the property (5). A Hermitian operator A is called *decomposable* [1, 3] iff $A = C + D^\tau$, where $C, D \geq 0$. When (5) holds, one can easily prove that (7) is equivalent to decomposability of A . Thus we arrive at the following conclusion.

Proposition 1. *Let X, Y be finite-dimensional linear spaces over \mathbb{R} . Let \mathcal{W} be the set of blockpositive, indecomposable operators on $X \otimes Y$ which are symmetric with respect to transposition and partialtransposition. Denote with \mathcal{P} the set of positiverealpolynomials of the form $A_{ab,cd}x^ay^bx^cy^d$ which are notSOS. There is a linear isomorphism between \mathcal{W} and \mathcal{P} .*

Proof. The isomorphism in question is $\Pi : \mathcal{W} \ni A \mapsto A_{ab,cd}x^ay^bx^cy^d \in \mathcal{P}$. We still need to show that Π is one-to-one. To this end, we assume the equality $\sum_{a,b,c,d} A_{ab,cd}x^ay^bx^cy^d = \sum_{a,b,c,d} B_{ab,cd}x^ay^bx^cy^d$ for some operators $A, B \in \mathcal{W}$. Choose some $a, c \in \{1, 2, \dots, M_1\}, b, d \in \{1, 2, \dots, M_2\}$. Considering the coefficients at $x^ay^bx^cy^d$ in the two polynomials, we obtain $A_{ab,cd} + A_{ad,cb} = B_{ab,cd} + B_{ad,cb}$. Thanks to the partial transpose symmetry of A and B , we get $A = B$. This tells us that Π is injective. On the other hand, every polynomial of the form $A_{ab,cd}x^ay^bx^cy^d$ is an image by Π of the partial transpose symmetric operator $\frac{1}{2}(A + A^\tau)$. The operator $\frac{1}{2}(A + A^\tau)$ must be an element of \mathcal{W} for $A_{ab,cd}x^ay^bx^cy^d$ to be an element of \mathcal{P} (cf the discussion above). Thus we conclude that Π is surjective. \square

It was demonstrated by Choi [6] and Størmer [20] that there exist positive maps which are not decomposable. The example by Choi can easily be used to show that there exist, by proposition 1, positive polynomials of the form $A_{ab,cd}x^ay^bx^cy^d$ which are not SOS [6]. Proposition 1 gives a general motivation to investigate block positive operators over \mathbb{R} on account of their connection to sums of squares. It may also be expedient to study the real case in order to develop intuitions about block positivity over \mathbb{C} . It should, however, be kept in mind that (1) and (2) are not the same. Despite an apparent similarity, the block positivity over \mathbb{C} should not be perceived as a simple generalization of the block positivity over \mathbb{R} . In

general both definitions of block positivity do not coincide, what can be demonstrated by the following example of a real symmetric matrix:

$$A = \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{2} \\ 0 & 1 & \frac{3}{2} & 0 \\ 0 & \frac{3}{2} & 1 & 0 \\ -\frac{1}{2} & 0 & 0 & 1 \end{bmatrix}. \tag{8}$$

This matrix represents an operator on $\mathbb{C}^2 \otimes \mathbb{C}^2$ written in the standard product basis, $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$. It is easy to show that A satisfies inequality (2), but it does not satisfy condition (1). Hence the matrix A in (8) is block positive over \mathbb{R} , but is not block positive over \mathbb{C} . Moreover, when considering operators with unit trace, one can easily show that the set of such block positive operators over \mathbb{C} is compact whereas block positivity over \mathbb{R} does not imply compactness. In spite of this basic difference between the two notions of block positivity, there exist families of matrices for which conditions (1) and (2) turn out to be equivalent—see section 4.

3. Block positivity and quantifier elimination

Although the block positivity condition (1) is simple to understand, it does not seem easy to check. The early papers by Jamiołkowski [18, 19] suggest that the problem can be solved effectively. Even though this conclusion is in some sense true, we show a weak point of the argument presented in these papers.

For convenience of the reader, let us recall the details of the reasoning presented in [18]. First, we write condition (1) in index notation,

$$A_{\alpha\beta,\gamma\delta} \bar{u}^\alpha \bar{v}^\beta u^\gamma v^\delta \geq 0 \quad \forall_{\{u^\alpha\}_{\alpha=1}^{N_1}, \{v^\beta\}_{\beta=1}^{N_2} \subset \mathbb{C}}. \tag{9}$$

Next, we introduce *blocks*,

$$(A_v^{(1)})_{\alpha\gamma} := A_{\alpha\beta,\gamma\delta} \bar{v}^\beta v^\delta, \quad (A_u^{(2)})_{\beta\delta} := A_{\alpha\beta,\gamma\delta} \bar{u}^\alpha u^\gamma. \tag{10}$$

We can interpret them simply as matrices or as operators on \mathcal{H}_1 and \mathcal{H}_2 , respectively. Block positivity condition (9) can be rewritten as

$$A_v^{(1)} \geq 0 \quad \forall_{v \in \mathcal{H}_2} \quad \text{or as} \quad A_u^{(2)} \geq 0 \quad \forall_{u \in \mathcal{H}_1}, \tag{11}$$

where ‘ \geq ’ refers to semipositive definiteness. We shall concentrate on the right-hand side of (11). Semipositivity of $A_u^{(2)}$ is equivalent to the following set of inequalities,

$$W_l(u) := \sum_{1 \leq i_1 < i_2 < \dots < i_l \leq N_2} \Delta_{i_1 i_2 \dots i_l} (A_u^{(2)}) \geq 0 \quad \forall_{u \in \mathcal{H}_1} \forall_{l=1 \dots N_2}, \tag{12}$$

where $\Delta_{i_1 i_2 \dots i_l} (A_u^{(2)})$ is the minor of $A_u^{(2)}$ involving the columns and the rows with the numbers i_1, \dots, i_l . It follows from the discussion in [18] that the functions W_l are homogeneous real polynomials of an even degree in the variables $\{\text{Re}(u^\alpha)\}_{\alpha=1}^{N_1}, \{\text{Im}(u^\gamma)\}_{\gamma=1}^{N_2}$. Thus (12) is a set of positivity conditions for real homogeneous polynomials of an even degree. If we could solve these conditions explicitly, we would answer the question whether a given matrix is block positive.

That was the idea presented in [18] by Jamiołkowski, who suggested considering $\sum_{i_1, i_2, \dots, i_n} C_{i_1 i_2 \dots i_n} X_1^{i_1} X_2^{i_2} \dots X_n^{i_n}$ ($C_{i_1 i_2 \dots i_n} \in \mathbb{R}$) as a polynomial in the variable X_n with coefficients in $\mathbb{R}[X_1, \dots, X_{n-1}]$. He obtained positivity conditions for such a polynomial in a disjunctive normal form,

$$\forall_{\{x_1, \dots, x_{n-1}\} \subset \mathbb{R}} \bigvee_i \bigwedge_j D_j^i(x_1, \dots, x_{n-1}) \geq 0, \tag{13}$$

where $D_j^i \in \mathbb{R}[X_1, \dots, X_{n-1}] \forall_{i,j}$. Because the same procedure could be applied to any of the D_j^i 's (considered as elements of $\mathbb{R}[X_1, \dots, X_{n-2}][X_{n-1}]$), it was claimed that the number of variables in (13) can be iteratively reduced so as to yield quantifier free formulae. The problem with this argument is that equation (13) does not turn out to be equivalent to

$$\bigvee_i \bigwedge_j \forall_{\{x_1, \dots, x_{n-1}\} \subset \mathbb{R}} D_j^i(x_1, \dots, x_{n-1}) \geq 0, \tag{14}$$

so one cannot use the procedure iteratively.

To the best of our knowledge, no simple method is known to check the positivity of a general multivariate polynomial. It is in principle possible to eliminate quantifiers [21] from formulae such as $\forall_{\{x_1, \dots, x_{n-1}\} \subset \mathbb{R}} \sum_{i_1, i_2, \dots, i_n} C_{i_1 i_2 \dots i_n} x_1^{i_1} \dots x_n^{i_n} \geq 0$, but the outcome involves *zeros of univariate polynomials of an arbitrary high degree*, which cannot in general be expressed in terms of the coefficients of the polynomials. The known quantifier elimination procedures are laborious and should not be expected to provide a constructive solution to the problem. Thus we have to conclude this section by repeating the accepted statement that the question of explicit conditions for block positivity remains open.

4. A three-parameter family of block positive matrices

Fortunately, there exist some particular cases for which positivity conditions (12) turn out to be useful in checking block positivity. Let $a, b, c \in \mathbb{C}$. Consider the following family of matrices,

$$F = \begin{bmatrix} F_{00,00} & F_{00,01} & F_{00,10} & F_{00,11} \\ F_{01,00} & F_{01,01} & F_{01,10} & F_{01,11} \\ F_{10,00} & F_{10,01} & F_{10,10} & F_{10,11} \\ F_{11,00} & F_{11,01} & F_{11,10} & F_{11,11} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & a & 0 & 0 \\ \bar{a} & \frac{1}{2} & b & 0 \\ 0 & \bar{b} & \frac{1}{2} & c \\ 0 & 0 & \bar{c} & \frac{1}{2} \end{bmatrix}, \tag{15}$$

which represent operators on $\mathcal{H}_1 \otimes \mathcal{H}_2 = \mathbb{C}^2 \otimes \mathbb{C}^2$.

We are going to test condition (1) using the method suggested in the previous section. The blocks (10) with respect to the subsystem described by \mathcal{H}_2 are

$$F_u^{(2)}(a, b, c) = \begin{bmatrix} \frac{1}{2}(|u_1|^2 + |u_2|^2) & a|u_1|^2 + c|u_2|^2 + \bar{b}u_1\bar{u}_2 \\ \bar{a}|u_1|^2 + \bar{c}|u_2|^2 + b\bar{u}_1u_2 & \frac{1}{2}(|u_1|^2 + |u_2|^2) \end{bmatrix}. \tag{16}$$

Obviously, $F_u^{(2)}(a, b, c)$ is semipositive definite for all $u \in \mathbb{C}^2$ if and only if $\det F_u^{(2)}(a, b, c) \geq 0 \forall_{u \in \mathbb{C}^2}$. That is,

$$\left(\frac{1}{2}(|u_1|^2 + |u_2|^2)\right)^2 - |a|u_1|^2 + c|u_2|^2 + \bar{b}u_1\bar{u}_2|^2 \geq 0 \quad \forall_{u_1, u_2 \in \mathbb{C}}. \tag{17}$$

Keeping $|u_1|$ and $|u_2|$ constant, we can maximize the absolute value of the term $a|u_1|^2 + c|u_2|^2 + \bar{b}u_1\bar{u}_2$ by choosing the phases of u_1 and u_2 such that the phase of $\bar{b}u_1\bar{u}_2$ is the same as the phase of $a|u_1|^2 + c|u_2|^2$. So done, we see that condition (17) is equivalent to

$$\left(\frac{1}{2}(x^2 + y^2)\right)^2 - (|ax^2 + cy^2| + |b|xy)^2 \geq 0 \quad \forall_{x, y \in \mathbb{R}^+}. \tag{18}$$

In inequality (18), we substituted x for $|u_1|$ and y for $|u_2|$. It is now easy to see that (18) is the same as

$$\frac{1}{2}(x^2 + y^2) - |ax^2 + cy^2| - |b|xy \geq 0 \quad \forall_{x, y \in \mathbb{R}}. \tag{19}$$

We extended the domain of x, y in (19) to \mathbb{R} , which is permissible because $|b|xy$ does not increase if we change the sign of x or y from plus to minus. Substituting $x \rightarrow r \cos \frac{\varphi}{2}, y \rightarrow r \sin \frac{\varphi}{2}$ in (19) we obtain

$$1 - |\alpha + \gamma \cos \varphi| - |b| \sin \varphi \geq 0 \quad \forall_{\varphi \in \mathbb{R}}, \tag{20}$$

where $\alpha := a + c, \gamma := a - c$. Condition (20) can easily be solved in the two following situations:

- (a) $\text{Re}(\alpha\bar{\gamma}) = 0 \iff |a| = |c|$
- (b) $\text{Re}(\alpha\bar{\gamma}) = \pm|\alpha||\gamma| \iff a = rc, r \in \mathbb{R}$.

In case (a), condition (20) simplifies to

$$1 - \sqrt{|\alpha|^2 + |\gamma|^2 \cos^2 \varphi} - |b| \sin \varphi \geq 0 \quad \forall \varphi \in \mathbb{R}. \tag{21}$$

We observe that $|\alpha|^2 + |\gamma|^2 \leq 1$ must hold in order that (21) be true. Keeping this in mind, we can rewrite (21) as

$$\left| \frac{b}{\gamma} \right|^2 \lambda^2 - \lambda + \left(1 - \left| \frac{b}{\gamma} \right|^2 (|\alpha|^2 + |\gamma|^2) \right) \geq 0 \quad \forall \lambda \in [|\alpha|, \sqrt{|\alpha|^2 + |\gamma|^2}], \tag{22}$$

where we substituted $\sqrt{|\alpha|^2 + |\gamma|^2 \cos^2 \varphi} \rightarrow \lambda$. As a positivity condition for a quadratic function, (22) can easily be solved explicitly. Together with the condition on $|\alpha|^2 + |\gamma|^2$, we obtain

$$|\alpha|^2 + |\gamma|^2 \leq 1 \wedge |\alpha| + |b|^2 \leq 1 \wedge \{2|b|^2|\alpha| \leq |\gamma|^2 \vee 2|b|^2\sqrt{|\alpha|^2 + |\gamma|^2} \geq |\gamma|^2\}, \tag{23}$$

which is an equivalent form of (22). In case (b), it is even simpler to get the conditions on α, γ and b equivalent to (20). We have $|\alpha + \gamma \cos \varphi| \leq |\alpha| + |\gamma| |\cos \varphi|$. Either for φ or for $\varphi \rightarrow \pi - \varphi$, we obtain $|\alpha + \gamma \cos \varphi| = |\alpha| + |\gamma| |\cos \varphi|$ and $\sin \varphi$ is not changed by the substitution $\varphi \rightarrow \pi - \varphi$. Hence we can rewrite (20) as

$$1 - |\alpha| - |\gamma| |\cos \varphi| - |b| \sin \varphi \geq 0 \quad \forall \varphi \in \mathbb{R}. \tag{24}$$

This is equivalent to $(1 - |\alpha| - |\gamma| \cos \varphi - |b| \sin \varphi) \geq 0 \forall \varphi \in \mathbb{R}$, which is easy to solve explicitly in terms of α, γ and b . We obtain

$$1 - |\alpha| - \sqrt{|\gamma|^2 + |b|^2} \geq 0. \tag{25}$$

In the case of general a, b and c , condition (20) is equivalent to the following system of four inequalities,

(1)

$$|\alpha| + |\gamma| \leq 1, \tag{26}$$

(2)

$$|\gamma|^2 - |b|^2 \leq |\text{Re}(\alpha\bar{\gamma})|, \tag{27}$$

(3)

$$|1 - |\alpha|^2 - |\gamma|^2| \geq 2|\text{Re}(\alpha\bar{\gamma})|, \tag{28}$$

(4)

$$\begin{aligned} & (|\gamma|^2 + |b|^2)^2 \cos^4 \psi - 4(|\gamma|^2 + |b|^2)\text{Re}(\alpha\bar{\gamma}) \cos^3 \psi \\ & + (4\text{Re}(\alpha\bar{\gamma})^2 - 2(1 - |\alpha|^2 - |b|^2)(|\gamma|^2 + |b|^2) - 4|\gamma|^2) \cos^2 \psi \\ & - 4(3 - |\alpha|^2 - |b|^2)\text{Re}(\alpha\bar{\gamma}) \cos \psi + (1 - |\alpha|^2 - |b|^2)^2 - 4|\alpha|^2 \geq 0, \end{aligned} \tag{29}$$

which have to be satisfied for all real $\psi = 2\varphi$. The expression on the left-hand side of condition (iv) is a polynomial $P(\cos \psi)$ of degree 4 in the variable $\cos \psi$. This condition means that P is nonnegative in the interval $[-1, 1]$. Given particular values of a, b and c , non-negativity of P in $[-1, 1]$ can easily be checked using the Sturm sequences [22]. It is also

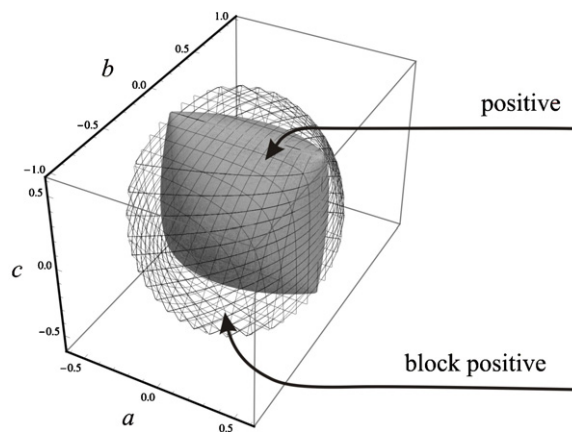


Figure 1. The gray set of positive semidefinite matrices defined by equation (30) is contained inside the set of block positive matrices determined by (25). In this case the block positivity over \mathbb{C} is equivalent to the block positivity over \mathbb{R} . It is assumed here that $a, b, c \in \mathbb{R}$, but formulae (24) and (25) apply also for a, b, c complex, provided that $a = rc$ with $r \in \mathbb{R}$.

possible to produce general conditions on a, b, c in this way, but the resulting formulae would be too complicated to reproduce them here and not suitable for further analysis.

An analogous problem of block positivity over \mathbb{R} can also be solved for the family of matrices (15). Most of the work has already been done above. We only need to observe that the passage from (17) to (18) is possible also when a, b, c and u_1, u_2 are real numbers. This is true because the maximal value of $|a|u_1|^2 + c|u_2|^2 + bu_1u_2|$ for fixed $|u_1|, |u_2|$ is $|au_1^2 + cu_2^2| + |b||u_1||u_2|$. Thus the condition (18) turns out to be equivalent to block positivity over \mathbb{R} of the matrices of the form (15) with $a, b, c \in \mathbb{R}$. Later analysis follows as in the case (b) discussed above. In this way we arrive at two important conclusions. First, symmetric matrices of the form (15) are block positive over \mathbb{R} if and only if they are block positive over \mathbb{C} . Second, the block positivity condition takes the form (25) with $\alpha = a + c, \gamma = a - c$. On the other hand, positivity conditions for the family of matrices (15) are easily obtained:

$$\frac{1}{16} - \frac{|a|^2}{4} - \frac{|b|^2}{4} - \frac{|c|^2}{4} + |a|^2|c|^2 \geq 0 \wedge \frac{1}{2} - |a|^2 - |b|^2 - |c|^2 \geq 0. \quad (30)$$

We can compare them with the block positivity condition (25) in a picture.

It is clear from figure 1 that conditions (30) and (25) are not equivalent, and the set of positive matrices of the family (15) forms a proper subset of the set of block positive matrices.

A similar investigation can be performed for a related family of matrices,

$$E(s, p, q, r) = \begin{bmatrix} \frac{1}{2} & s & 0 & r \\ s & \frac{1}{2} & p & 0 \\ 0 & p & \frac{1}{2} & q \\ r & 0 & q & \frac{1}{2} \end{bmatrix}, \quad (31)$$

with real parameters s, p, q and r . The block positivity conditions for $E(s, p, q, r)$ can be obtained using the methods presented in this section. In particular, taking $E(a, \frac{b}{2}, c, \frac{b}{2})$ with a, b, c real, we get a symmetrization of the family (15)

$$F'(a, b, c) := \frac{F(a, b, c) + F(a, b, c)^T}{2} = \begin{bmatrix} \frac{1}{2} & a & 0 & \frac{b}{2} \\ a & \frac{1}{2} & \frac{b}{2} & 0 \\ 0 & \frac{b}{2} & \frac{1}{2} & c \\ \frac{b}{2} & 0 & c & \frac{1}{2} \end{bmatrix}. \quad (32)$$

Deriving conditions for positivity and block positivity of the matrices $F'(a, b, c)$, it turns out that in this case both properties do coincide, unlike in the example discussed above. In the light of proposition 1, this fact can be understood as a consequence of the following theorem [23].

Theorem 2 (Calderón). *Let $m \in \mathbb{N}$, $x^1, x^2 \in \mathbb{R}$, $\{y^j\}_{j=1}^m, \{A_{abcd}\}_{a,b,c,d} \subset \mathbb{R}$*

$$A_{abcd}x^a y^b x^c y^d \geq 0 \forall_{x^1, x^2, \{y^j\}_{j=1}^m} \implies A_{abcd}x^a y^b x^c y^d = \sum_i (B_{ab}^i x^a y^b)^2. \quad (33)$$

That is, any positive biquadratic form in $2 \times m$ variables is the sum of squares of quadratic forms.

According to proposition 1 and Calderón’s result, any operator A on $X \otimes Y \cong \mathbb{R}^2 \otimes \mathbb{R}^m$ which is symmetric with respect to partial transpose and block positive over \mathbb{R} , is decomposable as well. More than that, we know from the discussion preceding proposition 1 that A can be written in the form $(B + B^\tau)/2$ with $B \geq 0$. In the case of $A = F'(a, b, c)$, B must be of the form (31) with $s = a, q = b$ and $p + r = b$. As can be checked by direct computation, the characteristic polynomials of $E(s, p, q, r)$ and $E(s, r, q, p)$ are the same. It follows that $E \geq 0 \iff E^\tau \geq 0$, which in turn leads us to the conclusion that the matrix $F' = \frac{1}{2}(F + F^\tau) = \frac{1}{2}(E + E^\tau)$ is block positive if and only if it is positive.

To explain our observation about F' , we could also have used the Størmer–Woronowicz theorem [1, 3], which implies that an operator A on $\mathbb{R}^2 \otimes \mathbb{R}^2$ (or on $\mathbb{R}^2 \otimes \mathbb{R}^3$) is block positive if and only if it is decomposable. This suggests a possible connection between the Calderón and the Størmer–Woronowicz theorems. On the other hand, the first theorem holds for all $\mathbb{R}^2 \otimes \mathbb{R}^m$ ($m \in \mathbb{N}$) whereas the latter works for $m \leq 3$ only.

The theorem of Calderón allows us to find some further implications for the subject of positive maps.

Proposition 3. *Let $m \in \mathbb{N}$. Either all block positive operators on $\mathbb{C}^2 \otimes \mathbb{C}^m$ with real matrices are decomposable or there exists an operator A on $\mathbb{C}^2 \otimes \mathbb{C}^m$ with real matrix elements $A_{ab,cd}$ such that $A_{ab,cd}x^a y^b x^c y^d$ is the sum of squares of bilinear forms, but A is not decomposable.*

Proof. Let A be an operator on $\mathbb{C}^2 \otimes \mathbb{C}^m$ with real matrix elements. If A is block positive on $\mathbb{C}^2 \otimes \mathbb{C}^m$, it must be block positive on $\mathbb{R}^2 \otimes \mathbb{R}^m$. From Calderón’s theorem it follows that $A_{ab,cd}x^a y^b x^c y^d$ is the sum of squares of bilinear forms. If this implies decomposability of A , any block positive operator on $\mathbb{C}^2 \otimes \mathbb{C}^m$ with real matrix elements is decomposable. If not, there exists an indecomposable operator A such that $A_{ab,cd}x^a y^b x^c y^d$ is SOS. \square

Both the mutually exclusive possibilities in proposition 3 are interesting and it will be good to know which of them is true for which m (of course, the answer is known for $m = 1, 2, 3$ —every positive map is decomposable). We hope that stronger results of similar kind can also be obtained and they should give better insights into the structure of positive and indecomposable maps.

5. Block positivity of 4×4 matrices over \mathbb{R}

We want to illustrate the abstract discussion presented in section 3 with a concrete example. To that aim, following [25], we derive sufficient and necessary conditions for an arbitrary operator A on $\mathbb{R}^2 \otimes \mathbb{R}^2$ to be block positive. Let the matrix elements of A be $A_{ab,cd}$ ($a, b, c, d \in \{1, 2\}$). The blocks with respect to the first subsystem have the matrix elements $(A_y^{(1)})_{ac} = A_{a1,c1}(y^1)^2 + (A_{a1,c2} + A_{a2,c1})y^1 y^2 + A_{a2,c2}(y^2)^2$. Positivity of $A_y^{(1)}$ is equivalent

to the requirements that $\text{Tr}A_y^{(1)} \geq 0$ and $\det A_y^{(1)} \geq 0$. Non-negativity of the trace of $A_y^{(1)}$ for all $y = (y^1, y^2) \in \mathbb{R}^2$ means that

$$\sum_{i=1}^2 A_{i1,i1}(y^1)^2 + \sum_{j=1}^2 (A_{j1,j2} + A_{j2,j1})y^1y^2 + \sum_{k=1}^2 A_{k2,k2}(y^2)^2 \geq 0 \quad \forall_{y^1, y^2 \in \mathbb{R}}. \quad (34)$$

Obviously, (34) is a positivity condition for a quadratic form on \mathbb{R}^2 and we can write it explicitly as

$$\sum_{i,j=1}^2 A_{ij,ij} \geq 0 \wedge \sum_{i=1}^2 A_{i1,i1} \sum_{k=1}^2 A_{k2,k2} - \frac{1}{4} \left(\sum_{j=1}^2 (A_{j1,j2} + A_{j2,j1}) \right)^2 \geq 0. \quad (35)$$

The expression for the determinant of $A_y^{(1)}$ reads

$$\det A_y^{(1)} = c_4x^4 + c_3x^3z + c_2x^2z^2 + c_1xz^3 + c_0z^4, \quad (36)$$

where we substituted x for y^1 , z for y^2 and we introduced

$$c_0 = A_{12,12}A_{21,21} - A_{11,21}A_{21,11}, \quad (37)$$

$$c_1 = A_{22,22}(A_{12,11} + A_{11,12}) + A_{12,12}(A_{22,21} + A_{21,22}) - A_{22,12}(A_{12,21} + A_{11,22}) - A_{11,21}(A_{22,11} + A_{21,12}), \quad (38)$$

$$c_2 = A_{11,11}A_{22,22} + A_{21,21}A_{12,12} + (A_{11,12} + A_{12,11})(A_{21,22} + A_{22,21}) - A_{11,21}A_{21,11} - A_{12,22}A_{22,12} - (A_{11,22} + A_{12,21})(A_{21,12} + A_{22,11}), \quad (39)$$

$$c_3 = A_{11,11}(A_{21,22} + A_{22,21}) + (A_{11,12} + A_{12,11}) - A_{11,21}(A_{21,12} + A_{22,11}) - A_{22,12}(A_{11,22} + A_{12,21}), \quad (40)$$

$$c_4 = A_{11,11}A_{21,21} - A_{11,21}A_{21,11}. \quad (41)$$

The c_i 's are homogeneous polynomials in the matrix elements $A_{ab,cd}$. It is easy to see that non-negativity of (36) for all $x, z \in \mathbb{R}$ is equivalent to

$$c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0 \geq 0 \quad \forall_{x \in \mathbb{R}}. \quad (42)$$

Thus we showed that in the case of a symmetric matrix A of order 4 condition (3) is equivalent to (35) plus (42). The inequalities (35) are explicit conditions on the matrix elements $A_{ab,cd}$, but in (42) we need some additional work to dispose of the quantifier $\forall_{x \in \mathbb{R}}$. There is no single method of doing it, but the one which seems most economical to us is using the following theorem [24].

Theorem 4 (Sturm). Let $f = f_0$ be a real univariate polynomial with no multiple roots in \mathbb{R} . Let f_1 be the first derivative of f . Define

$$f_{n+1} := \text{rem}(f_{n-1}, f_n), \quad (43)$$

where $\text{rem}(h, g)$ is the remainder obtained when dividing h by g . Define $N(r)$ as the number of sign changes in the sequence

$$f_0(r), f_1(r), -f_2(r), -f_3(r), f_4(r), f_5(r), -f_6(r), \dots \quad (44)$$

with zeros skipped. Assume $\alpha, \beta \in \mathbb{R}, \alpha < \beta, f_0(\alpha) \neq 0$ and $f_0(\beta)$. The number of zeros of f_0 in the interval (α, β) equals $N(\alpha) - N(\beta)$,

$$N(\alpha) - N(\beta) = \#\{r \in (\alpha; \beta) | f(r) = 0\}. \quad (45)$$

The sequence of functions (43) is the same as in Euclid’s algorithm applied to f and f' . When the signs are changed as in (44), the sequence is called the *Sturm sequence* of f . We know that $\{f_n\}_{n=0,1,\dots}$ must terminate at some $f_m \in \mathbb{R} \setminus \{0\}$, which is the greatest common divisor of f and f' . If we go to the limits $\alpha = -\infty, \beta = +\infty$ in theorem 4, we easily obtain the number of real roots of f .

Corollary 5. *Let $f = f_0$ be a real univariate polynomial with no multiple roots in \mathbb{R} and f_1 its first derivative. Let $f_n (n = 2, 3 \dots)$ be defined like in (43) and assume*

$$f_n(r) = a_{n,k_n} r^{k_n} + a_{n,k_n-1} r^{k_n-1} + \dots + a_{0,n}, \tag{46}$$

where $k_n \geq 0, a_{n,k_n} \neq 0 \forall_n$. Denote with $N(+\infty)$ the number of sign changes in the sequence

$$a_{0,k_0}, a_{1,k_1}, -a_{2,k_2}, -a_{3,k_3}, a_{4,k_4}, \dots, \pm a_{m,k_m} \tag{47}$$

and with $N(-\infty)$ the number of sign changes in

$$(-)^{k_0} a_{0,k_0}, (-)^{k_1} a_{1,k_1}, (-)^{k_2+1} a_{2,k_2}, (-)^{k_3+1} a_{3,k_3}, (-)^{k_4} a_{4,k_4}, \dots, \pm a_{m,k_m}. \tag{48}$$

The number of real zeros of f equals $N(-\infty) - N(+\infty)$,

$$N(-\infty) - N(+\infty) = \#\{x \in \mathbb{R} | f(x) = 0\}. \tag{49}$$

Let us take for f the polynomial $c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0$ which appears in (42). We shall now assume that it has no multiple roots in \mathbb{R} . Then we can use Corollary 5 to check the positivity of f .

The sequence $\{f_n\}_{n=0,1,\dots}$ consists of at most five polynomials,

$$\begin{aligned} f &= f_0 = c_4x^4 + c_3x^3 + c_2x^2 + c_1x + c_0, \\ f_1 &= 4c_4x^3 + 3c_3x^2 + 2c_2x + c_1, \\ f_2 &= a_{2,2}x^2 + a_{2,1}x + a_{2,0}, \\ f_3 &= a_{3,1}x + a_{3,0}, \\ f_4 &= a_{4,0}. \end{aligned} \tag{50}$$

If we make an additional *normality* assumption, which says that the degrees of f_0, \dots, f_4 drop one by one in the successive lines of (50), it is easy to write the positivity conditions for f ,

$$c_4 > 0 \wedge (a_{2,2} > 0 \vee a_{3,1} > 0) \wedge a_{4,0} > 0. \tag{51}$$

The expressions for $a_{2,2}, a_{3,1}$ and $a_{4,0}$ can also be easily obtained in the present situation. We obtain

$$a_{2,2} = \frac{1}{16c_4} \sigma_1, \quad a_{3,1} = \frac{32c_4}{\sigma_1^2} \sigma_2, \quad a_{4,0} = -\frac{\sigma_1^2}{64c_4\sigma_2^2} \sigma_3, \tag{52}$$

where

$$\sigma_1 := c_2c_4 - 3c_3^2, \tag{53}$$

$$\sigma_2 := c_1c_3^3 - 14c_1c_2c_3c_4 - c_3^2(c_2^2 - 6c_0c_4) + 2c_4(2c_2^3 + 9c_1^2c_4 - 8c_0c_2c_4), \tag{54}$$

$$\begin{aligned} \sigma_3 &:= c_1(2c_1^2 - 9c_0c_2)c_3^3 + 2c_1c_3c_4(-9c_1^2c_2 + 40c_0c_2^2 + 96c_0^2c_4) \\ &\quad + c_0^2c_3^4 + c_3^2(-c_1^2c_2^2 + 4c_0c_2^3 + 6c_0c_1^2c_4 - 144c_0^2c_2c_4) \\ &\quad + c_4(4c_1^2c_2^3 + 27c_1^4c_4 + 128c_0^2c_2^2c_4 - 256c_0^3c_4^2 - 16c_0c_2(c_2^3 + 9c_1^2c_4)). \end{aligned} \tag{55}$$

According to (52), the normality assumption is equivalent to $c_4 \neq 0 \wedge \sigma_1 \neq 0 \wedge \sigma_2 \neq 0 \wedge \sigma_3 \neq 0$. If these conditions hold, we can rewrite (51) as

$$c_4 > 0 \wedge (\sigma_1 > 0 \vee \sigma_2 > 0) \wedge \sigma_3 < 0. \tag{56}$$

This is an explicit condition for f to be positive. Of course, it was obtained under the assumption that $\{f_n\}_{n=0}^4$ are normal. Nevertheless, we can use (56) as a starting point for an all-purpose non-negativity test for polynomials of degree less than or equal to four. Indeed, suppose that the sequence $\{f_n\}_{n=0}^4$ is not normal. That is, at least one of the numbers $c_4, \sigma_1, \sigma_2, \sigma_3$ happens to be zero. If $c_4 = 0$, the non-negativity question becomes trivial. We get that $c_3x^3 + c_2x^2 + c_1x + c_0$ is non-negative if and only if

$$c_3 = 0 \wedge c_2 \geq 0 \wedge c_1^2 - 4c_2c_0 \leq 0. \tag{57}$$

The case in which $c_4 \neq 0$ but $\sigma_1\sigma_2\sigma_3 = 0$ can be analyzed using a little more sophisticated techniques (see Appendix B). All in all, we arrive at the following non-negativity conditions for f ,

$$\{c_4 > 0 \wedge (\sigma_1 \geq 0 \vee \sigma_2 \geq 0) \wedge \sigma_3 < 0\} \\ \vee \{c_4 = 0 \wedge c_3 = 0 \wedge c_2 \geq 0 \wedge c_1^2 - 4c_2c_0 \leq 0\}, \tag{58}$$

where $\sigma_3 < 0$ means $\exists_{\xi > 0} \forall_{\xi' < \xi} (\sigma_3(c_4, c_3, c_2, c_1, c_0 + \xi') < 0)$. We can write $\sigma_3 < 0$ explicitly as

$$\sigma_3 < 0 \vee (\sigma_3 = 0 \wedge (\kappa_1 < 0 \vee (\kappa_1 = 0 \wedge \kappa_2 \leq 0))), \tag{59}$$

where

$$\kappa_1 = 4c_3^2c^3 - 18c_3^3c_2c_1 + 80c_4c_3c_2^2c_1 + 6c_4c_3^2c_1^2 - 16c_4c_2(c_2^3 + 9c_4c_1^2), \tag{60}$$

$$\kappa_2 = 27c_3^4 - 144c_4c_3^2c_2 + 128c_4^2c_2^2 + 192c_4^2c_3c_1. \tag{61}$$

Obviously, conditions (58) and (35) together with the definitions (59), (60), (61), (53), (54), (55), (37), (38), (39), (40) and (41) provide us with a method to test block positivity over \mathbb{R} of 4×4 matrices. We see that lengthy calculations are involved, even though the studied example is the simplest possible one. It is also clear that the iterative procedure proposed in [18] could not work with conditions such as (56), let alone (58).

6. Conclusions

We have re-examined the method [18] of establishing the positivity of a map with the help of multivariate polynomials and we conclude that in the general case this problem remains open. The same can be said about the equivalent problem of checking whether a given operator acting on a composite Hilbert space is block positive. Nevertheless, for certain family of operators checking the positivity of the associated polynomials allowed us to find a concrete criterion for block positivity. Such concrete examples are provided in sections 4 and 5. By giving example (8), we touched upon the relation between the block positivity conditions over \mathbb{C} and over \mathbb{R} .

We also outlined connections between block positivity, indecomposability and the sums of squares (propositions 1 and 3, theorem 2). Proposition 3 opens a discussion about the two mutually exclusive possibilities concerning indecomposable maps on $\mathbb{C}^2 \otimes \mathbb{C}^m$ (cf section 4).

Finally, we tried to show that polynomials, which have been thoroughly studied by mathematicians and engineers, may deserve more respect of physicists working on quantum information or on open quantum systems. In particular, the separability problem itself can be formulated as a set of polynomial equalities [26]. Techniques like the calculation of a Gröbner basis of an ideal are widely used to solve polynomial equations and they could be of importance in physical problems such as the separability problem.

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Appendix A. Equality between operators \tilde{A} and A

Consider the symmetric bilinear forms $\Phi : (X \otimes Y)^2 \ni (w_1, w_2) \mapsto w_1 \cdot A(w_2) \in \mathbb{R}$, $\tilde{\Phi} : (X \otimes Y)^2 \ni (w_1, w_2) \mapsto w_1 \cdot \tilde{A}(w_2) \in \mathbb{R}$. We know that $\Phi(x \otimes y) = \tilde{\Phi}(x \otimes y)$ for arbitrary $x \in X, y \in Y$. From (5) and (6) it follows that $\Phi, \tilde{\Phi}$ are symmetric with respect to partial transposition,

$$\Phi(x_1 \otimes y_1, x_2 \otimes y_2) = \Phi(x_1 \otimes y_2, x_2 \otimes y_1), \tag{A.1}$$

$$\tilde{\Phi}(x_1 \otimes y_1, x_2 \otimes y_2) = \tilde{\Phi}(x_1 \otimes y_2, x_2 \otimes y_1). \tag{A.2}$$

Let us choose $x \in X$ and consider the following maps, $\Phi_x : Y^2 \ni y \mapsto \Phi(x \otimes y_1, x \otimes y_2) \in \mathbb{R}$, $\tilde{\Phi}_x : Y^2 \ni y \mapsto \tilde{\Phi}(x \otimes y_1, x \otimes y_2) \in \mathbb{R}$. From (A.1) and (A.2) we know that $\Phi_x, \tilde{\Phi}_x$ are symmetric bilinear forms on Y . As a consequence of $\Phi(x \otimes y) = \tilde{\Phi}(x \otimes y)$, $\Phi_x(y, y) = \tilde{\Phi}_x(y, y)$ for arbitrary $y \in Y$. Hence the quadratic forms corresponding to Φ and $\tilde{\Phi}$ are equal. This implies $\Phi_x = \tilde{\Phi}_x$, so we obtain

$$\Phi(x \otimes y_1, x \otimes y_2) = \tilde{\Phi}(x \otimes y_1, x \otimes y_2) \quad \forall x \in X, y_1, y_2 \in Y. \tag{A.3}$$

Now we consider the maps $\Phi_{y_1, y_2} : X^2 \ni (x_1, x_2) \mapsto \Phi(x_1 \otimes y_1, x_2 \otimes y_2)$, $\tilde{\Phi}_{y_1, y_2} : X^2 \ni (x_1, x_2) \mapsto \tilde{\Phi}(x_1 \otimes y_1, x_2 \otimes y_2)$. From the symmetry of $\Phi, \tilde{\Phi}$ and the properties (A.1), (A.2), we see that $\Phi_{y_1, y_2}, \tilde{\Phi}_{y_1, y_2}$ are symmetric bilinear forms on X . As a consequence of (A.3), $\Phi_{y_1, y_2}(x, x) = \tilde{\Phi}_{y_1, y_2}(x, x)$ for all $x \in X$. This implies $\Phi_{y_1, y_2}(x_1, x_2) = \tilde{\Phi}_{y_1, y_2}(x_1, x_2)$ for arbitrary $x_1, x_2 \in X$. In this way we get $\Phi(x_1 \otimes y_1, x_2 \otimes y_2) = \tilde{\Phi}(x_1 \otimes y_1, x_2 \otimes y_2) \forall x_1, x_2 \in X, y_1, y_2 \in Y$, which is the same as

$$(x_1 \otimes y_1) \cdot A(x_2 \otimes y_2) = (x_1 \otimes y_1) \cdot \tilde{A}(x_2 \otimes y_2) \quad \forall x_1, x_2 \in X \forall y_1, y_2 \in Y. \tag{A.4}$$

Of course, (A.4) implies $A = \tilde{A}$.

Appendix B. Non-negative polynomials with $\sigma_1 \sigma_2 \sigma_3 = 0$ and $c_4 \neq 0$

Our aim is to figure out all the sign configurations of $c_4, \sigma_1, \sigma_2, \sigma_3$ such that they meet the constraints $c_4 \neq 0 \wedge \sigma_1 \sigma_2 \sigma_3 = 0$ and they correspond to nonnegative polynomials $f = c_4 x^4 + c_3 x^3 + c_2 x^2 + c_1 x + c_0$. We also have to check that the remaining sign configurations can never give a non-negative f . Of course, $c_4 < 0$ implies that $f(x)$ be negative for some x , so we only need to consider c_4 positive. First we show that $\sigma_3 > 0$ cannot happen for a non-negative f . Suppose $\sigma_3 > 0$. We know that $\sigma_1 = 0$ or $\sigma_2 = 0$. Let us first consider $\sigma_1 = 0$. Because $\sigma_1 = 8c_2 c_4 - 3c_3^2$ and $c_4 > 0$, we can increase c_2 by $\varepsilon > 0$ and get $\sigma_1 > 0$ for sure. If σ_2 turns out to be zero after this operation, we additionally increase c_0 by $\xi > 0$, which must give us $\sigma_2 \neq 0$ because $\sigma_2 = \tilde{\sigma}_2 - 2c_0 c_4 \sigma_1$ where $\tilde{\sigma}_2$ does not depend on c_0 . The numbers ε, ξ can be made arbitrarily small, so as not to influence the sign of σ_3 . Hence we see that $f + \varepsilon x^2 + \xi$ has a normal Sturm sequence and it does not satisfy (56) because σ_3 corresponding to $f + \varepsilon x^2 + \xi$ is positive. But $f + \varepsilon x^2 + \xi \neq 0$ implies $f \not\geq 0$, so f cannot

be non-negative. We conclude that $f \geq 0$ is impossible for $c_4 > 0, \sigma_3 > 0$ and $\sigma_1 = 0$. For $c_4 > 0, \sigma_3 > 0$ and $\sigma_2 = 0$, we only need to increase f by a sufficiently small ξ to get to the conclusion $f \not\geq 0$. Our observations mean that $\sigma_3 > 0$ always implies $f \not\geq 0$. Let us now consider the polynomials f for which conditions

$$c_4 > 0 \wedge (\sigma_1 = 0 \vee \sigma_2 = 0) \wedge \sigma_3 < 0. \tag{B.1}$$

are satisfied. If σ_1 vanishes, we can get $\sigma_1 > 0$ by increasing c_2 ($c_2 \rightarrow c_2 + \varepsilon$). If σ_2 turns out to be 0 afterwards, any change of c_0 ($c_0 \rightarrow c_0 + \xi$) will give us $\sigma_2 \neq 0$ (cf the discussion above). We can take ε and ξ arbitrarily small, which allows us to avoid changing the sign of σ_3 . After all, we get a polynomial $f + \varepsilon x^2 + \xi$ which has a normal Sturm sequence and it is positive since $c_4 > 0 \wedge \sigma_1 > 0 \wedge \sigma_3 > 0$ for the corresponding c_4, σ_1 and σ_3 . Because ε and ξ can arbitrarily be small, we see that f is a pointwise limit of a sequence of positive polynomials. Hence f is non-negative. The same conclusion can be drawn for $c_4 > 0, \sigma_1 \neq 0, \sigma_2 = 0$ and $\sigma_3 < 0$, so we should add (B.1) to our set of non-negativity conditions. We can write (B.1) and (56) as a single condition,

$$c_4 > 0 \wedge (\sigma_1 \geq 0 \vee \sigma_2 \geq 0) \wedge \sigma_3 < 0. \tag{B.2}$$

The only situation which is left to analyze is that of $\sigma_3 = 0$. To that end, let us write σ_3 as a polynomial in c_0 ,

$$\sigma_3 = \kappa_3 c_0^3 + \kappa_2 c_0^2 + \kappa_1 c_0 + \kappa_0, \tag{B.3}$$

where

$$\kappa_0 = -c_3^2 c_2^2 c_1^2 + 4c_4 c_2^3 c_1^2 + 4c_3^3 c_1^3 - 18c_4 c_3 c_2 c_1^3 + 27c_4^2 c_1^4, \tag{B.4}$$

$$\kappa_1 = 4c_3^2 c^3 - 18c_3^3 c_2 c_1 + 80c_4 c_3 c_2^2 c_1 + 6c_4 c_3^2 c_1^2 - 16c_4 c_2 (c_3^3 + 9c_4 c_1^2), \tag{B.5}$$

$$\kappa_2 = 27c_3^4 - 144c_4 c_3^2 c_2 + 128c_4^2 c_2^2 + 192c_4^2 c_3 c_1, \tag{B.6}$$

$$\kappa_3 = -256c_4^3. \tag{B.7}$$

Because of the assumption $c_4 > 0$, we know that σ_3 is not constant with respect to c_0 . The idea now is to infinitesimally increase c_0 and see what the outcome is. If σ_3 becomes positive, we conclude that $f \not\geq 0$. If it turns out to be negative (we denote this with $f < 0$), we go back to the initial values of c_i and ask about the signs of σ_1, σ_2 . If $\sigma_1 > 0$, we choose $\xi > 0$ so small that $\sigma_1 > 0$ holds when we increase c_0 by ξ . Then (B.2) is true for $f + \xi$, so $f + \xi$ is non-negative. Since the ξ in $f + \xi$ can be made arbitrarily small, we get $f \geq 0$. If $\sigma_1 = 0$, we increase c_0 by ξ to get $\sigma_3 < 0$ and then we increase c_2 by ε so as to get $\sigma_1 > 0$ and not to violate $\sigma_3 > 0$. After that (B.2) holds for $f + \varepsilon x^2 + \xi$ and again we get to the conclusion that $f \geq 0$. Therefore we can add

$$c_4 > 0 \wedge \sigma_1 \geq 0 \wedge \sigma_3 < 0 \tag{B.8}$$

to our list of non-negativity conditions for f . Now we only need to analyze the case $c_4 > 0 \wedge \sigma_1 < 0 \wedge \sigma_2 \geq 0 \wedge \sigma_3 < 0$ to finish our work. If $\sigma_2 > 0$, we choose $\xi > 0$ so small that the σ_2 corresponding to $f + \xi$ is also positive. Then $f + \xi > 0$ and we get $f \geq 0$. The case $\sigma_2 = 0$ is also simple to analyze. Because $\sigma_1 < 0$, increasing c_0 causes $\sigma_2 = \tilde{\sigma}_2 - c_4 c_0 \sigma_1$ to become positive, so we get $\sigma_2 > 0 \wedge \sigma_3 < 0$ for $f + \xi$ and again this leads us to $f \geq 0$. Thus we can add

$$c_4 > 0 \wedge \sigma_1 < 0 \wedge \sigma_2 \geq 0 \wedge \sigma_3 < 0 \tag{B.9}$$

to our non-negativity conditions for f . It is convenient to write (B.2) and (B.8) in a single formula,

$$c_4 > 0 \wedge (\sigma_1 \geq 0 \vee \sigma_2 \geq 0) \wedge \sigma_3 < 0. \tag{B.10}$$

Because of the particular form (B.3) of σ_3 , we explicitly write the condition $\sigma_3 \leq 0$ as

$$\sigma_3 < 0 \vee (\sigma_3 = 0 \wedge (\kappa_1 < 0 \vee (\kappa_1 = 0 \wedge \kappa_2 \leq 0))), \quad (\text{B.11})$$

since condition $c_4 > 0$ implies that $\kappa_3 < 0$.

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