Extremal spacings between eigenphases of random unitary matrices and their tensor products

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Extremal spacings between eigenphases of random unitary matrices of size N pertaining to circular ensembles are investigated. Explicit probability distributions for the minimal spacing for various ensembles are derived for $N = 4$. We study ensembles of tensor product of k random unitary matrices of size n which describe independent evolution of a composite quantum system consisting of k subsystems. In the asymptotic case, as the total dimension $N = n^k$ becomes large, the nearest neighbor distribution $P(s)$ becomes Poissonian, but statistics of extreme spacings $P(s_{\min})$ and $P(s_{\max})$ reveal certain deviations from the Poissonian behavior.

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I. INTRODUCTION

Random unitary matrices are useful for describing spectra of periodic quantum systems, the classical analogs of which are chaotic [1,2]. The choice of a specific ensemble of matrices is dictated by the symmetry properties of the investigated physical system. If the system possesses no time-reversal symmetry, the circular unitary ensemble (CUE) of matrices distributed according to the Haar measure of the unitary group is appropriate [3]. For systems with a generalized time reversal symmetry the circular orthogonal ensemble (COE) describes properly statistical properties of spectra if we neglect additional subtleties caused by specific rotational symmetry features of systems with half-integer spin, which are of no concern for investigations reported in this paper. In the case of classically regular dynamics the spectrum of the evolution operator displays level clustering characteristic of the circular Poissonian ensemble (CPE) of diagonal random unitary matrices. To describe intermediate statistics one uses interpolating ensembles of unitary matrices [4–6] or composed ensembles of unitary matrices [7]. In the case of emerging chaos, in which the chaotic layer covers only a fraction of the phase space of the classical system, one may apply the distribution of Berry and Robnik, originally used for autonomous systems [8].

To characterize statistical properties of spectra of a random matrix one often uses the nearest neighbor spacing distribution $P(s)$ [3,9]. The random variable s is the distance between adjacent eigenphases (phases of eigenvalues) normalized by assuming that the mean spacing is equal to unity.

In this work we investigate the distribution of yet another random variable, the minimal spacing $s_{\min}$ between two eigenphases. Similar to the standard statistics of nearest level spacings, the distribution $P(s_{\min})$ also encodes information about properties of the spectrum. Observe that for any unitary matrix U the size of its minimal spacing $s_{\min}$ provides information about to what extent the investigated matrix U is close to being degenerate. For completeness we are also going to study the size of the largest spacing $s_{\max}$ defined analogously.

The statistics of the minimal spacings in spectra of random Hermitian matrices was analyzed by Le Caër *et al.* [10] and was also discussed in the book by Forrester [9]. Our current approach is somewhat similar but is different because we investigate extremal gaps between eigenvalues of unitary matrices distributed along the unit circle and study tensor products of unitary matrices. After part of our project was completed, we learned about a relevant work by Arous and Bourgade [11] in which the distribution of extremal spacings was studied for random matrices of circular unitary ensemble.

This paper is organized as follows. For exemplary ensembles of random matrices of size $N = 4$ we derive in Sec. II exact forms of the distributions of minimal spacings. The chosen dimension allows exact calculations, which become rather complicated for larger matrices. Moreover, this is the minimal dimension in which results for CUE and CPE can be compared with those for the ensemble consisting of tensor products of two CUE random matrices of size $N = 2$. Such an ensemble corresponds to a generic local dynamics in a two-qubit system [12].

The case of large matrices is studied in Sec. III. We recall the heuristic argument put forward, e.g., in [9] (see Exercise 14.6.5, p. 697) justifying that for a random unitary matrix of size N the size of the minimal gap scales as $s_{\min} \approx N^{-\frac{1}{1+\beta}}$, where $\beta = 0,1$, and 2 for the Poissonian, orthogonal, and unitary circular ensembles, respectively. Analogously, we approach the asymptotic scaling of the maximal gap $s_{\max}$. We also provide some numerical results confirming our nonrigorous predictions concerning the order of the mean values of the extremal spacings $\langle s_{\min}\rangle$ and $\langle s_{\max}\rangle$ and the distribution of the minimal spacing $s_{\min}$. Recently, the latter has been rigorously studied in [11] and [13]. It was considered for the first time in [14].

Furthermore, we analyze extremal spacings for products of k independent random unitary matrices, each of them of size n. If the total dimension of the matrix, $N = n^k$, is large, the level spacing distribution $P(s)$ becomes Poissonian [12]. This property holds also for a tensor product of two random unitary matrices of different sizes [15]. However, in the case of a large number of one-qubit systems ($n = 2$ and k large), the statistics of the minimal spacing $s_{\min}$ displays significant deviations from the predictions for the Poisson ensemble, reviewed in the Appendix.

We use the following notation. For a single unitary or orthogonal matrix A of size N we consider its spectrum to be $\{\exp(i\varphi_j)\}_{j=1}^N$, where $(\varphi_1,\ldots,\varphi_N)$ represents the vector of the eigenphases ordered nondecreasingly, $0 \leqslant \varphi_1 \leqslant \cdots \leqslant \varphi_N < 2\pi$. We order nondecreasingly the spacings $\varphi_2-\varphi_1,\ldots,\varphi_N-\varphi_{N-1},2\pi+\varphi_1-\varphi_N$ between neighboring eigenphases, divide them by the average spacing $2\pi/N$, and denote the obtained sequence by

$$s_{\min} := s_1 \leqslant \cdots \leqslant s_N =: s_{\max}. \tag{1}$$

The standard level spacing distribution $P(s)$ is given by the average $\frac{1}{N}\sum_{m=1}^N P_m(s_m)$, where P_m is the density of the rescaled mth spacing $s_m = (\varphi_{m+1}-\varphi_m)N/2\pi$.

II. CASE STUDY: MINIMAL SPACINGS FOR TWO-QUBIT SYSTEM

Our first goal is to derive exact probability distributions of the minimal spacing $P_{\min}$ for exemplary ensembles of random unitary matrices of size $N = 4$. Besides the Poissonian and the unitary ensemble we analyze also the tensor product of two independent random matrices of size $N = 2$. This ensemble, denoted for brevity as $\text{CUE}_{2\otimes2}$, describes dynamics of two independent quantum subsystems [12]. In the quantum information literature such a case is called a two-qubit system.

To derive the desired distribution we calculate the tail distribution $T(t) = \mathbb{P}(s_{\min} > t)$ and take the derivative of T. We have

$$\begin{aligned} T(t) &= \mathbb{P}(s_{\min} > t)\\ &= \mathbb{P}(\varphi_2-\varphi_1,\varphi_3-\varphi_2,\varphi_4-\varphi_3,2\pi+\varphi_1-\varphi_4 > \pi t/2)\\ &= \int_{\{\varphi_2-\varphi_1,\varphi_3-\varphi_2,\varphi_4-\varphi_3,2\pi+\varphi_1-\varphi_4>\pi t/2\}} \\ &\quad P^{\text{ord}}(\varphi_1,\varphi_2,\varphi_3,\varphi_4)d(\varphi_1,\varphi_2,\varphi_3,\varphi_4), \end{aligned} \tag{2}$$

where P^{ord} is the joint probability distribution of ordered eigenphases, which can be obtained from the joint probability distribution for a given ensemble. After changing variables, $\psi_1 = \varphi_1, \psi_2 = \varphi_2 - \varphi_1, \psi_3 = \varphi_3 - \varphi_2$, and $\psi_4 = \varphi_4 - \varphi_3$, the integration domain splits into two tetrahedrons. Standard but tedious calculations yield, in each case, the tail distribution function $T(t)$, which leads to the corresponding probability density, $P(s_{\min}) = -\frac{d}{dt}T(t)|_{t=s_{\min}}$.

(a) For $\text{CUE}_{2\otimes2}$,

$$\begin{aligned} P^U_{2\otimes2}(s_{\min}) = \frac{1}{4\pi}\{&2\pi(1-s_{\min})[4-\cos(\pi s_{\min}/2)]\\ &-3\sin(\pi s_{\min}/2)+8\sin(\pi s_{\min})\\ &-3\sin(3\pi s_{\min}/2)\}. \end{aligned} \tag{3}$$

(b) For CUE_4

$$\begin{aligned} P^U_4(s_{\min}) = \frac{1}{72\pi^2}&\sin^2(\pi s_{\min}/4)\{666+720\pi^2(1-s_{\min})^2\\ &+36[11+16\pi^2(1-s_{\min})^2]\cos(\pi s_{\min}/2)\\ &+18[8\pi^2(1-s_{\min})^2-13]\cos(\pi s_{\min})\\ &-100\cos(3\pi s_{\min}/2)-608\cos(2\pi s_{\min})\\ &-380\cos(5\pi s_{\min}/2)+234\cos(3\pi s_{\min})\\ &+74\cos(7\pi s_{\min}/2)-58\cos(4\pi s_{\min})\\ &+10\cos(9\pi s_{\min}/2)\\ &+24\pi(1-s_{\min})[60\sin(\pi s_{\min}/2)\\ &+63\sin(\pi s_{\min})+22\sin(3\pi s_{\min}/2)\\ &+2\sin(2\pi s_{\min})-4\sin(5\pi s_{\min}/2)]\}. \end{aligned} \tag{4}$$

(c) For CPE_4

$$P^P_4(s_{\min}) = 3(1-s_{\min})^2. \tag{5}$$

These three distributions are presented in Fig. 1. The behavior of the densities around zero encodes some information concerning level repulsion and level clustering. The variable $s_{\min}$ is the smallest distance between two neighboring eigenphases. Therefore, the fact that its density is separated from zero, say, $P(s_{min}) > 1$ for s_{min} close to zero, means that for a small $\epsilon > 0$ the probability that two phases are at a distance closer than ϵ is $\mathbb{P}(s_{\min} < \epsilon) = \int_0^\epsilon P(s_{min})ds > \epsilon$. In the cases of CPE_4 and CUE_4 these features are consistent with level clustering and level repulsion observed in the distribution of spacings $P(s)$. Figure 1 shows that the eigenphases of the tensor product $\text{CUE}_{2\otimes2}$ tend to accumulate in a spectacular contrast to the case of a single random unitary matrix form CUE [12].

Numerical results show that for large N the distributions of the mth spacing $P(s_m)$ are close to the level spacing distribution

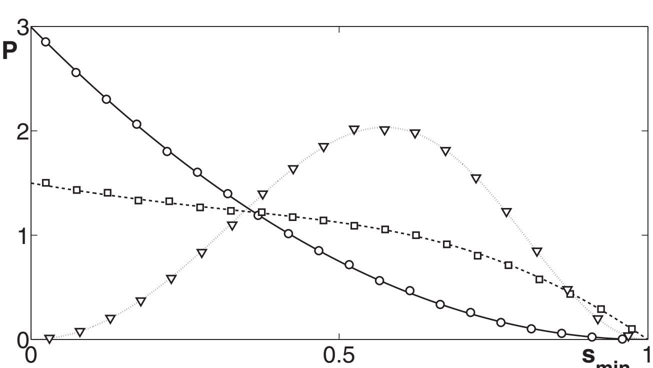

FIG. 1. Probability densities of the minimal spacing $s_{\min}$ for random matrices of size $N = 4$ pertaining to CUE_4 (downward triangles), $\text{CUE}_{2\otimes2}$ (squares), and CPE_4 (circles). Symbols denote numerical results obtained for 2^{14} independent matrices, while the curves represent distributions (3), (4), and (5), respectively.

$P(s)$ for $m \approx N/2$. However, for any N the distributions of the smallest spacing $s_{\min} = s_1$ and of the largest spacing $s_{\max} = s_N$ differ considerably. We shall analyze these distributions of extremal spacings, which can be used as auxiliary statistical tools to characterize ensembles of random matrices.

III. EXTREMAL STATISTICS FOR LARGE MATRICES

In this section we analyze extremal gaps in the spectra of circular ensembles of random matrices of a large size, $N \gg 1$, giving the numerical evidence to support some simple heuristic arguments (the subject for the CUE ensemble has been rigorously studied, however; see, e.g., [11]). As usual, we parameterize canonical ensembles by the level repulsion parameter β, equal to 0,1, and 2 for Poissonian, orthogonal, and unitary ensembles, respectively. The relevant quantities are labeled by the index $\beta = 0,1,2$. For instance $P_\beta(s)$ represents the level spacing distribution for the corresponding ensemble of random unitary matrices. We shall start with the Poissonian ensemble described by the case $\beta = 0$. Some basic properties of the Poissonian process are reviewed in the Appendix.

A. Asymptotes of the extreme spacings for Poisson process

We are interested in asymptotic properties of spectra of diagonal random unitary matrices. We choose at random N points from the unit circle $\{z \in \mathbb{C}, |z| = 1\}$, selecting each independently according to the uniform distribution. The arguments of these points ordered nondecreasingly will be called $0 \leqslant \theta_1 \leqslant \cdots \leqslant \theta_N < 2\pi$. We define a point process Ξ_N of the rescaled eigenphases of a diagonal random unitary matrix $D_N = \mathrm{diag}(e^{i\theta_1}, \ldots, e^{i\theta_N})$ pertaining to CPE$_N$,

$$\Xi_N = \{(N/2\pi)\theta_1, \ldots, (N/2\pi)\theta_N\}. \tag{6}$$

Moreover, we define the spacings s_i, $s_{\min}$, and $s_{\max}$ according to (1). Note that the scaling is chosen so that the mean spacing $\langle s \rangle$ is fixed to unity.

For the standard Poisson process $\Pi = \{X_1, X_2, \ldots\}$ (see the Appendix), where its points are labeled in the nondecreasing order $0 \leqslant X_1 \leqslant X_2 \leqslant \ldots$, we also define the spacings

$$Y_1 = X_1, \quad Y_2 = X_2 - X_1, \quad Y_3 = X_3 - X_2, \ldots . \tag{7}$$

It is known that for large N the process Ξ_N becomes Poissonian, as the correlation functions converge to the constant functions equal to unity characteristic of the Poisson process Π.

We would like to address the question of the asymptotic behavior of the variables $s_{\min}$ and $s_{\max}$. Since for a diagonal unitary matrix of CPE the process (6) becomes Poissonian, the variables $\min_{j \leqslant N} Y_j$ and $\max_{j \leqslant N} Y_j$ satisfy

$$\begin{aligned} \sup_{t\in\mathbb{R}} \left|\mathbb{P}(s_{\min} \leqslant t) - \mathbb{P}\left(\min_{j\leqslant N} Y_j \leqslant t\right)\right| \xrightarrow[N\to\infty]{} 0, \\ \sup_{t\in\mathbb{R}} \left|\mathbb{P}(s_{\max} \leqslant t) - \mathbb{P}\left(\max_{j\leqslant N} Y_j \leqslant t\right)\right| \xrightarrow[N\to\infty]{} 0. \end{aligned} \tag{8}$$

In view of (8) we arrive at the desired conclusions regarding $s_{\min}$ and $s_{\max}$. These quantities are of order

$$\langle s_{\min} \rangle_{\mathrm{CPE}} \sim 1/N, \quad \langle s_{\max} \rangle_{\mathrm{CPE}} \sim \ln N. \tag{9}$$

After rescaling, $s_{\min}$ converges to a random variable y with exponential density,

$$N s_{\min} \xrightarrow{d} e^{-y} \mathbf{1}_{\{y>0\}}, \tag{10}$$

where by $\mathbf{1}_Y$ we denote the characteristic function of the set Y. The maximal spacing $s_{\max}$ converges to a constant,

$$s_{\max}/\ln N \xrightarrow{d} 1, \tag{11}$$

where $\xrightarrow{d}$ denotes the convergence in distribution.

The fluctuations of the rescaled variable $s_{\max}/\ln N$ around 1 are of order $1/\ln N$, and they are described by the Gumbel distribution,

$$s_{\max} - \langle s_{\max} \rangle \xrightarrow{d} P(x) \sim e^{-(x+\gamma) - e^{-(x+\gamma)}}. \tag{12}$$

Here and throughout, we denote by $\gamma \approx 0.5772$ Euler's constant.

B. Mean minimal spacing

For the sake of convenience, we recall here the heuristic reasoning leading to the estimate of the mean of the minimal gap (Exercise 14.6.5 in [9]). In the next section we follow this idea to deal with the maximal gap.

To get an estimation of the behavior of the mean minimal spacing of a random unitary matrix of size N let us assume that spacings s_j, $j = 1, \ldots, N$, are independent random variables. For small spacing one has $P_\beta(s) \sim s^\beta$, so the integrated distribution $I_\beta(s) = \int_0^s P_\beta(s')ds'$ behaves as $I_\beta(s) \sim s^{1+\beta}$. A matrix U of size N yields N spacings s_j. Thus, the minimal spacing $s_{\min}$ occurs on average for such an argument of the integrated distribution where $I_\beta(s_{\min}) \approx 1/N$. This implies that $(s_{\min})^{1+\beta} \approx 1/N$, which allows us to estimate the average minimal spacing

$$\langle s_{\min} \rangle \approx N^{-\frac{1}{1+\beta}}. \tag{13}$$

In the case of $\beta = 2$ corresponding to CUE this statement is consistent with the rigorous results [11] of Arous and Bourgade. As shown in Fig. 2, the above heuristic reasoning provides the correct value of the exponent in the dependence

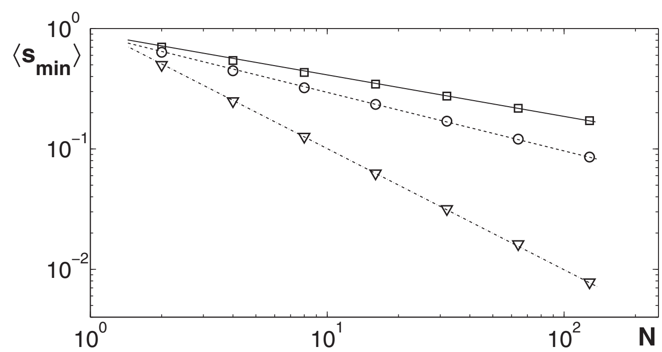

FIG. 2. Mean minimal spacing $\langle s_{\min} \rangle$ as a function of the matrix size $N = 2^m$ for CPE (downward triangles), COE (squares), and CUE (circles) for $m = 1, \ldots, 7$. Symbols denote numerical results obtained for 2^{14} independent random matrices. Solid, dashed, and dash-dotted lines are plotted with slopes implied by the estimation (13) and equal to -1, $-1/2$, and $-1/3$, respectively. The linear fit to the numerical data yields slopes -0.98, -0.48, and -0.33, respectively.

of the mean minimal spacing $\langle s_{\min}\rangle$ on the matrix size N for CPE ($\beta = 0$), COE($\beta = 1$), and CUE ($\beta = 2$).

C. Mean maximal spacing

We study the average maximal spacing $\langle s_{\max}\rangle$ for random unitary matrices of the circular orthogonal ensemble. A matrix of size N yields N spacings s_j. In analogy to the previous reasoning we shall assume that all spacings are independent random variables described by the Wigner surmise,

$$P(s) = \frac{\pi}{2} s e^{-\pi s^2/4}. \tag{14}$$

Thus, the integrated distribution $I(s) = \int_0^s P(s')ds'$ reads $I(s) = 1 - e^{-\pi s^2/4}$. The maximal spacing $s_{\max}$ occurs on average for an argument of the integrated distribution function such that $1 - I(s_{\max}) \approx 1/N$. This implies that $e^{-\pi s_{\max}^2/4} \approx 1/N$, which allows us to estimate the average maximal spacing,

$$\langle s_{\max}\rangle^2_{\mathrm{COE}} \approx \frac{4}{\pi} \ln N. \tag{15}$$

This implies that $\langle s_{\max}\rangle^2$ grows proportionally with the matrix size N to $\frac{4}{\pi}\ln N$, which is demonstrated in Fig. 3.

Let us deal now with the circular unitary ensemble. We employ here the Wigner formula for the level spacing distribution of a large CUE matrix, $P_2(s) = \frac{32}{\pi^2} s^2 e^{-4s^2/\pi}$. By the same reasoning as above we obtain an estimate that the maximal spacing $s_{\max}$ occurs on average for an argument of the integrated distribution function $I(s) = \int_0^s P(s')ds'$ such that $1 - I(s_{\max}) \approx 1/N$. Thus,

$$\frac{1}{N} \approx \int_{s_{\max}}^{\infty} \frac{32}{\pi^2} s^2 e^{-4s^2/\pi} ds. \tag{16}$$

We change the variable setting $u = 4s^2/\pi$ and obtain $\frac{1}{N} \approx \int_{4s_{\max}^2/\pi}^{\infty} \frac{2}{\sqrt{\pi}} u^{1/2} e^{-u} du$. Therefore, supposing $s_{\max}$ is large, we get

$$\frac{1}{N} \approx \frac{4}{\pi} s_{\max} e^{-4s_{\max}^2/\pi}. \tag{17}$$

Now we take the logarithm of both sides, neglect $\ln s_{\max}$ as it is of lower order than $s_{\max}^2$ for large $s_{\max}$, and arrive at

$$\langle s_{\max}\rangle^2_{\mathrm{CUE}} \approx \frac{\pi}{4} \ln N. \tag{18}$$

In the case of a Poissonian spectrum the level spacing distribution displays an exponential tail, $P(s) \sim e^{-s}$. Thus, the integrated distribution function $I(s) = \int_0^s P(s')ds'$ behaves as $I(s) = 1 - e^{-s}$. For a matrix of size N the maximal spacing $s_{\max}$ occurs on average for an argument such that $1 - I(s_{\max}) \approx 1/N$. This implies that $e^{-s_{\max}} \approx 1/N$ and enables us to estimate the average maximal spacing for the circular Poisson ensemble as

$$\langle s_{\max}\rangle_{\mathrm{CPE}} \approx \ln N. \tag{19}$$

Analyzing estimations following from Eqs. (15), (18), and (19), one obtains slopes $A_{\mathrm{COE}} = \frac{4}{\pi} \approx 1.27$, $A_{\mathrm{CUE}} = \frac{\pi}{4} \approx 0.77$, and $A_{\mathrm{CPE}} = 1$, which are comparable with the numerical results $A_{\mathrm{COE}} \approx 1.33$, $A_{\mathrm{CUE}} \approx 0.84$, and $A_{\mathrm{CPE}} \approx 0.97$, presented in Fig. 3.

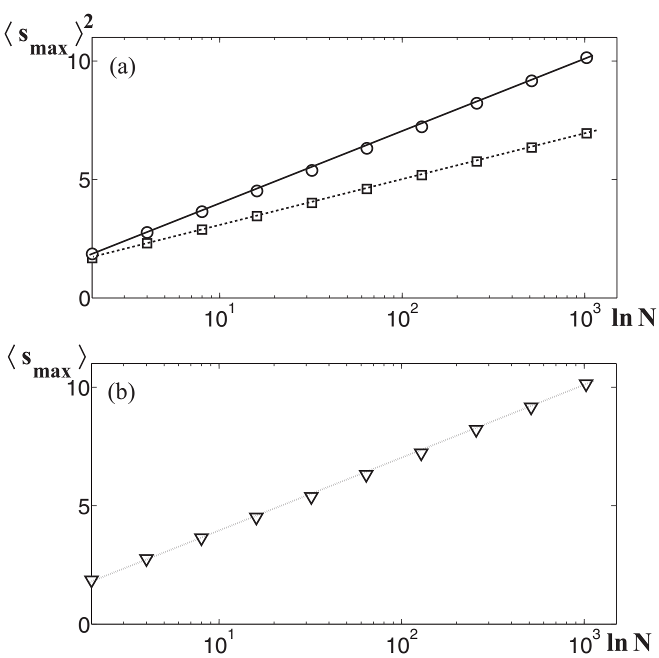

FIG. 3. Mean maximal spacing $\langle s_{\max}\rangle$ as a function of the matrix size $N = 2^k$, with $k = 1,\ldots,10$, plotted for (a) CUE (circles), $\beta = 2$, and COE (squares), $\beta = 1$, and (b) CPE (downward triangles), $\beta = 0$. Symbols denote numerical results obtained for 2^{14} independent random matrices. Solid, dashed [in (a)], and dotted [in (b)] lines are plotted with slopes implied by estimations (15), (18), and (19), respectively.

D. Distribution of extremal spacings

To study the distributions of the minimal spacing $s_{\min}$ we introduce a rescaled variable suggested by (13),

$$x_{\min}^{(\beta)} := A_{(\beta)} N^{\frac{1}{1+\beta}} s_{\min}, \tag{20}$$

where $A_{(\beta)}$ is a constant that is, in general, different for CPE, COE, and CUE.

The case of the unitary ensemble was recently studied by Arous and Bourgade [11], who derived the following expression for the asymptotic distribution of the minimal spacing:

$$P(x_{\min}) = 3x_{\min}^2 e^{-x_{\min}^3} \tag{21}$$

in the rescaled variable $x_{\min} = (\pi/3)^{2/3} N^{1/3} s_{\min}$. This result suggests the following general form of the distribution of minimal spacing for all three ensembles considered that are labeled by the level repulsion parameter β:

$$P^{(\beta)}(x_{\min}) := (\beta + 1) x_{\min}^{\beta} e^{-x_{\min}^{\beta+1}}, \tag{22}$$

which agrees with the numerical data (see Fig. 4).

The above formula has the structure $F(x) := \frac{df(x)}{dx} e^{-f(x)}$, which helps us determine the normalization. Numerical results suggest that constants read $A_{(0)} = 1$ for CPE, $A_{(1)} = \langle s\rangle$ for COE, and $A_{(2)} = (\pi/3)^{2/3}$ for CUE.

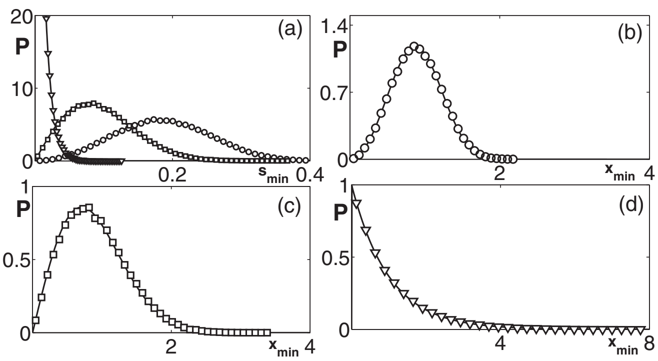

FIG. 4. (a) Probability distributions $P(s_{min})$ for random unitary matrices of CUE (downward triangles), $\beta = 2$; COE (squares), $\beta = 1$; and CPE (circles), $\beta = 0$. The same data are shown for variable $x_{\min}$ rescaled according to (20) for (b) CUE, (c) COE, and (d) CPE. Symbols denote numerical results obtained for 2^{17} independent matrices of size $N = 100$, while solid curves represent asymptotic predictions (22).

Returning to the original variable $s_{\min}$, we obtain the distributions $P^{(\beta)}(s_{\min})$,

$$P^{(0)}(s_{\min}) = A_{(0)} N e^{-N s_{\min}}, \tag{23}$$

$$P^{(1)}(s_{\min}) = 2A_{(1)}^2 N s_{\min} e^{-A_{(1)}^2 N s_{\min}^2}, \tag{24}$$

$$P^{(2)}(s_{\min}) = 3A_{(2)}^3 N s_{\min}^2 e^{-A_{(2)}^3 N s_{\min}^3}. \tag{25}$$

The distributions of the minimal spacing obtained numerically for Poisson, orthogonal, and unitary circular ensembles of random matrices of the size $N = 100$ are presented in Fig. 4.

IV. EXTREMAL SPACINGS FOR TENSOR PRODUCTS OF RANDOM UNITARY MATRICES

In this section we study the eigenphases of tensor products of random unitary matrices. We shall need the following easy observation.

Lemma 1. Let $A_1,\ldots,A_k$ be unitary matrices of size $n_1,\ldots,n_k$ with eigenphases $\{\psi_{1,j}\}_{j=1}^{n_1},\ldots,\{\psi_{k,j}\}_{j=1}^{n_k}$. Then the eigenphases of the tensor product $A_1 \otimes \cdots \otimes A_k$ read

$$\sum_{i=1}^{k} \psi_{i,j_i} \mod 2\pi, \qquad j_1 \leqslant n_1,\ldots,j_k \leqslant n_k.$$

Proof. The proof is obvious as the eigenvalues of tensor products are the products of the eigenvalues of their factors (see Theorem 4.2.12 in [16]). ■

We are interested in two cases.

(A) *Two-qunit system.* Given two independent CUE matrices, U_A,U_B of size n with eigenphases $\{\psi_j\}_{j=1}^n$, $\{\phi_j\}_{j=1}^n$, respectively, define the point process Ξ_n of the rescaled eigenphases of the tensor product $U_A \otimes U_B$,

$$\Xi_n = (n^2/2\pi)\{(\psi_i + \phi_j) \mod 2\pi,\ i,j = 1,\ldots,n\}. \tag{26}$$

(B) *k-qubit system.* Given k independent CUE matrices of order 2, $V_1,\ldots,V_k$ with eigenphases $\{\psi_{m,1},\psi_{m,2}\}$, $m = 1,\ldots,k$, respectively, define the point process Ψ_k of the rescaled eigenphases of the tensor product $V_1 \otimes \cdots \otimes V_k$,

$$\Psi_k = (2^k/2\pi)\left\{\sum_{m=1}^{k} \psi_{m,\epsilon_m} \mod 2\pi,\ \epsilon_k,\ldots,\epsilon_k \in \{1,2\}\right\}. \tag{27}$$

It has recently been shown that both the process Ξ_n and Ψ_k asymptotically behave as the standard Poisson point process Π (see [12] and the Appendix). Therefore, one might expect that the extremal spacings of the processes Ξ_n and Ψ_k also exhibit the asymptote of the extremal spacings of the Poisson process Π.

We have studied the problem numerically. To investigate the asymptotic regime we analyzed large matrices, which cannot be diagonalized directly. In case B, for instance, to deal with a 20-qubit system one has to work with matrices of size $N = 2^{20} > 10^6$. To obtain eigenphases and, as a consequence, the desired distribution of level spacings, we adopted another strategy summarized in the following algorithm.

(1) Take an ensemble of k random unitary matrices U_j of size 2 distributed according to the Haar measure [6,17].

(2) Diagonalize them to obtain their spectra, $\{e^{i\varphi_{jm}}\}$, where $j = 1,\ldots,k$ labels the number of the matrix, while $m = 1,2$ labels eigenvalues of the jth matrix.

(3) Construct $N = 2^k$ eigenphases of the tensor product $U = U_1 \otimes \cdots \otimes U_k$ by summing all combinations of phases from different matrices, $\psi_{m_1,\ldots m_k} = \sum_{j=1}^{k} \varphi_{jm_j}|_{\mathrm{mod} 2\pi}$, where $m_j = 1,2$.

(4) Order nondecreasingly the spectrum of U containing $N = 2^k$ eigenphases, $0 \leqslant \psi_1 \leqslant \cdots \leqslant \psi_N \leqslant 2\pi$.

(5) Compute spacings between neighboring eigenphases, $s_1 = (\psi_2 - \psi_1)N/2\pi,\ldots,s_{N-1} = (\psi_N - \psi_{N-1})N/2\pi, s_N = (2\pi + \psi_1 - \psi_N)N/2\pi$, order them nondecreasingly, and find the minimal spacing $s_{\min}$ and the maximal spacing s_{max}.

Note that Lemma 1 justifies point 3 of this algorithm.

Such a procedure allowed us to achieve N above 10^6 with a minor numerical effort (see Fig. 5). A similar procedure was be used in case A corresponding to the two-qunit system. Taking two independent random unitary matrices U_1 and U_2 of size $n = 1000$, diagonalizing them, and adding the phases modulo 2π, we construct the spectrum of the tensor product, $U = U_1 \otimes U_2$ of size n^2. In this way we computed averages taken over the ensemble of tensor product matrices of order $N = 10^6$.

The dependence of the mean extremal spacings on the matrix size N for tensor products in case A (two qunits) and case B (k qubits) are shown in Fig. 5. Figure 5(a) shows the average *minimal* spacing $\langle s_{\min}\rangle$. Note that the scaling of the minimal spacing for the two subsystems of size n (squares) agrees with the Poissonian predictions. On the other hand, in the case of the system consisting of k qubits, the scaling exponent is close to -0.6 and differs considerably from the value -1 characteristic of the Poissonian ensemble. As shown in Fig. 5(b), the behavior of the average maximal spacing for the tensor products corresponding to $N = n \times n$ and $N = 2^k$ systems is closer to the prediction of the Poisson ensemble, $\langle s_{\max}\rangle \sim \ln N$.

A. Minimal spacings for tensor products

To analyze the distribution of the minimal spacing $P(s_{\min})$ for the tensor products of random unitary matrices it is

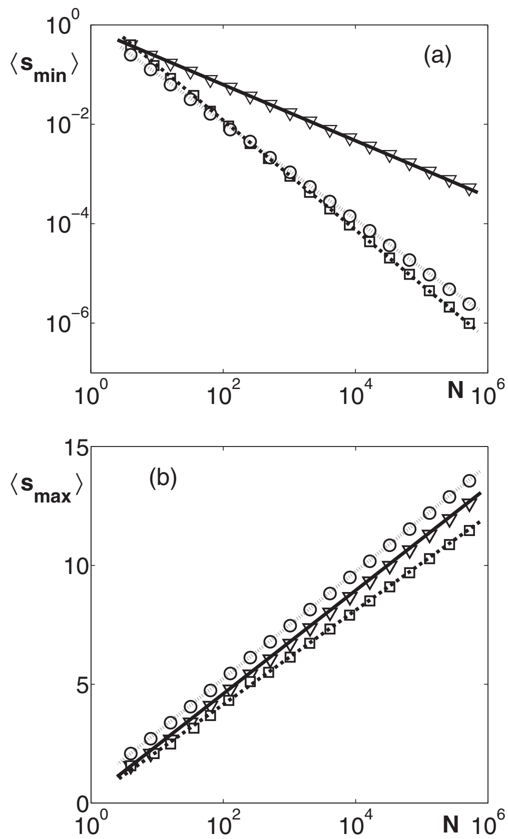

FIG. 5. Dependence of the mean extremal spacing on the matrix size N. (a) Mean minimal spacing $\langle s_{\min}\rangle$, assumed to behave as N^{η}, is plotted in log-log scale, and the fitted exponents read $\eta_{(P)} = -0.98$ for CPE_N (circles), $\eta_{(2)} = -1.09$ for $\mathrm{CUE}_{n\otimes n}$ (squares), and $\eta_{(k)} = -0.58$ for $\mathrm{CUE}_2^{\otimes k}$ (downward triangles). (b) Mean maximal spacing $\langle s_{\max}\rangle$ assumed to behave as $\kappa \log N$ and plotted in log-linear scale, with fitted prefactors $\kappa_{(P)} = 0.98$ for CPE_N (circles), $\kappa_{(2)} = 0.85$ for $\mathrm{CUE}_{n\otimes n}$ (squares), and $\kappa_{(k)} = 0.95$ for $\mathrm{CUE}_2^{\otimes k}$ (downward triangles). Symbols denote numerical results obtained for 2^{14} independent random matrices. Solid, dashed, and dotted lines represent the fitted lines.

convenient to introduce an auxiliary variable $y_{\min} = s_{\min}/\langle s_{\min}\rangle$. The probability distribution $P(y_{\min})$ is presented in Fig. 6 for the $n \times n$ systems with $n = 2,3$, and 8. Numerical

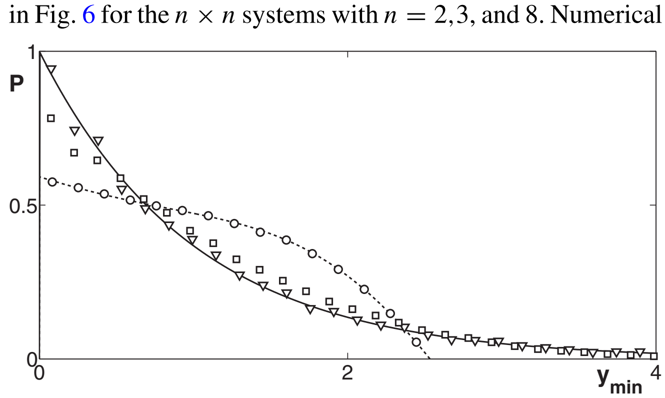

FIG. 6. Probability densities $P(y_{\min})$ of the rescaled minimal spacing $y_{\min} = s_{\min}/\langle s_{\min}\rangle$ for tensor products of CUE random unitary matrices $\mathrm{CUE}_n \otimes \mathrm{CUE}_n$ for $n = 2$ (circles), $n = 3$ (squares), and $n = 8$ (downward triangles). The symbols denote numerical results obtained for 2^{14} independent matrices; the solid curve represents the Poissonian distribution, while the dashed line corresponds to Eq. (3).

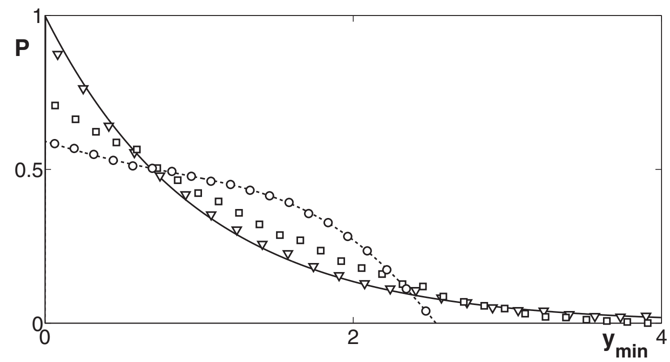

FIG. 7. As in Fig. 6 for tensor products of k independent Haar random unitary matrices of order 2, $\mathrm{CUE}_2^{\otimes k}$ for $k = 2$ (circles), $k = 3$ (squares), and $k = 8$ (downward triangles).

results for $n = 2$ agree with the explicit analytical prediction (3). Due to the tensor product structure of the ensemble the effect of level repulsion, characteristic of CUE, is washed out.

For larger n the opposite effect of level clustering (large probability at small values of the minimal spacing) becomes stronger, and already for $n = 8$ the probability distribution can be approximated by the exponential distribution, $P(y_{\min}) = \exp(-y_{\min})$, typical of the Poissonian distribution. A similar transition from distribution (3) to the Poisson distribution occurs in the case of k-qubit systems, as shown in Fig. 7.

B. Maximal spacings for tensor products

As in Sec. III D we rescale the maximal spacing $s_{\max}$ and analyze the rescaled deviation from the expectation value

$$z_{\max} = \frac{\pi}{\sqrt{6\mathrm{Var}(s_{\max})}}(s_{\max} - \langle s_{\max}\rangle). \tag{28}$$

The normalization factor is adjusted to predictions for the Poissonian process, for which the distribution of the variable z is asymptotically described by the Gumbel distribution,

$$P(z) = e^{-(z+\gamma)-e^{-(z+\gamma)}}. \tag{29}$$

Recall that $\gamma \approx 0.5772$ denotes Euler's constant, while the variance of the Gumbel distribution equal to $\pi^2/6$ suggests the

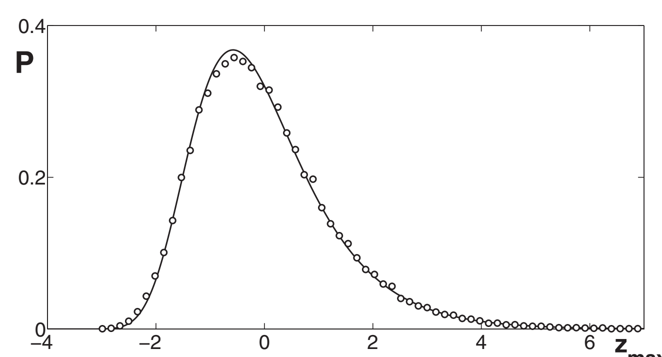

FIG. 8. Distribution $P(z_{\max})$ of the deviations of the rescaled maximal spacing from the expected value $z_{\max} = \alpha(s_{\max} - \langle s_{\max}\rangle)$, with $\alpha = \pi/\sqrt{6\mathrm{Var}(s_{\max})}$, for an ensemble of $\mathrm{CUE}_{2^6\otimes 2^6}$ matrices (circles). Numerical data were obtained from 2^{16} realizations, and the solid line denotes the Gumbel distribution (29).

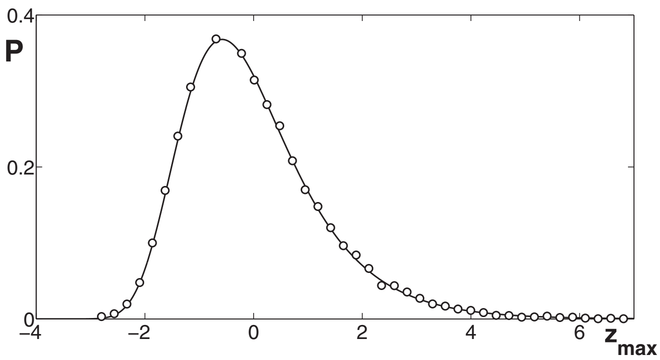

FIG. 9. As in Fig. 8 for a sample of 10^5 realizations of tensor products of $k = 22$ random unitary matrices of order 2.

convenient prefactor in the definition (28). Numerical results on the distributions of the variable $z_{\max}$ characterizing the distribution of the maximal spacings for the tensor products corresponding to two qunits and several qubits are presented in Figs. 8 and 9, respectively. In the asymptotic limit of a large matrix size numerical data seem to agree with predictions (29) of the Poisson ensemble.

V. CONCLUDING REMARKS

A significant and spectacular difference between the Poissonian ensemble on one side and COE and CUE on the other, concerning the degree of "repulsion" between adjacent levels, can be effectively analyzed in terms of distributions of the extremal spacings. We analyzed the average minimal spacing for several ensembles of random unitary matrices. On the basis of numerical results we propose a general form of the probability distribution $P(s_{min})$ of the minimal spacing for the standard ensembles of random unitary matrices. For CUE this distribution coincides with the recent result derived by Arous and Bourgade [11], while for COE it corresponds to the distributions analyzed for real symmetric matrices in [9,10].

The key part of this work concerned tensor products of random unitary matrices. In the case of k independent random matrices of order n distributed according to the Haar measure the tensor product leads asymptotically to a spectrum with a Poissonian level spacing distribution [12,15]. However, we report here a different behavior for the statistics of the extreme spacings. Even though the mean largest spacing $\langle s_{max}\rangle$ can be described by predictions obtained for the Poisson ensemble of diagonal random unitary matrices of size $N = n^k$, this is not the case for the mean minimal spacings.

In particular, in the case of k noninteracting qubits, described by the tensor product $\mathrm{CUE}^{\otimes k}$, the mean minimal spacing $\langle s_{min}\rangle$ displays significant deviations with respect to the predictions of the Poisson ensemble. In the simplest case of a two-qubit system we have shown that the eigenphases of the tensor product $\mathrm{CUE}_{2\otimes 2}$ show weaker repulsion than in the case of random CUE matrices of order $N = 4$.

Our study leaves several questions open. In particular, numerical results encourage one to derive an unknown scaling law of the average minimal spacing $\langle s_{\min}\rangle$ in the case of a k-qubit system. Furthermore, our observations suggesting that the distributions of the extremal spacing for ensembles of random matrices corresponding to two-qunit or k-qubit systems are asymptotically governed by the Poisson and the Gumbel distributions, respectively, should be confirmed by an analytical proof.

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APPENDIX: BASIC PROPERTIES OF THE POISSON PROCESS

By a *point process* Ξ on the real half line $\mathbb{R}_+ = [0,\infty)$ we mean a countable collection of random nonnegative numbers. For instance, a set $\Xi_U = \{(N/2\pi)\theta_1,\ldots,(N/2\pi)\theta_N\}$ of the rescaled eigenvalues of a random unitary matrix U can be viewed as a point process on $\mathbb{R}_+$.

A key example is a homogeneous *Poisson point process* Π on $\mathbb{R}_+$ with a parameter $\lambda > 0$ which is characterized by the following: (i) for any pairwise disjoint and measurable subsets $B_1,\ldots,B_n$ of $\mathbb{R}_+$ the number of points in these subsets forms independent random variables, and (ii) for any measurable subset B of $\mathbb{R}_+$ the number of points contained inside is described by the Poisson distribution with the parameter $\lambda|B|$, where $|B|$ denotes the Lebesgue measure of B. A detailed treatment of this process can be found in a classical monograph [18]. In this work we set the parameter λ to 1 and call it the standard Poisson point process.

One of the fundamental properties of the Poisson process is that its spacings are independent and are described by exponential distributions. We read in [18] the following.

Theorem 1. Let $\Pi = \{X_1,X_2,\ldots\}$ be the standard Poisson point process, where the points are labeled so that they do not decrease. Define its spacings $Y_1,Y_2,\ldots$ by (7). Then the variables $Y_1,Y_2,\ldots$ are independent and identically distributed with density e^{-y}, $y > 0$.

Knowing this, we are able to examine the asymptotes of the extreme gaps $Y_{\min} = \min_{j\leqslant N} Y_j$ and $Y_{\max} = \max_{j\leqslant N} Y_j$.

Theorem 2. Let $Y_1,Y_2,\ldots$ be a sequence of random variables which are independent identically distributed with density $P(y) = e^{-y}$ for $y > 0$. Then,

$$\langle Y_{\min}\rangle = \Big\langle \min_{j\leqslant N} Y_j\Big\rangle = 1/N, \qquad \langle Y_{\max}\rangle = \Big\langle \max_{j\leqslant N} Y_j\Big\rangle = \sum_{k=1}^{N} 1/k \sim \ln N. \tag{A1}$$

If we rescale the variables to set the mean to unity, $y = NY$, asymptotically, they behave exponentially and concentrate, respectively, as

$$NY_{\min} \xrightarrow{d} e^{-y}\mathbf{1}_{\{y>0\}}, \tag{A2}$$

$$Y/\langle Y_{\min}\rangle \xrightarrow{d} 1, \tag{A3}$$

where $\xrightarrow{d}$ denotes the convergence in the distribution.

Furthermore, the fluctuations of $Y/\langle Y_{\min}\rangle$ around 1 are governed at the scale $\langle \max_{j\leqslant N} Y_j\rangle \sim \ln N$ by the Gumbel distribution,

$$Y - \langle Y_{\min}\rangle \xrightarrow{d} P(z) = e^{-(z+\gamma)-e^{-(z+\gamma)}}, \tag{A4}$$

where $\gamma := \lim_{n\to\infty}(\sum_{k=1}^{n} 1/k - \ln n) \approx 0.5772$ is Euler's constant.

Given the fact that the distribution functions are easily calculable,

$$\mathbb{P}\Big(\min_{j\leqslant N} Y_j > t\Big) = e^{-Nt}, \quad t > 0,$$
$$\mathbb{P}\Big(\max_{j\leqslant N} Y_j \leqslant t\Big) = (1-e^{-t})^N, \quad t > 0,$$

Theorem 2 can be proved by a direct computation.

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