REGULAR ARTICLES

Entropy computing via integration over fractal measures

Wojciech Słomczyński[a)]
Instytut Matematyki, Uniwersytet Jagielloński, ul. Reymonta 4, 30–059, Kraków, Poland

Jarosław Kwapień[b)]
Instytut Fizyki Ją̨drowej im. H. Niewodniczańskiego, ul. Radzikowskiego 152, 31–305, Kraków, Poland

Karol Życzkowski[c)]
Instytut Fizyki im. M. Smoluchowskiego, Uniwersytet Jagielloński, ul. Reymonta 4, 30–059, Kraków, Poland

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We discuss the properties of invariant measures corresponding to iterated function systems (IFSs) with place-dependent probabilities and compute their Rényi entropies, generalized dimensions, and multifractal spectra. It is shown that with certain dynamical systems, one can associate the corresponding IFSs in such a way that their generalized entropies are equal. This provides a new method of computing entropy for some classical and quantum dynamical systems. Numerical techniques are based on integration over the fractal measures. © *2000 American Institute of Physics.* [S1054-1500(99)01204-5]

Dedicated to the memory of Marin Poźniak

In order to characterize quantitatively properties of a given nonlinear system, one often uses the notion of the dynamical entropy. It describes the asymptotic changes of the system entropy in time. Since analytical computing of this quantity is possible only for a limited number of simple models, it is important to develop efficient numerical techniques for this purpose. In this article we propose a method of computing the dynamical entropy by averaging the static Boltzmann-Shannon entropy. The integration is performed over a suitably chosen measure, which in the general case displays fractal properties.

I. INTRODUCTION

Chaos in a *classical* dynamical system can be defined by the positiveness of the *Kolmogorov-Sinai (KS) dynamical entropy*.[1] This quantity, characterizing dynamical properties of a system, and defined via an asymptotic limit (time tending to infinity) is in general not easy to obtain analytically. On the other hand, numerical computing of dynamical entropy from time series requires advanced techniques.[2,3] Also in the *quantum* case estimating, so called, *coherent states* (CS) *quantum entropy*[4–7] is not a simple task. In the present paper we propose a method of computing dynamical entropy of a system by finding an appropriate iterated function system with the same entropy.

An *iterated function system* (IFS) consists of a certain number k of functions $F_i, i=1,\ldots,k$, which act randomly with given probabilities $p_i, i=1,\ldots,k$. An IFS may, therefore, be concerned as a combination of deterministic and stochastic dynamics. For sufficiently contracting functions one can prove (under some irreducibility conditions) that IFS generates a unique invariant measure (see Sec. II). Generically[8,9] this measure is localized on a fractal set. As it was described, e.g., in the elegant book of Barnsley[10] IFSs may be used to produce interesting fractal images, or to encode and transmit graphics via computer. For the majority of commonly analyzed and applied IFSs the probabilities p_i are constant. For example, such IFSs have been used to construct multifractal energy spectra of certain quantum systems[11] and to investigate the one-dimensional random-field Ising model[12] or second order phase transitions.[13] On the other hand, with some classical and quantum dynamical systems one can associate in a natural way IFSs with place-dependent probabilities.[6,14–19] In the present paper such IFSs will be called *iterated function systems of the second kind*, on the analogy of position-dependent gauge transformations.[20]

We estimate the Kolmogorov–Sinai and Rényi dynamical entropies of certain IFSs of the second kind, using various numerical methods, which can be also applied in the general case. We use similar procedures to analyze the properties of the invariant measures of these IFSs and demonstrate their multifractal character. Eventually, we show that one can calculate the entropy of certain dynamical systems constructing IFSs with the same entropy. We give several examples that illustrate this new method of computing entropy.

[a)]Electronic mail: slomczyn@im.uj.edu.pl
[b)]Electronic mail: kwapien@castor.if.uj.edu.pl
[c)]Also at: Centrum Fizyki Teoretycznej PAN, Al. Lotników 32/46, 02-668 Warszawa, Poland. Electronic mail: karol@tatry.if.uj.edu.pl

This paper is organized as follows. In the next section the definitions of IFSs of the first and second kind are recalled and their basic properties are considered. In Sec. III we discuss briefly several methods of analytical and numerical computing of dynamical entropy of an IFS. In Sec. IV we study generalized dimensions of measures which are invariant under the action of a one-dimensional IFS. Section V presents a detailed analysis of a family of IFSs of the second kind and their invariant measures. Certain integrals over these measures are calculated. Moreover we compute the Rényi entropies, generalized dimensions and multifractal spectra of these measures. In the subsequent section we investigate the connection between one-dimensional dynamical systems and IFSs of the second kind. In particular, IFSs associated to asymmetric tent map, logistic map, and ''hut map'' are analyzed and their entropies are calculated. We also show how one can apply this method to compute CS-quantum entropy. Concluding remarks are contained in the last section.

In this paper we present only the results and numerical calculations. For the proofs we refer the reader to a forthcoming publication.

II. ITERATED FUNCTION SYSTEMS AND THEIR INVARIANT MEASURES

An iterated function system (IFS) is specified by k functions transforming a metric space into itself and k place-dependent probabilities which characterize the likelihood of choosing a particular map at each step of the evolution of the system. Under certain contractivity and irreducibility conditions one can prove the existence of a unique attractive invariant measure for an IFS, as well as ergodic and central limit theorems. Miscellaneous results of this type have been established since the late thirties (some of them have been proved independently by several authors)—see for instance Refs. 14, 15, 21–33, and references therein.

In the present paper we study IFSs $\mathcal{F}=\{F_i,p_i:i=1,\ldots,k\}$ that fulfill the following (rather strong) assumptions which guarantee veracity of the above mentioned theorems:

General assumption:

(1) X is a compact metric space;

(2) $F_i:X\to X,\ i=1,\ldots,k$ are Lipschitz functions with the Lipschitz constants $L_i<1$;

(3) $p_i:X\to[0,1],\ i=1,\ldots,k$ are Hölder continuous functions fulfilling $\sum_{i=1}^k p_i(x)=1$ for each $x\in X$;

(4) $p_i(x)>0$ for every $x\in X$ and $i=1,\ldots,k$.

Such IFSs are often called *hyperbolic*. Unless otherwise stated we assume that all IFSs under consideration are hyperbolic.

Let us recall briefly several basic facts on IFSs.

The IFS $\mathcal{F}=\{F_i,p_i:i=1,\ldots,k\}$ generates the following *Markov operator V* acting on $M(X)$ (the space of all probability measures on X):

$$(V\nu)(B)=\sum_{i=1}^{k}\int_{F_i^{-1}(B)}p_i(\lambda)\,d\nu(\lambda), \tag{2.1}$$

where $\nu\in M(X)$ and B is a measurable subset of X. This operator describes the *evolution of probability measures* under the action of $\mathcal{F}$. The related *Markov process* can be defined in the following way. As a probability space we take the *code space* $\Omega=\{1,\ldots,k\}^{\mathbb{N}}$ and we put P_x for the probability measure on Ω given by

$$\begin{aligned}P_x(i_1,\ldots,i_n)&:=P_x(\{\omega\in\Omega:\omega(j)=i_j,j=1,\ldots,n\})\\&:=p_{i_1}(x)p_{i_2}(F_{i_1}(x))\cdots\\&\quad\times p_{i_n}(F_{i_{n-1}}(F_{i_{n-2}}(\ldots(F_{i_1}(x))))),\end{aligned} \tag{2.2}$$

where $x\in X, i_j=1,\ldots,k, j=1,\ldots,n; n\in\mathbb{N}$. Then the formulas

$$Z_n^x(\omega)=F_{\omega(n)}(F_{\omega(n-1)}(\ldots(F_{\omega(1)}(x)))),\quad Z_0^x(\omega)=x, \tag{2.3}$$

for $x\in X,\omega\in\Omega,n\in\mathbb{N}$ define the requested Markov stochastic process on $(\Omega,\{P_x\}_{x\in X})$.

One can show that for an IFS which fulfills our assumption there exists a unique *invariant probability measure* μ satisfying the equation $V\mu=\mu$ (the proof of this claim can be found in Ref. 14). This measure is *attractive*, i.e., $V^n\nu$ converges weakly to μ for every $\nu\in M(X)$ if $n\to\infty$ or, in other words, $\int_X u\ dV^n\nu$ tends to $\int_X u\ d\mu$ for every continuous $u:X\to\mathbb{R}$. Thus, in order to obtain the exact value of $\int_X u\ d\mu$, it is sufficient to find the limit of the sequence $\int_X u\ dV^n\nu$ for an arbitrary initial measure ν. For instance, taking ν equal to a Dirac delta measure δ_x for some $x\in X$ we obtain the integral of u over the invariant measure μ as the limit of the sequence

$$U_n:=\sum_{i_1,\ldots,i_n=1}^{k}P_x(i_1,\ldots,i_n)u(x_{i_1,\ldots,i_n}), \tag{2.4}$$

where $x_{i_1,\ldots,i_n}:=F_{i_n}(F_{i_{n-1}}(\ldots(F_{i_1}(x))))$. After Barnsley[10] (see also Ref. 34) we call this method of computing integrals over the invariant measure *deterministic algorithm*. To find the integral numerically we can also employ the ergodic theorem for IFSs.[15,21,22,26,31,32] Any initial point $x\in X$ iterated by the IFS generates a random sequence $(z_0=x,z_1,\ldots,z_n,\ldots)$, where $z_i:=Z_i^x$. Then

$$\int_X u(x)\,d\mu(x)=\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}u(z_i), \tag{2.5}$$

with probability one, i.e., except of a set of measure P_x zero. Moreover, if u fulfills the Lipschitz condition, we can evaluate the rate of convergence in the ergodic theorem utilizing the central limit theorem for IFS.[15,30] This leads to a probabilistic (Monte Carlo) numerical method of computing integrals over the invariant measure which was called *random iterated algorithm* by Barnsley.[10] We have successfully applied both techniques to compute numerically various integrals, including those necessary to estimate the dynamical entropy of IFS (see Sec. III).

Finally, let us look at the evolution of densities under the action of IFS. If m is a finite measure on X and $F_i(i=1,\ldots,k)$ are *nonsingular* [i.e., $m(A)=0$ implies

$m(F_i^{-1}(A))=0$ for each measurable $A\subset X$], then the IFS $\mathcal{F}$ generates a Markov operator on the space of densities (with respect to m) on X, also called the *Frobenius–Perron operator*.[35] It is so if, for instance, X is an interval in $\mathbb{R}$ and $\{F_i : i=1,\ldots,k\}$ are diffeomorphisms. It follows from Eq. (2.1) that the Frobenius–Perron operator M associated with $\mathcal{F}$ is given in this case by the formula

$$M[\gamma](x)=\sum_i p_i(F_i^{-1}(x))\gamma(F_i^{-1}(x))\left|\frac{dF_i^{-1}(x)}{dx}\right|, \quad (2.6)$$

where the sum goes over all $1\leqslant i\leqslant k$ such that $x\in F_i(X)$, for γ a density and $x\in X$.

If probabilities p_i are constant then we will say that an IFS is *of the first kind*. IFSs with place-dependent probabilities will be called IFSs *of the second kind* (they also appear in the literature under the name of *learning systems*).

III. ENTROPY OF IFS

Let μ be the attractive invariant measure for the IFS $\mathcal{F}=\{F_i,p_i : i=1,\ldots,k\}$. We define the probability measure P_μ on the code space $\Omega=\{1,\ldots,k\}^{\mathbb{N}}$ by

$$P_\mu(i_1,\ldots,i_n):=P_\mu(\{\omega\in\Omega:\omega(j)=i_j, j=1,\ldots,n\})$$
$$:=\int_X P_x(i_1,\ldots,i_n)\,d\mu(x), \quad (3.1)$$

for $i_j=1,\ldots,k, j=1,\ldots,n; n\in\mathbb{N}$.

It is easy to show that this measure is invariant with respect to the shift on Ω.

Now we can define the *partial entropies* as

$$H(n):=-\sum_{i_1,\ldots,i_n=1}^{k} P_\mu(i_1,\ldots,i_n)\ln P_\mu(i_1,\ldots,i_n), \quad (3.2)$$

and the *relative entropies* by

$$G(1):=H(1),\quad G(n)=H(n)-H(n-1),\quad \text{for } n>1. \quad (3.3)$$

The dynamical entropy of Kolmogorov and Sinai can be extracted from both sequences, i.e., $K_1=\lim_{n\to\infty}G(n)=\lim_{n\to\infty}H(n)/n$. The usage of relative entropies is often advantageous, since the convergence of $H(n)/n$ is slow (usually as $1/n$), while in many cases the sequence $G(n)$ converges to the KS-entropy exponentially fast.[2,36] Note that the entropy of a stochastic system like an IFS, can be defined in several different ways.[37] Here we are interested in the dynamics induced by an IFS in the k-symbols code space, which leads to the entropy finite and bounded by $\ln k$.

The concept of dynamical KS-entropy is based on the notion of the Boltzmann–Shannon entropy function which can be multifariously generalized.[38] In this paper we discuss the two versions of Rényi entropy defined in Ref. 6 for any real $q\neq 1$. The first one, often used in the literature, corresponds to the limit of *partial entropies*

$$\tilde{K}_q:=\limsup_{n\to\infty}\frac{1}{n}\frac{1}{1-q}\ln\left[\sum_{i_1,\ldots,i_n=1}^{k}[P_\mu(i_1,\ldots,i_n)]^q\right]. \quad (3.4)$$

The other one, based on the notion of Rényi conditional entropy,[39] is defined via *relative entropies*:

$$K_q:=\limsup_{n\to\infty}\frac{1}{1-q}\ln\left[\sum_{i_1,\ldots,i_n=1}^{k}P_\mu(i_1,\ldots,i_n)\times(P_\mu(i_1,\ldots,i_n)/P_\mu(i_1,\ldots,i_{n-1}))^{q-1}\right]. \quad (3.5)$$

For $q=0$ both quantities are equal to the topological entropy $K_0=\ln k$ and the KS-entropy is obtained for both quantities in the limit $q\to 1$. On the other hand, in general, the two versions of Rényi entropies are different (see Secs. III and V). The computation of entropy K_q is more straightforward than $\tilde{K}_q$ and an analytical treatment is possible in some cases, on the other hand, its relation to the thermodynamical formalism seems to be less clear.

Both definitions give us some form of the Rényi dynamical entropy for the dynamics generated by an IFS in the k-symbols code space and for the specific partition of this space into k rectangles labeled by the first symbol. Note, however, that if one defines the (partition independent) Rényi dynamical entropy taking simply the supremum over all finite partitions (as for the KS-entropy), this leads to trivial dependence: $K_q=\infty,\ q<1;\quad K_q=K_1,\ q\geqslant 1$.[40] Consequently, from now on, we shall discuss only the Rényi entropy for the above mentioned k-elements partition.

For an IFS of the first kind (with constant probabilities p_i) both Rényi entropies are equal and can be written down explicitly[41,42]

$$K_q=\tilde{K}_q=\frac{1}{1-q}\ln(p_1^q+p_2^q+\cdots+p_k^q), \quad (3.6)$$

for $q\neq 1$. The KS-entropy is obtained by calculating the limit $\lim_{q\to 1}K_q$, which gives $K_1=\tilde{K}_1=-\sum_{i=1}^{k}p_i\ln p_i$. Observe that for an IFS of the first kind, the value of the entropy does not depend on the character of functions F_i.

For an IFS of the second kind one cannot directly apply formula (3.6), since the probabilities are place dependent. A natural generalization for this case is possible,[6,43] viz., one has to average the Rényi entropy performing an integral over the invariant measure μ

$$K_q=\frac{1}{1-q}\ln\int_X\sum_{i=1}^{k}(p_i(x))^q\,d\mu(x). \quad (3.7)$$

In the limit $q\to 1$, corresponding to KS-entropy, this formula gives

$$K_1=-\int_X\sum_{i=1}^{k}p_i(x)\ln[p_i(x)]\,d\mu(x). \quad (3.8)$$

Moreover, one can show that the relative entropies converge to the limiting value K_q exponentially.[43] Now, to compute the entropy, it suffices to apply one of the two methods of calculating the integral over the invariant measure of an IFS presented in Sec. II.

To estimate entropy $\tilde{K}_q$ one may consider the modified IFS $\mathcal{F}_q=\{F_i,\tilde{p}_i(q): i=1,\ldots,k\}$ with the probabilities $\tilde{p}_i(q)$ proportional to p_i^q, that is, given by the formula

$$\tilde{p}_i(q)(x)=(p_i(x))^q \Big/ \sum_{j=1}^{k} (p_j(x))^q \tag{3.9}$$

for $x\in X, i=1,\dots,k$, and $q\in\mathbb{R}$.

Note that a similar method was used for one-dimensional IFSs of the first kind in Ref. 12. It is easy to prove that the IFS $\mathcal{F}_q$ satisfies our general assumption, and hence, a unique invariant probability measure μ^q exists. Then one can derive[43] the following inequality:

$$(1-q)\tilde{K}_q \geqslant \int_X \ln\sum_{i=1}^{k} (p_i(x))^q d\mu^q(x), \tag{3.10}$$

which provides a lower bound for entropy $\tilde{K}_q$ for $q<1$, and an upper bound for $q>1$. In examples we analyze (see Secs. V C and VI B) this bound is actually very close to the exact value of the entropy $\tilde{K}_q$ calculated numerically. Furthermore, the integral on the right-hand side of Eq. (3.10) can be relatively easily computed (see Sec. II), whereas the convergence in Eq. (3.4) seems to be rather slow, namely as n^{-1}.

IV. DIMENSIONS OF INVARIANT MEASURE FOR IFS

In this section we assume that a one-dimensional $(X\subset\mathbb{R})$ IFS $\mathcal{F}=\{F_i, p_i : i=1,\dots,k\}$ is given, where F_i are diffeomorphisms fulfilling the general assumption from Sec. II and the following *separation condition*: $\operatorname{int} F_i(X)\cap \operatorname{int} F_j(X)=\emptyset$ for $i\neq j$, $i,j=1,\dots,k$, where $\operatorname{int} F_i(X)$ denotes the interior of the set $F_i(X)$. Our aim is to calculate the *generalized dimensions* D_q of the invariant measure for $\mathcal{F}$. These quantities were introduced and analyzed by Grassberger, Hentschel, and Procaccia[44,45] (for more information see Refs. 46–48), and D_0 is just the Hausdorff dimension of the invariant measure. The correlation dimension D_2 for certain IFSs has been recently studied by Chin, Hunt, and Yorke.[49]

Let us consider the following *pressure function*:

$$P(q,\tau)(x)=:\limsup_{n\to\infty}\frac{1}{n}\ln\Bigg[\sum_{i_1,\dots,i_n=1}^{k} P_x(i_1,\dots,i_n)^q \times|(F_{i_n}\circ\dots\circ F_{i_1})'(x)|^{-\tau}\Bigg], \tag{4.1}$$

for $q\geqslant 0$ and $\tau\in\mathbb{R}$.

In the sequel we assume that the limit in Eq. (4.1) does not depend on x and the generalized dimensions D_q are given by the formula

$$D_q=\frac{\tau(q)}{q-1}, \tag{4.2}$$

where $\tau(q)=\tau$ is the only solution of the equation $P(q,\tau)=0$. For the IFS of the first kind this assumption was heuristically verified by Halsey *et al.*[50] (see also Ref. 47). Moreover, Bohr and Rand[51,52] showed that it holds for the IFS generated by expanding maps on the interval (‘‘cookie-cutters’’).

To estimate the generalized dimension D_q we can use the technique already introduced in the preceding section. Namely, we consider the modified IFS $\mathcal{F}_{q,\tau}=\{F_i, \tilde{p}_i(q,\tau): i=1,\dots,k\}$ with the probabilities $\tilde{p}_i(q,\tau)$ given by

$$\tilde{p}_i(q,\tau)(x)=(p_i(x))^q|F_i'(x)|^{-\tau} \Big/ \sum_{j=1}^{k} (p_j(x))^q|F_j'(x)|^{-\tau} \tag{4.3}$$

for $x\in X, i=1,\dots,k$, $q>0$, and $\tau\in\mathbb{R}$.

Again it is easy to prove that the IFS $\mathcal{F}_{q,\tau}$ fulfills our general assumption, and hence, admits a unique invariant probability measure $\mu^{q,\tau}$. Then one can show[43] that the following inequality holds:

$$(q-1)D_q\leqslant(q-1)\overline{D_q}, \tag{4.4}$$

where $(q-1)\overline{D_q}=\overline{\tau}$ is the solution of the equation

$$\int_X \ln\sum_{i=1}^{k} (p_i(x))^q|F_i'(x)|^{-\overline{\tau}} d\mu^{q,\overline{\tau}}(x)=0, \tag{4.5}$$

for $q>0$.

This provides a lower bound for the generalized dimension D_q for $q<1$, and an upper bound for $q>1$. In all the cases we study in Secs. V C and VI B this bound (which can be relatively easily computed) is actually very close to the value of the dimension D_q calculated numerically. In order to calculate the generalized dimensions D_q we use the ‘‘box-counting’’ algorithm, which in this case yields better results than the algorithm of Grassberger and Procaccia[3,45] applied to the time series extracted from the IFS. Note that if $|F_i'(x)|\equiv L>0$ for all $i=1,\dots,k$, then the generalized entropies and dimensions are related by a simple formula $\tilde{K}_q=-D_q\ln L$ (a relation between both quantities for IFSs of the first kind was examined in Ref. 12).

Scaling properties of the invariant measure could be described with the aid of its multifractal spectrum $f(\alpha)=\inf_q\{\alpha q+(1-q)D_q\}$ (for more information on multifractal spectrum see Refs. 3, 46, 47, and 53).

V. MULTIFRACTALS GENERATED BY IFS OF THE SECOND KIND

A. Cantor measures

Let us consider a family of IFSs $\{X=[0,1], k=2;\ F_1(x)=x/3, F_2(x)=(x+2)/3;\ p_1(x)=(1-a)+(2a-1)x,\ p_2(x)=a+(1-2a)x$ for $x\in X\}$, where $a\in[0,1]$. It is easy to see that these IFSs fulfill our general assumption for $a\in(0,1)$, which guarantees the existence of a unique invariant measures μ_a and veracity of the other results mentioned in Secs. II, III, and IV. For $a=1$ one can prove the existence of a unique attractive invariant measure as well as the ergodic theorem (but not the central limit theorem) using more refined results which may be found in Ref. 15 or Ref. 26. On the other hand, for $a=0$, the IFS attracts every measure into a linear combination of two Dirac deltas localized at points 0 and 1. Hence, there exists a whole family of invariant measures $\{r\delta_0+(1-r)\delta_1 : r\in[0,1]\}$ in this case.

An IFS of the first kind is obtained for $a=1/2$, since the probabilities $p_1(x)=p_2(x)\equiv 1/2$ do not depend on x. The invariant measure $\mu_{1/2}$ is spread uniformly over the Cantor set. The generalized fractal dimension is constant $D_q=D_0$

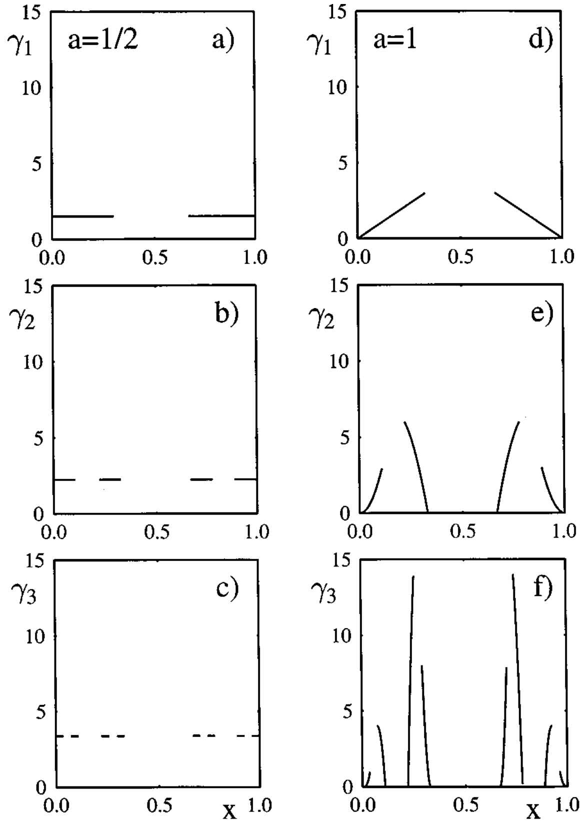

FIG. 1. First three iterations of the uniform density on $X=[0,1]$ by two IFSs $\{F_1(x)=x/3, F_2(x)=(x+2)/3\}$ attracting to the Cantor set. (a)–(c) are obtained for IFS ($a=1/2$) with constant probabilities $p_1(x)=p_2(x)=1/2$, while (d)–(f) for IFS ($a=1$) with place-dependent probabilities $p_1(x)=x, p_2(x)=1-x$.

$=\ln 2/\ln 3$, which implies a singular multifractal spectrum concentrated at $\alpha_1=\ln 2/\ln 3$ with $f(\alpha_1)=\alpha_1$. The generalized Rényi entropy for this IFS can be directly obtained from Eq. (3.6). It gives $K_q=\ln 2$ for all $q\in\mathbb{R}$. The Cantor measure $\mu_{1/2}$ can thus be called both *uniform* (constant generalized dimension) and *balanced* (constant Rényi entropy).[41]

In the case $a=1$ the probabilities are place-dependent ($p_1(x)=x; p_2(x)=1-x$) and define an IFS of the second kind. In order to understand the nature of the measure μ_1, let us consider the iterations $\gamma_n=M_1(\gamma_{n-1})$ of the initially uniform density γ_0 with respect to the Frobenius–Perron operator M_1 given by Eq. (2.6).

We simplify the notation by introducing the ''box' functions $x\rightarrow\Theta_{a,b}(x):=\Theta(x-a)\Theta(b-x)$, with the Heaviside function Θ given by $\Theta(y)=0$, for $y<0$, and $\Theta(y)=1$, for $y\geqslant 0$. The uniform density in X can thus be written as $\gamma_0=\Theta_{0,1}$.

Formula (2.6) allows us to obtain for instance the first two iterations of γ_0

$$\gamma_1(x)=9x\Theta_{0,1/3}(x)+9(1-x)\Theta_{2/3,1}(x) \tag{5.1}$$

and

$$\begin{aligned}\gamma_2(x)=&3^5x^2\Theta_{0,1/9}(x)+3^4(x-3x^2)\Theta_{2/9,3/9}(x)\\&+3^4(3x-2)(1-x)\Theta_{6/9,7/9}(x)\\&+3^5(1-x)^2\Theta_{8/9,1}(x),\end{aligned} \tag{5.2}$$

for $x\in[0,1]$.

Similarities and differences between densities approximating the standard Cantor measure $\mu_{1/2}$ and the measure μ_1 are displayed in Fig. 1. Due to constant probabilities, in the first case the measure $\mu_{1/2}$ covers uniformly the Cantor set [Figs. 1(a)–1(c)]. On the other hand, for the IFS of the second kind, the place-dependent probabilities induce a highly non-uniform distribution of the measure μ_1 [Figs. 1(d)–1(f)]. For example, in each connected component of the support of the density γ_n it can be expressed as a polynomial in x of nth degree. Note that, if γ_n achieves its maximum at x_n, then $\gamma_{n+1}(x_n)=0$. One may expect, therefore, that the measure μ_1 is multifractal.

B. Integration over fractal measures

Let us now calculate the integrals of certain functions u over the invariant measures $\mu_{1/2}$ and μ_1. Let us find, for example, the mean ($u_A(x)=x$) and the mean square ($u_B(x)=x^2$) for these measures. Iterating the uniform density γ_0 by the Frobenius–Perron operator $M_{1/2}$ we get the sequences of integrals ${U_{A_n}}^{(1/2)}=\int_X u_A(x)\gamma_n(x)dx=1/2$ (independently of n) and ${U_{B_n}}^{(1/2)}=\int_X u_B(x)\gamma_n(x)dx=3(1-3^{-2n-2})/8$. Consequently, two integrals in question read $\int_X x\,d\mu_{1/2}(x)=1/2$ and $\int_X x^2 d\mu_{1/2}(x)=3/8$. Computing integrals over the measure μ_1 it is advantageous to start with an initially singular measure. To demonstrate the convergence rate explicitly we take a one parameter family of measures consisting of a combination of two delta peaks localized in both ends of the unit interval: $\kappa_r=r\delta_0+(1-r)\delta_1$, where $r\in[0,1]$. Iterating this measure with respect to the Markov operator V_1 given by Eq. (2.1) we compute the r-dependent integrals of both functions ${U_{A_n}}^{(1)}=\frac{1}{2}[1+(-\frac{1}{3})^n]-r(-\frac{1}{3})^n\rightarrow\frac{1}{2}$ and ${U_{B_n}}^{(1)}=\frac{1}{3}[1+2(-\frac{1}{3})^n]-r(-\frac{1}{3})^n\rightarrow\frac{1}{3}$. Both sequences tend to their limits independently of the parameter r, which contributions into the integral decay with n as 3^{-n}. Since both measures $\mu_{1/2}$ and μ_1 are symmetric with respect to $x=1/2$, the first moments $\langle x\rangle$ are equal, however, already the second moments reveal the difference.

In a similar way an integral of a function over the Cantor set may be expressed as a limit of the sum of 2^n terms (multiplied by the appropriate weights), which probe the function on the ends of the intervals composing the Cantor set. In some cases this result can be put into a form of an infinite product. For example the characteristic function of the uniform Cantor measure $\mu_{1/2}$ is given by Ref. 54 (see also Ref. 55)

$$\int_0^1 e^{itx}d\mu_{1/2}(x)=e^{it/2}\prod_{n=1}^{\infty}\cos\left(\frac{t}{3^n}\right). \tag{5.3}$$

Due to fast convergence this form is particularly useful for numerical evaluation. In general, computing the integrals over multifractal measures generated by IFSs of the second kind one has to rely on numerical methods described in Sec. II. For IFSs with small number of functions the deterministic algorithm based on Eq. (2.4) provides more precise results than the random iterated algorithm (2.5). The latter seems to be more efficient for IFSs consisting of many functions.

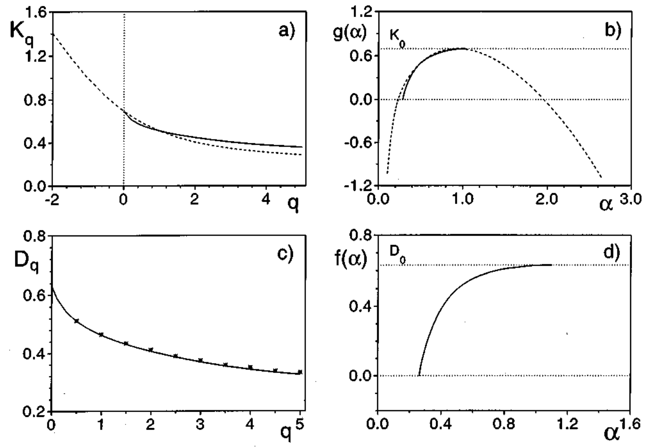

FIG. 2. Quantities characterizing the invariant measure for ''Cantor'' IFS of the second kind ($a=1$): (a) Rényi entropies K_q (dashed line), $\tilde{K}_q$ (solid line); (b) scaling spectra $g(\alpha)$ (dashed line), $\tilde{g}(\alpha)$ (solid line); (c) fractal dimensions $\bar{D}_q$ (solid line), D_q (stars); (d) multifractal spectrum $f(\alpha)$.

C. Computing of entropies and dimensions via integration over the fractal measures

The entropy of the IFSs can be expressed as an integral of the Rényi (or Boltzmann–Shannon) entropy function over the invariant measure μ_a [see Eqs. (3.7) and (3.8)] for K_q, or estimated by the respective integral over the measure μ_a^q [see Eq. (3.10)] for $\tilde{K}_q$. Note that, comparing the latter case with the former, the natural logarithm interchanges with the integral over X.

Numerically computed Rényi entropies K_q and $\tilde{K}_q$ of the measure μ_1 are displayed in Fig. 2(a) [however, formula (3.10) is valid only for $q>0$ in this case]. As expected, both entropies depend substantially on the Rényi parameter q, which means that the invariant measure μ_1 is not balanced. Note that the inflection point of the curve K_q is situated not at $q=0$ but at some negative q_c. Making use of the integrals $U_{A_n}(1)$ and $U_{B_n}(1)$ we obtain analytical results $K_2=(\ln 3)/2$ and $K_3=(\ln 2)/2$ directly from Eq. (3.7). Rényi entropies allow one to compute the scaling spectra via the Legendre transform: $g(\alpha)=\inf_q\{\alpha q+(1-q)K_q\}$ and $\tilde{g}(\alpha)=\inf_q\{\alpha q+(1-q)\tilde{K}_q\}$ [see Fig. 2(b)]. The common maximum of the scaling spectra gives the topological entropy $K_0=\ln 2$. Observe that the spectrum $g(\alpha)$ acquires also negative values. This does not contradict the interpretation of the scaling spectrum given by Bohr and Rand,[51] which is applicable for $\tilde{g}(\alpha)$.

Let us recall that the Hausdorff dimension D_0 of μ_1 is the same as for the standard Cantor measure $\mu_{1/2}$ (or any other invariant measure μ_a for $a>0$) and equals $D_0=\ln 2/\ln 3\approx 0.631$. The generalized dimensions are given by $D_q=\tilde{K}_q/\ln 3$ and can be fairly approximated by $\bar{D}_q$ (see Sec. IV). In Fig. 2(c) we compare these quantities with those obtained by the box-counting algorithm and observe that the difference is very small. As expected, the generalized dimension decreases with the Rényi parameter q (for example, $D_1\approx 0.47$ and $D_2\approx 0.41$), which confirms the multifractal property of the measure μ_1 [see also Fig. 2(d)]. For this measure we observed that the numerical algorithm, providing reliable results for $q\geqslant 1$, definitely ceases to work for negative q.

VI. DYNAMICAL SYSTEM AND IFS

Let us consider a dynamical system (quantum or classical) endowed with an invariant measure and a partition of the phase space. We shall look for an IFS with the entropy equal to the entropy of the dynamical system with respect to the given partition. This IFS represents, in a sense, the backward evolution of the system.[43] Having such an IFS we could apply formulas for the entropy of IFS given in Sec. III and so we would obtain a new method of computing the dynamical entropy of the system. We illustrate this procedure on two examples: The Rényi entropy of certain 1D (one-dimensional) dynamical systems and the coherent states (CS) entropy of certain quantum systems.

A. One-dimensional dynamical systems

Let us consider a piecewise continuously differentiable map $f:[0,1]\to[0,1]$. We assume that there exist subintervals A_i $(i=1,\dots,k)$ such that $[0,1]=\cup_{i=1}^k A_i$, $f(A_i)=[0,1]$, and $|f'|>0$ in the interior of A_i, for each i. Let us suppose that f admits an absolutely continuous invariant measure μ and let us denote its density by ρ. The partition $\{A_i\}_{i=1}^k$ is generating in this case, i.e., the dynamical entropy with respect to this partition is equal to the dynamical entropy of the system. With the map f we can associate an IFS $\mathcal{F}=\{F_i,p_i:i=1,\dots,k\}$ given by

$$F_i(x)=f|_{A_i}^{-1}(x) \tag{6.1}$$

and

$$p_i(x)=\frac{\rho(F_i(x))}{\rho(x)}|F_i'(x)|, \tag{6.2}$$

for $x\in[0,1]$ and $i=1,\dots,k$ (this is a particular case of the general construction from Refs. 17 and 18). Note that the functions $(F_i)_{i=1}^k$ are just continuous branches of the inverse of f (see Fig. 3).

It is well known that the measure μ is also invariant for the IFS $\mathcal{F}$.[14,15,18] Clearly, the generalized entropies [given by Eqs. (3.4) and (3.5)] are in both cases equal, as the probabilities $P_\mu(i_1,\dots,i_n)$ are the same. In general, the most difficult stage in this construction is to show that the IFS $\mathcal{F}$ satisfies the assumptions which guarantee the truthfulness of formulas (3.7) and (3.8).

In the present paper we analyze three examples: asymmetric *''tent'' map*, *''igloo'' map* (better known as the *logistic map*) and *''hut'' map* given by:

(a) *Tent map*: $f(y)=y/r$ for $0\leqslant y<r$ and $f(y)=(1-y)/(1-r)$ for $r\leqslant y\leqslant 1$ (where $r\in(0,1)$ is a parameter) with the constant invariant density $\rho=1$. Then $F_1(x)=rx$, $F_2(x)=(r-1)x+1$, $p_1(x)=r$, and $p_2(x)=1-r$ for $x\in[0,1]$ [Figs. 3(a) and 3(b)];

(b) *Igloo map*: $f(y)=4y(1-y)$ for $y\in[0,1]$. In this case the invariant density has the form $\rho(y)$

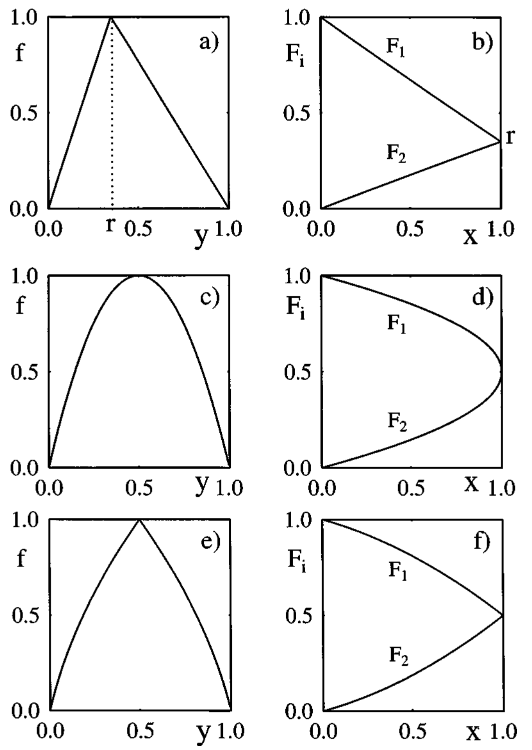

FIG. 3. Attaching an IFS to a 1D dynamical system: (a) The tent map, and (b) functions F_1 and F_2 of the corresponding IFS; (c) and (d) analogous pictures for the igloo (logistic) map; (e) and (f) analogous pictures for the hut map.

$=1/(\pi\sqrt{y(1-y)})$ for $y\in[0,1]$. Then $F_1(x)=(1-\sqrt{1-x})/2$, $F_2(x)=(1+\sqrt{1-x})/2$, and $p_1(x)=p_2(x)=1/2$ for $x\in[0,1]$ [Figs. 3(c) and 3(d)];

(c) *Hut map*: $f(y)=(-1+\sqrt{9-16|y-1/2|})/2$ for $y\in[0,1]$. The invariant density is given by $\rho(y)=y+1/2$ for $y\in[0,1]$. Then $F_1(x)=(x^2+x)/4$, $F_2(x)=1-((x^2+x)/4)$, $p_1(x)=(x^2+x+2)/8$, and $p_2(x)=(6-x^2-x)/8$ for $x\in[0,1]$ [Figs. 3(e) and 3(f)].

It is easy to show that all the required assumptions are satisfied here and we can use formulas (3.7) and (3.8) to calculate the entropy:

(a) *Tent map*: $K_q=\tilde{K}_q=(\ln(r^q+(1-r)^q))/(1-q)$ for $q\neq 1$, and $K_1=-(r\ln r+(1-r)\ln(1-r))$;

(b) *Igloo map*: $K_q=\tilde{K}_q=\ln 2$ for $q\in\mathbb{R}$;

(c) *Hut map*: $K_q=(\ln(4^{-q}(3^{q+1}-1)/(q+1)))/(1-q)$ for $q\neq 1$ and $K_1=1/2+2\ln 2-(9/8)\ln 3$.

Note that, for the hut map, one can hardly obtain such an analytical formula for the alternative version of Rényi entropy $\tilde{K}_q$.

Clearly, $K_0=\ln 2$ for each of the three maps. In the cases (a) and (c) $D_q=1$ for each q. The dependence of D_q on q in the case (b) is presented, e.g., in Ref. 3.

A similar technique can be applied to other classes of 1D maps like, for example, cookie-cutters introduced by Bohr and Rand in Refs. 51 and 52. Let us consider, e.g., a repeller given on the unit interval by $f(y)=3y$ for $y\in[0,2/3]$ and $f(y)=3y-2$ for $y\in[2/3,1]$, for which typical (with respect to the Lebesgue measure) trajectories eventually leave the interval with probability one. Then the measure ''uniformly'' localized on the Cantor set is the invariant measures for this system. The corresponding IFS given by $\{F_1(x)=x/3,\ F_2(x)=(x+2)/3\}$ for $x\in[0,1]$ with constant probabilities $p_1=p_2\equiv 1/2$ is just the IFS we discussed in Sec. V.

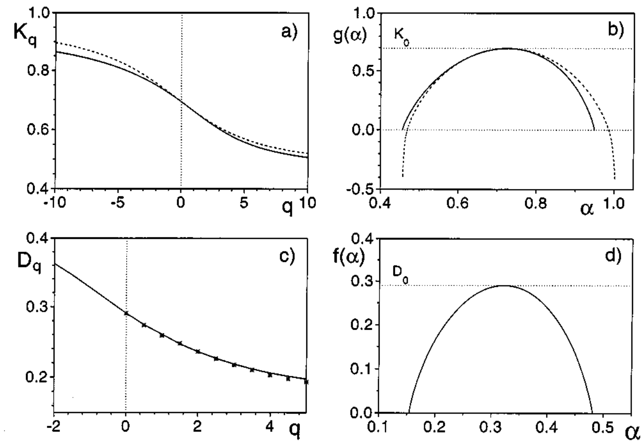

FIG. 4. Quantities characterizing the invariant measure of the IFS related to the quantum system: (a) Rényi entropies K_q (dashed line), $\tilde{K}_q$ (solid line); (b) scaling spectra $g(\alpha)$ (dashed line), $\tilde{g}(\alpha)$ (solid line); (c) fractal dimensions $\overline{D}_q$ (solid line), D_q (stars); (d) multifractal spectrum $f(\alpha)$.

B. Quantum systems

In papers[4,5] we introduced the notion of *coherent states (CS) quantum entropy*. This quantity may be used to characterize chaos in quantum dynamical systems. Out of entire spectrum of Rényi-type quantum entropies K_q,[6] a special meaning may be attached to K_1. Namely, the CS–entropy K_1 corresponds to the classical KS-entropy. The average of K_1 over the set of all structureless quantum systems, represented by unitary matrices distributed uniformly with respect to the Haar measure on $U(N)$, diverges with the matrix size N as $\ln(N)$.[7] This result provides an argument in favor of the ubiquity of chaos in classical mechanics (which corresponds to the limit $N\to\infty$).

The method of computing the CS–entropy based on the notion of IFS was proposed in Refs. 6 and 19 (but see also Ref. 56). Again we have shown that one can associate with a quantum system and a partition of the phase space an IFS with the same entropy.

Here we present an exemplary IFS obtained for the family of spin coherent states, the identity evolution operator, the quantum number $j=1/2$, and the partition of the phase (which is the two-dimensional sphere in this case) into two hemispheres (see Ref. 6 for details). For this IFS we have: $X=[-1,1]$, $F_1(x)=(-3+2x)/(6-3x)$, $F_2(x)=(3+2x)/(6+3x)$, $p_1(x)=1/2-x/4$, and $p_2(x)=1/2+x/4$ for $x\in X$. Large contraction coefficient characteristic for this IFS ensures fast convergence of integrals performed over the measures approximating corresponding invariant measure. It enables us to evaluate numerically the entropy with an enormous precision. For example, the entropy $K_1\approx 0.661\,314\,332\,711\,30$, being a quantum counterpart of the classical KS-entropy, is evaluated by the deterministic algorithm (2.4) with 14 significant digits. Such precision could be hardly obtained either with random iterated algorithm (2.5) or with standard techniques of time series analysis.[57,58] The quantities characterizing the invariant measure of the IFS: (a) Rényi entropies: K_q, $\tilde{K}_q$; (b) scaling spectra: $g(\alpha)$, $\tilde{g}(\alpha)$; (c) fractal dimensions: D_q, $\underline{D}_q$; and (d) multifractal spectrum $f(\alpha)$ are presented in Fig. 4. The common maximum of the scaling spectra gives the topological entropy $K_0 = \ln 2$, while the same curves intersects the bisectrix at the KS-entropy K_1. Fractal dimensions D_q are computed with the aid of the box-counting algorithm and compared with the quantities $\overline{D}_q$ defined in Sec. IV.

VII. CONCLUDING REMARKS

We have analyzed properties of IFSs with place-dependent probabilities and showed that their invariant measures often posses the *multifractal property*, i.e., the fractal dimension D_q depends substantially on the Rényi parameter q.

We have described a method of computing the generalized entropy for such IFSs by integrating the entropy function over their invariant measures. For numerical evaluation of the entropy one can apply the deterministic algorithm (useful for small number of functions) or random iterated algorithm (advantageous for large number of functions in IFS). Numerical calculations performed for generalized Cantor measures have shown superiority of both methods with respect to the standard method of computing entropy from time series generated by IFSs.[57,58] The entropy and the dimension of some IFSs of the second kind studied here display nontrivial scaling properties. The invariant measure for such an IFS may be thus neither uniform nor balanced.

It is possible to attach an IFS of the second kind to certain dynamical systems in such a way that the generalized entropies of their invariant measures are equal. This idea allows us to propose a new method of computing entropy for dynamical systems. In this work we demonstrated its usefulness for some classical (Rényi-type entropy of asymmetric tent map, logistic map, and hut map) and quantum (CS-measurement entropy for two hemispheres, $j = 1/2$) systems. The method of computing entropy by integration over fractal measure has been recently applied to other dynamical systems. The tent map with a gap, related to physical problem of communication with chaos, was studied in Ref. 59, while the fractal structure of an exemplary repelling system has been analyzed in Ref. 60. Moreover, we used a similar method to compute the dynamical entropy of some systems with stochastic perturbations.[61] This technique may be extended for a wider class of classical and quantum dynamical systems (or even for Markov chains). Such results will be presented in a forthcoming publication.[43]

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