

Convex set of quantum states with positive partial transpose analysed by hit and run algorithm

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Abstract

The convex set of quantum states of a composite $K \times K$ system with positive partial transpose is analysed. A version of the *hit and run* algorithm is used to generate a sequence of random points covering this set uniformly and an estimation for the convergence speed of the algorithm is derived. For $K \geq 3$ this algorithm works faster than sampling over the entire set of states and verifying whether the partial transpose is positive. The level density of the PPT states is shown to differ from the Marchenko–Pastur distribution, supported in $[0, 4]$ and corresponding asymptotically to the entire set of quantum states. Based on the shifted semi–circle law, describing asymptotic level density of partially transposed states, and on the level density for the Gaussian unitary ensemble with constraints for the spectrum we find an explicit form of the probability distribution supported in $[0, 3]$, which describes well the level density obtained numerically for PPT states.

Keywords: separable states, positive partial transpose, spectral density, Marchenko–Pastur distribution, gaussian unitary ensemble

(Some figures may appear in colour only in the online journal)

1. Introduction

The structure of the set of quantum states is a subject of a current interest from the perspective of pure mathematics [1] and theoretical physics [2–5]. A special attention is paid to the bipartite systems, in which separable and entangled states can be distinguished [3] as entanglement plays a key role in the theory of quantum information processing [6].

It is well known that any separable state of a bipartite, $K \times K$ system has positive partial transpose (PPT). In the case of a two-qubit system, $K = 2$, every PPT state is separable, while for higher systems it is not the case [7]. Description of the set of separable states for an arbitrary K is difficult, while it is straightforward for the set of PPT states.

Up till now, we are not aware of any dedicated algorithm to obtain a random PPT states with respect to the flat (Hilbert–Schmidt) measure. It is not difficult to generate a random density matrix from the entire set Ω_N of quantum states of size N using a matrix G of the Ginibre ensemble of complex square random matrices of order N , as the matrix $\rho = GG^\dagger / \text{Tr}GG^\dagger$ is positive and normalized and forms thus a legitimate random quantum state [8, 9]. However, as the relative volume of the subset of PPT states decreases exponentially with the dimension [10–12], is not efficient to generate random points in the entire set Ω_N of states of size $N = K^2$ and to check *a posteriori*, if the PPT property is satisfied.

The goal of this work is to analyse the set of PPT states by generating a sequence of random points from this set according to the flat measure. In this way we are in position to analyse the level density of random PPT states and to show the difference with respect of the Hilbert–Schmidt density, which corresponds to uniform sampling of the entire set of quantum states and asymptotically tends to the Marchenko–Pastur distribution. Note that analytical analysis of the set of PPT states is not an easy task, as this set does not enjoy the symmetry with respect to unitary transformations. However, based on the fact that the level density of partially transposed quantum states is asymptotically described by the shifted semi-circle law [13, 14] and making use of the level density of for the gaussian unitary ensemble (GUE) with constraints for the spectrum [15], we are in position to conjecture the level density which describes well numerical results obtained by sampling the set of PPT states.

A parallel aim of the paper is to introduce an effective algorithm of generating random points in a convex high dimensional set and to analyse its features. We propose a variant of the *hit and run* algorithm [16–19] and establish its speed of convergence. In each iteration of the algorithm one selects a random direction by choosing a random point from a hypersphere. In the next step one finds the longest interval belonging to the set which passes through the initial point and is parallel to this direction and jumps to a random point drawn from this interval.

Results obtained are easily applicable for all convex sets for which it is computationally efficient to find two points in which a given line enters and exists the set. Hence this approach, suitable for the set of PPT states, is not easy to apply for the set of separable states of higher dimensions, for which the problem of finding out whether a given state is separable is ‘hard’ [21].

This paper is organised as follows. In section 2 we present the algorithm and provide a general estimation of the convergence speed. Usage of the algorithm for simple bodies in \mathbb{R}^d is illustrated in section 3. In section 4 we discuss the set of quantum states of composite systems and its subset containing the states with positive partial transform. Making use of the algorithm we generate sequences of random points in this set and analyze numerical results for the level density for such states. Considerable deviations from the known distributions for the entire set of states are reported and it is conjectured that the asymptotic level density for the PPT states is described by the distribution the (4.6), different from the Marchenko–Pastur distribution [22].

2. Hit and run algorithm and its convergence speed

Consider a given convex, compact set $X \subset \mathbb{R}^d$. To generate a sequence of random points distributed according to the uniform Lebesgue measure in X we will use the following version of the ‘hit and run’ algorithm [17, 20].

- (1) Choose an arbitrary starting point $x_0 \in X$,
- (2) Draw randomly a unit vector $e \in S^{d-1}$ according to the uniform distribution on the hypersphere,
- (3) Find boundary points $x_1^{\min}(e), x_1^{\max}(e) \in \partial X$ along the direction e : there exist $a, b \geq 0$ such that $x_1^{\min}(e) = x_0 - ae$ and $x_1^{\max}(e) = x_0 + be$.
- (4) Select a point x_1 randomly with respect to the uniform measure in the interval $[x_1^{\min}, x_1^{\max}]$.
- (5) Repeat the steps (ii)–(iv) to find subsequent random points x_2, x_3, \dots

Note that each time the direction e is taken randomly and that the direction used in step $i + 1$ does not depend on the direction used in step i .

A first result of this work consists in the following estimation of the convergence speed in the total variation norm. The total variation norm of two probability measures is defined by the formula

$$\|\mu_1 - \mu_2\|_{\text{TV}} := \hat{\mu}^+(X) + \hat{\mu}^-(X),$$

where $\hat{\mu}^+ - \hat{\mu}^-$ is the Jordan decomposition of the signed measure $\mu_1 - \mu_2$.

Let Φ be a Markov chain [24] described by the above algorithm and let S be its transition kernel.

Theorem 2.1. *Let $X \subset \mathbb{R}^d$ be a compact convex set. Assume that there exist $r, R > 0$ such that for some $x_0 \in X$ one has $B(x_0, r) \subset X \subset B(x_0, R)$. Then the chain Φ satisfies the condition*

$$\|S^n(x, \cdot) - \mathcal{L}_d\|_{\text{TV}} \leq (1 - \theta)^n \quad \text{for any } x \in X, \tag{2.1}$$

where

$$\theta = (2/d)[(R/r + 1)^{d-1}(R/r)]^{-1}. \tag{2.2}$$

(Here \mathcal{L}_d denotes the d -dimensional Lebesgue measure.)

Proof. At the very beginning of the proof we show that the Lebesgue measure \mathcal{L}_d restricted to X is invariant for the considered algorithm. To do this observe that for any $x \in X \setminus \partial X$ and a Borel set $A \subset X$ we have

$$S(x, A) = \Upsilon \int_{S^{d-1}} \frac{1}{\|x^{\max}(e) - x^{\min}(e)\|} \int_{\mathbb{R}} \mathbf{1}_A(x + \lambda e) d\lambda \mathcal{L}_{d-1}(de),$$

where

$$\Upsilon = (\mathcal{L}_{d-1}(S^{d-1}))^{-1}.$$

Therefore, by the Fubini theorem we have

$$\begin{aligned}
 \int_X S(x, A) \mathcal{L}_d(dx) &= \int_X \Upsilon \int_{S^{d-1}} \frac{1}{\|x^{\max}(e) - x^{\min}(e)\|} \int_{\mathbb{R}} \mathbf{1}_A(x + \lambda e) d\lambda \mathcal{L}_{d-1}(de) \mathcal{L}_d(dx) \\
 &= \Upsilon \int_{S^{d-1}} \frac{1}{\|x^{\max}(e) - x^{\min}(e)\|} \int_{\mathbb{R}} \int_X \mathbf{1}_A(x + \lambda e) \mathcal{L}_d(dx) d\lambda \mathcal{L}_{d-1}(de) \\
 &= \Upsilon \int_{S^{d-1}} \frac{1}{\|x^{\max}(e) - x^{\min}(e)\|} \int_{\mathbb{R}} \int_X \mathbf{1}_{A-\lambda e}(x) \mathcal{L}_d(dx) d\lambda \mathcal{L}_{d-1}(de) \\
 &= \Upsilon \int_{S^{d-1}} \frac{1}{\|x^{\max}(e) - x^{\min}(e)\|} \int_{\mathbb{R}} \mathcal{L}_d(A - \lambda e) d\lambda \mathcal{L}_{d-1}(de) \\
 &= \mathcal{L}_d(A) \Upsilon \int_{S^{d-1}} \frac{1}{\|x^{\max}(e) - x^{\min}(e)\|} \int_{\mathbb{R}} d\lambda \mathcal{L}_{d-1}(de) = \mathcal{L}_d(A) \Upsilon \Upsilon^{-1} = \mathcal{L}_d(A),
 \end{aligned}$$

which finishes the proof that the Lebesgue measure \mathcal{L}_d is invariant.

We are going to evaluate $S(x, \cdot)$ for any $x \in X$. First assume that $X = B(x, R)$. It is easy to see that the transition function $S(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure \mathcal{L}_d . Its density is equal to

$$f_x(u) = (c_d \|u - x\|^{d-1} R)^{-1} \quad \text{for } u \in B(x, R) \setminus \{x\}, \tag{2.3}$$

where $c_d = 2\pi^{d/2}/\Gamma(d/2)$ denotes the surface of the d -sphere of radius 1. Indeed, by symmetry argument we see that $f_x(u) = \tilde{f}(\|u - x\|)$ for some function $\tilde{f} : [0, R] \rightarrow \mathbb{R}$. Then we have

$$\int_0^u \tilde{f}(v) c_d v^{d-1} dv = u/R \quad \text{for any } u \in [0, R].$$

Differentiating both sides with respect to u we obtain (2.3).

Now let X be arbitrary. Since $X \subset B(x_0, R)$ we see that for any $x \in X$ the transition function $S(x, \cdot)$ is absolutely continuous with respect to the Lebesgue measure \mathcal{L}_d and its density f_x satisfies

$$f_x(u) \geq (c_d \|u - x\|^{d-1} 2R)^{-1} \quad \text{for } u \in X.$$

Thus

$$\begin{aligned}
 S(x, \cdot) &\geq \int_{\cdot \cap B(x_0, r)} f_x(u) \mathcal{L}_d(du) \geq (c_d (R+r)^{d-1} 2R)^{-1} \mathcal{L}_d(B(x, r)) \nu(\cdot) \\
 &\geq (b_d/c_d) [(R/r + 1)^{d-1} (R/r)]^{-1} \nu(\cdot),
 \end{aligned}$$

where $\nu(\cdot) = \mathcal{L}_d(\cdot \cap B(x_0, r))/\mathcal{L}_d(B(x_0, r))$ and $b_d = \pi^{d/2}/\Gamma(d/2 + 1)$ denotes the volume of a unit ball in \mathbb{R}^d . Since $b_d/c_d = 2/d$, we finally obtain

$$S(x, \cdot) \geq (2/d) [(R/r + 1)^{d-1} (R/r)]^{-1} \nu(\cdot).$$

Now the version of Doeblin's theorem finishes the proof—see proposition 3.1 in [20] (see also [25]). □

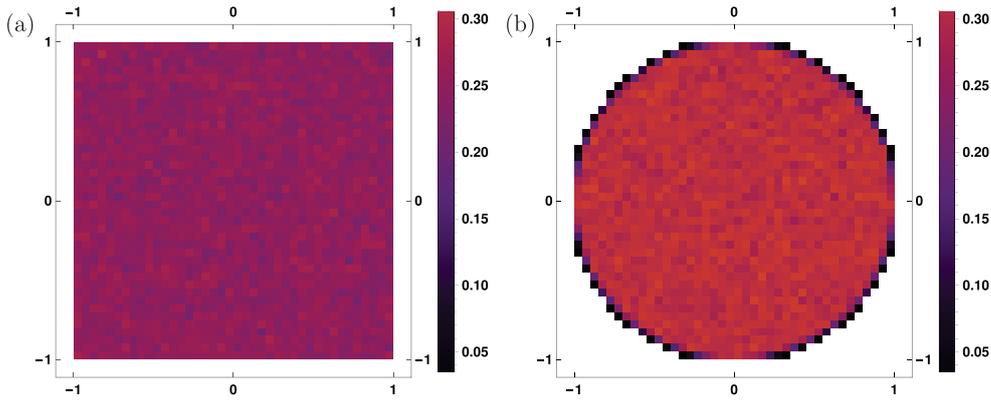


Figure 1. Histograms of sampled points in two simple 2-D cases: (a) square and (b) unit circle. For both plots the number of sampled points is 500 000, the bins are squares of side 0.05.

3. Balls, cubes and simplices in \mathbb{R}^d .

To show the convergence estimate (2.1) in action, in this section we apply it for simple bodies in \mathbb{R}^d .

For cubes and balls in \mathbb{R}^d it is straightforward to generate random points according to the uniform measure, so we will not advocate to use the above algorithm for this purpose. However, it is illuminating to compare estimations for the parameters determining the convergence rate according to equation (2.1).

For a unit **ball** B^d both radii coincide, $R = r$, so their ratio $\kappa = r/R$ is equal to unity. Estimation (2.2) gives $\theta = 2^{-d}/d$ where b_d denotes the volume of a unit d -ball. This implies the convergence rate

$$\alpha = 1 - \theta = 1 - 2^{-d}/d.$$

For an unit cube C^d the inscribed radius $r = 1/2$, and outscribed radius $R = \frac{1}{2}\sqrt{d}$ so the ratio reads $\kappa = r/R = 1/\sqrt{d}$. Hence the convergence rate reads,

$$\alpha = 1 - \theta = 1 - (2/d)[(\sqrt{d} + 1)^{d-1}\sqrt{d}]^{-1}.$$

To demonstrate usefulness of the algorithm we present in figure 1 the density of points it generated in a circle and a square.

The χ^2 test of the data obtained numerically does not suggests to reject the hypothesis that the data were generated according to the uniform distribution with a confidence level $p = 0.999$.

For an N -simplex Δ_N embedded in \mathbb{R}^d with $d = N - 1$ we have $R = \sqrt{(N-1)/N}$ and $r = 1/\sqrt{N(N-1)}$ so that the ratio $\kappa = 1/(N-1) = d^{-1}$. In this case the convergence rate reads,

$$\alpha = 1 - \theta = 1 - (2/d)[(d+1)^{d-1}d]^{-1}.$$

Note that the simplex Δ_N describes the set of classical states— N -point probability distributions.

4. Quantum states with positive partial transpose

Let Ω_N be the set of density matrices of size N , formed by complex Hermitian and positive operators, $\rho^\dagger = \rho \geq 0$, normalized by the trace condition $\text{Tr}\rho = 1$. Due to the normalization condition the dimension of the set is $d = N^2 - 1$.

The radius of the out-sphere, equal to the Hilbert–Schmidt distance between a pure state $\text{diag}(1, 0, \dots, 0)$ and the maximally mixed state $\rho_* = \mathbb{I}/N$, is $R = \sqrt{(N-1)/N}$. The radius of the inscribed sphere given by the distance between ρ_* and the center of a face, $\text{diag}(0, 1, \dots, 1)/(N-1)$ is equal to $r = 1/\sqrt{N(N-1)}$, so their ratio $\kappa = 1/(N-1) = 1/(\sqrt{d+1}-1) \sim d^{-1/2}$. Hence the convergence rate of the algorithm reads in this case

$$\alpha = 1 - \theta \sim 1 - (2/d)[(\sqrt{d+1})^{d-1}\sqrt{d}]^{-1}.$$

Consider now a composite dimension $N = K^2$, so the Hilbert space has a tensor product structure, $\mathcal{H}_N = \mathcal{H}_A \otimes \mathcal{H}_B$. In this case one defines a partial transposition, $\mathbf{T}_2 = \mathbb{I} \otimes \mathbf{T}$ and the set of PPT states (positive partial transpose) which satisfy $\rho^{\mathbf{T}_2} \geq 0$. This set forms a convex subset of Ω_N and it can be obtained as an intersection of Ω_N and its reflection $\mathbf{T}_2(\Omega_N)$ [5]. The set of PPT states can be decomposed into cones of the same height $r = 1/\sqrt{N(N-1)}$ [23], hence $\mu = 1/(N-1) \sim d^{-1/2}$.

In the case of set of PPT states the algorithm to generate random elements from this set according to the flat measure gives the same rate of convergence as in the case of quantum states, $\alpha = 1 - \theta \sim 1 - (2/d)[(\sqrt{d+1})^{d-1}\sqrt{d}]^{-1}$.

4.1. Two-qubit system

We start the discussion with the simplest composed system consisting of two qubits, setting $K = 2$ and $N = K^2 = 4$. To construct a random state from the entire 15 dimensional set Ω_4 we generated a random square Ginibre matrix G of order four with independent complex Gaussian entries and used the normalized Wishart matrix, writing $\rho = GG^\dagger/\text{Tr}GG^\dagger$. In this way one obtains random states generated according to the Hilbert–Schmidt measure in Ω_4 [8, 9].

Numerical data, obtained from 10^5 density matrices and represented in figure 2(a) by closed boxes, fit well to the analytical distribution $P_4(x)$ denoted in the plot by a solid curve. Here $x = N\lambda = 4\lambda$ denotes the rescaled eigenvalue of ρ . In general, for any finite N , the distribution $P_N(x)$ can be represented as a superposition of $2N$ polynomial terms,

$$P_N(x) = \sum_{m=2}^{2N} a_m (Nx)^{m-2} (1 - Nx)^{N^2-m} \quad (4.1)$$

with weights a_m given by the Euler gamma function [26]. Observe characteristic oscillations of the density and four maxima of the distribution related to the effect of level repulsion.

Given any bipartite quantum state ρ it is straightforward to diagonalize the partial transpose, $\rho^{\mathbf{T}_2} = (\mathbb{1} \otimes T)\rho$, to check, whether this matrix is positive. In the case of two qubits the relative volume of the set of states with positive partial transpose (PPT) with respect to the HS measure is conjectured to be equal to $8/33 \approx 0.242$ [27]. Thus verifying *a posteriori* the PPT property we could divide the random states into two classes and obtain histograms of the level density for the sets of states with positive or negative partial transpose (sets PPT and NPT, respectively)—see figure 2(a).

Furthermore, we studied the level density of the partially transposed operators $\rho^{\mathbf{T}_2}$ with eigenvalues χ . For the states sampled from the entire set Ω_4 , some partially transposed

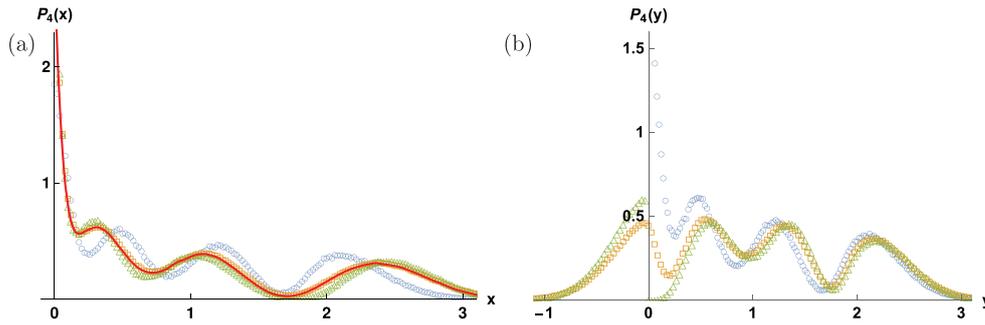


Figure 2. (a)—Spectral density $P(x)$ for two-qubit density matrices ρ where $x = 4\lambda$: numerical data representing sampling 10^5 times over entire set Ω_4 (light/yellow \square) compared with analytical distribution (solid, red curve). This distribution differs from level density corresponding to the set of PPT states (blue \circ), and the set of states with negative partial transpose states (green \triangle). Panel (b) shows analogous data for partially transposed states, ρ^{T_2} : (light/yellow \square) denotes data for entire set Ω_4 , (blue \circ) denotes transposed PPT states, while (green \triangle) denotes transposed states with negative partial transpose.

matrices are not positive, so the level density is supported also for negative values of the variable $y = 4\chi$ —see figure 2(a). It is known that for 2×2 systems only a single eigenvalue ρ^{T_2} can be negative and it is not smaller than $-1/2$ [28], so one sees a single peak of the density $P(y)$ containing the negative eigenvalues. For large systems the asymptotic density for the transposed states is known to converge to the shifted semicircle [13, 14], but no analytical expression for such a density in the case of $K = 2$ is known so far.

Figure 2(b) presents also histogram for the transposed states belonging to both classes of PPT and NPT states. Note that for the set of PPT states the level density $P(x)$ for the spectrum of ρ and $P(y)$ for the spectrum of ρ^{T_2} , are the same. This is a consequence of the fact that the partial transpose is a volume preserving involution, so the level density will not be altered, if statistics is restricted to PPT states only. Observe also that the spectrum corresponding to the PPT states of size $N = 4$ do not display a singularity at the origin.

4.2. Larger systems

In the asymptotic case of large systems, $N \gg 1$, the oscillations of the mean level density averaged over the entire set Ω_N of quantum states are smeared out and the distribution (4.1) tends to the Marchenko–Pastur distribution [22],

$$\text{MP}(x) = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}}, \quad (4.2)$$

where the rescaled eigenvalue $x = N\lambda$ belongs to the interval $[0, 4]$. Although there exists pure quantum states with the operator norm equal to unity, $\|\rho\| = 1$, a norm of a generic large dimension N mixed state is asymptotically almost surely bounded from above by $4/N$, since the support of the above distribution is bounded by $x_{\max} = 4$. Due to the effect of concentration of measure in high dimensions a typical quantum state has spectral distribution close to the Marchenko–Pastur law (4.2), while states with different level densities become unlikely [29].

To demonstrate the quality of the hit and run algorithm, we used it to generate samples of random states for the entire convex set Ω_N . Diagonalising the generated states we obtained

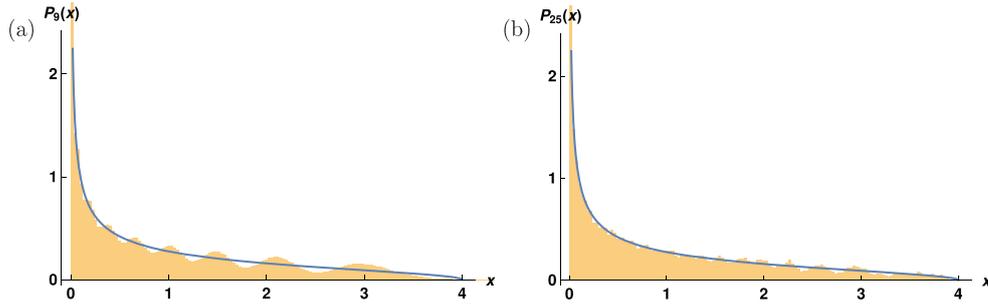


Figure 3. Spectral densities $P_N(x)$ averaged over entire set Ω_N of quantum states of size: (a) $N = 9$ and (b) $N = 25$. Spectral density (4.2) denoted by a solid (blue) line describes well the data already for $N = 25$.

their eigenvalues λ , and analyzed the level density $P(x)$ with $x = N\lambda$. Histograms obtained in this way for $N = 9$ and $N = 25$ coincide with the analytical expression (4.1)—see figure 3. Observe that the data for $N = 25$ are already close to the Marchenko–Pastur distribution valid asymptotically. Although in this case the random points are generated from the set Ω_{25} of dimension $d = N^2 - 1 = 624$, the average over a sample consisting of circa 10^5 random points provides reliable data.

Consider now a bipartite case of size $N = K \times K$, for which the notion of partial transpose is defined. It is was first shown by Aubrun [13] that level density of the partially transposed bipartite random states, written $\rho^{T^2} = (\mathbb{I} \otimes T)\rho$, asymptotically tends to the shifted semicircle law of Wigner,

$$P^T(x) \sim \frac{1}{2\pi} \sqrt{4 - (x - 1)^2}, \quad (4.3)$$

with support $x \in [-1, 3]$.

Up to a linear shift, $x \rightarrow x - 1$, this distribution is that of the ensemble of random Hermitian matrices of GUE—see e.g. [30].

An extended model of random GUE matrices with spectrum of the rescaled eigenvalue restricted to a certain interval $[0, L(z)]$ was analyzed by Dean and Majumdar [15]. They derived an explicit family of probability distributions, labeled by a parameter z , which determines the position of the ‘hard wall barrier’ for the corresponding Coulomb gas model,

$$h_z(y) = \frac{\sqrt{L(z) - y}}{2\pi\sqrt{y}} (L(z) + 2y + 2z), \quad (4.4)$$

where the upper edge $L(z)$ of the support of $h_z(y)$ reads

$$L(z) = \frac{2}{3} \left(\sqrt{z^2 + 6} - z \right). \quad (4.5)$$

Since partial transpose is a volume preserving involution, the level density averaged over the set of PPT states can be approximated by the density of states obtained from the shifted semicircle law of Aubrun (4.3), by imposing the restriction that all eigenvalues are non-negative. The approximation consists in an assumption that the ensemble of partially transposed density matrices asymptotically coincides with the shifted GUE.

Taking distribution (4.4), setting $z = 0$ and normalizing the rescaled variable $x = ay + b$ so that the expectation value is set to unity as required, $\langle x \rangle = \int xg(x) = 1$, we arrive at the normalized probability distribution

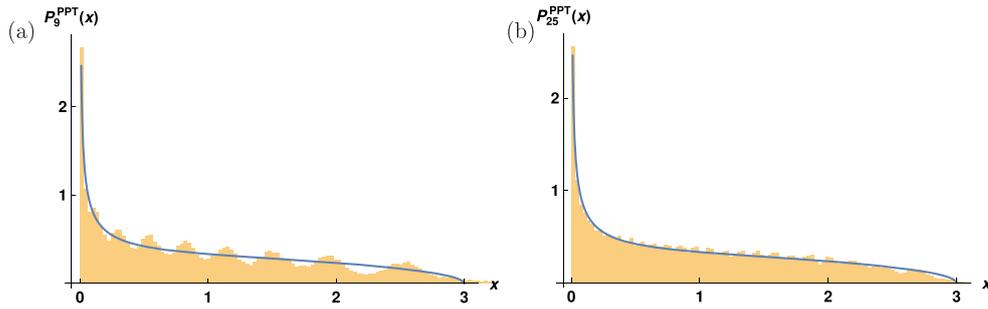


Figure 4. Spectral densities $P_N^{\text{PPT}}(x)$ for PPT states of bipartite systems of size: (a) $N = 3 \times 3 = 9$ and (b) $N = 5 \times 5 = 25$. Spectral density (equation (4.6)) denoted by a solid (blue) line describes well the data for $N = 25$.

$$g(x) = \frac{4}{27\pi} \sqrt{\frac{3-x}{x}} (3+2x), \tag{4.6}$$

supported in $[0, 3]$.

To demonstrate that this distribution can describe asymptotic level density of PPT states we need to rely on numerical computations. In the case of a larger bi-partite system of size $N = K^2$ the probability of finding a random PPT state decays exponentially with the dimension [10–12]. Therefore the simplest approach to get a PPT state by generating random states from the entire set Ω_N and verifying, whether its partial transpose is positive becomes ineffective.

However, the ‘hit and run’ algorithm, advocated in this work, is still suitable for the case of the set of PPT states. For any two points inside the set it is easy to find where the line joining them hits the boundary, provided the dimensionality $N = K^2$ is low enough. We have found the numerical procedure to be stable for $K \leq 5$. Level density for the subset of PPT states for bipartite $K \times K$ system is shown in figure 4. The number of random points generated, equal to 5×10^4 for $N = 9$ and 4.4×10^5 for $N = 25$, was found to give reliable results. Characteristic finite-size oscillations visible for $N = 9$ become less pronounced for $N = 25$. As for this dimension the exact expression (4.1) is already close to the Marchenko–Pastur distribution, our data obtained for the 5×5 system support the conjecture that the level density for the PPT states is asymptotically described by the distribution (4.6).

Observe that the right edge of the support of $P_N^{\text{PPT}}(x)$ is smaller than the upper bound for the MP distribution, $x_{\max} = 4$. Hence our numerical results support the conjecture that the operator norm of the PPT states is asymptotically almost surely smaller than the norm of a generic state taken from the entire set Ω_N , which typically behaves as $\|\rho\| \approx_{\text{a.s.}} 4/N$.

In view of these numerical results, it is natural to conjecture that as $N \rightarrow \infty$, the spectral distribution of ρ_{PPT} is different from the distribution obtained from a uniform sampling on all states. To prove such a statement one could show that there exists $\varepsilon > 0$ such that $\|\rho_{\text{PPT}}\| \leq_{\text{a.s.}} C_{\text{PPT}}/N \leq (4 - \varepsilon)/N$, as this would result in differentiating mathematically the uniform distribution on all states from the uniform distribution on PPT states. However, despite our efforts, we were not able to prove this result and we leave it as an open question.

It is slightly easier to deal with these sets of quantum states, which are invariant with respect to unitary transformations. There is a natural notion of APPT (absolute positive partial transpose), which means that a state satisfies the PPT property regardless of its (global) unitary evolution. In the case of 2×2 systems this property is equivalent to absolute separability [3], i.e. separability with respect to any choice of a two dimensional subspace embedded in \mathcal{H}_4 , which defines both subsystems. Hence the property of APPT of a given state cannot

depend on its eigenvectors but is only a function of its eigenvalues [31]. This feature holds also for higher dimensions and in some cases the boundary of this set are known [32, 33]. Interestingly, although for higher dimensions the set of PPT states has much larger volume than the set of separable states [34], it is conjectured that the set of absolutely separable states and APPT states do coincide [35]. Furthermore, proposition 8.2 from [36] concerning the largest eigenvalue of a quantum state belonging to the set of APPT states implies that the support of the level density for the states of this set is asymptotically bounded by $x = 3$. Observe that this fact is consistent with our observations concerning the generic behavior of the norm of a PPT state.

Note that the above questions of separating distributions through their typical largest eigenvalues can be partly addressed in the more general context of sampling from a uniform purification. It is known [8, 9] that if the ancilla space has the same dimension as the state space, this sampling method gives the uniform distribution. It is also interesting to study other regimes, where the ancilla space is bigger than the state space. Aubrun proved in [13] that the PPT property holds with probability 1 as $N \rightarrow \infty$ if, for $\varepsilon > 0$ the ancilla space is of dimension at least $(4 + \varepsilon)N$. Intuitively, a bigger ancilla space means that the sampling is more concentrated around the maximally mixed state. For even larger ancillas the generated states belong to the set of APPT states, as it was showed in [37] that the threshold is $(4 + \varepsilon)N^2$ and that this is essentially optimal. Unsurprisingly, this shows that the set of APPT states is much smaller than the set of PPT states. But the problem of comparing these sampling probabilities with the uniform PPT distribution or the uniform APPT distribution remains difficult to achieve formally.

5. Concluding remarks

In this paper we analysed a universal algorithm to generate random points inside an arbitrary compact set X in \mathbb{R}^d according to the uniform measure. Any initial probability measure μ transformed by the corresponding Markov operator converges exponentially to the invariant measure μ_* , uniform in X . Explicit estimations for the convergence rate are derived in terms of the ratio $\kappa = r/R$ between the radii of the sphere inscribed inside X and the sphere circumscribed on it.

Thus the algorithm presented here can be used in practice to generate, for instance, a sample of random quantum states. In the case of states of a composed quantum system, one can also generate a sequence of random states with positive partial transpose. Sampling random states satisfying a given condition and analyzing their statistical properties is relevant in the research on quantum entanglement and correlations in multi-partite quantum systems. A standard approach of generating random points from the entire set of quantum states with respect to the flat measure [8] and checking *a posteriori*, whether the partial transpose of the state constructed is positive, becomes inefficient for large dimensions, as the relative volume of the set of PPT states becomes exponentially small [10].

Obtained numerical results show that the level density for random states covering uniformly the subset of the PPT states differs considerably from the Hilbert-Schmidt level density corresponding to the entire set Ω_{K^2} of mixed states of a bipartite system. Making use of the shifted semicircle law of Aubrun (4.3), which describes the asymptotic density of partially transposed states, and imposing the restriction that all the eigenvalues are non-negative we arrived at the probability distribution (4.6). This distribution was compared with the level density obtained numerically with the ‘hit and run’ algorithm applied for the set of PPT states for the $K \times K$ system for $K = 3, 4, 5$. The larger dimension the better agreement of the numerical data with the distribution (4.6), so is tempting to conjecture that it describes the level density of the PPT states in the asymptotic limit.

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