Collectibility for mixed quantum states

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Bounds analogous to entropic uncertainty relations allow one to design practical tests to detect quantum entanglement by a collective measurement performed on several copies of the state analyzed. This approach, initially worked out for pure states only [Phys. Rev. Lett. **107**, 150502 (2011)], is extended here for mixed quantum states. We define collectibility for any mixed states of a multipartite system. Deriving bounds for collectibility for positive partially transposed states of given purity provides insight into the structure of entangled quantum states. In the case of two qubits the application of complementary measurements and coincidence based detections leads to a test of entanglement of pseudopure states.

I. INTRODUCTION

More than two decades ago the notion of the entanglement for mixed quantum states of a composite system was worked out by Werner [1]. Since then a lot of work has been done to develop efficient separability criteria and to design useful measures of quantum entanglement. Although several possible quantities have been proposed and analyzed [2–5] there is still a need for an efficient measure of quantum entanglement which could be accessible in a real-life experiments [6–8].

To get full information about the analyzed quantum state one can perform the scheme of quantum tomography, which allows one to determine the degree of quantum entanglement [9]. However, the full scheme of quantum tomography requires a large number of measurements; thus, it becomes impractical for higher-dimensional systems. Therefore, one can ask how to get the maximum information on the degree of entanglement of a given state by performing relatively few measurements.

Some progress in this direction was achieved in Ref. [10], in which a quantity based on collective measurements performed on several copies of the system investigated was proposed. This quantity, called *collectibility*, was defined for any pure state of a general composite quantum system containing an arbitrary number K of subsystems, each describing an N-level system. By deriving inequalities analogous to entropic uncertainty relations [11,12], we established separability criteria based on collectibility. To detect entanglement in the simplest two-qubit system we proposed a four-photon experiment [10] based on Hong-Ou-Mandel interference [13].

Entanglement criteria based on pure-state collectibility are reviewed in Sec. II. The main goal of the present paper is to generalize the notion of collectibility and the related separability criteria to the general case of mixed quantum states. To this end we shall propose two strategies.

First, to generalize the experimental part we modify the pure-state separability criteria to take into account contributions related to impurities of both copies of the investigated

state. We thus propose an entanglement test for "pseudopure" states that employs the minimal number of observables required for a scheme based on a single Hong-Ou-Mandel interferometer. In general, this test seems to be useful for quick but demanding tests of high-quality sources. In addition, it reports possible asymmetry in the character of the noise. In Sec. III these modified entanglement criteria are presented and the modified experimental setup is described. An extended analysis of the above results is presented in Sec. VI.

In the second strategy, we generalize the definition of pure-state collectibility and the corresponding entanglement criteria to the case of an arbitrary mixed state of a system consisting of K subsystems, each supported on N levels. This is done in Sec. IV. In Sec. V we investigate in more detail the bipartite case. Derivations of some formulas and proofs of certain lemmas are relegated to the Appendix.

II. PURE-STATE COLLECTIBILITY

A. Maximal collectibility for multipartite pure states

An entanglement test for pure states of K-quNit systems based on uncertainty relations was proposed in Ref. [10]. Consider a general case of a *K-partite Hilbert space* $\mathcal{H} = \mathcal{H}^A \otimes \mathcal{H}^B \cdots \otimes \mathcal{H}^K$ and for simplicity assume that $\dim(\mathcal{H}^A) = \cdots = \dim(\mathcal{H}^K) = N$. In a first step we chose a set of N separable pure states, $|\chi_j^{\text{sep}}\rangle = |a_j^A\rangle \otimes \cdots \otimes |a_j^K\rangle$, where $|a_j^I\rangle \in \mathcal{H}^I$ and both indices run $j = 1, \ldots, N$ and $I = A, \ldots, K$. A crucial property of the states $|a_j^I\rangle$ is that they are mutually orthogonal in each subspace, so that

$$\left|a_1^I\rangle, \ldots, |a_N^I\rangle \in \mathcal{H}^I, \quad \langle a_j^I | a_k^I \rangle = \delta_{jk}. \tag{1}$$

For a pure state of a composite system, $|\Psi\rangle \in \mathcal{H}$, obeying the normalization condition $\langle \Psi | \Psi \rangle = 1$, we introduce the following definition:

Definition 1. Maximal collectibility [10] of a pure state $|\Psi\rangle$ is

$$Y^{\max}[|\Psi\rangle] = \max_{|\chi^{\text{sep}}\rangle} \prod_{j=1}^{N} \left| \langle \Psi | \chi_j^{\text{sep}} \rangle \right|^2. \tag{2}$$

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To emphasize the fact that the above definition is valid for pure states only we shall also call $Y^{\max}[|\Psi\rangle]$ the *pure-state collectibility*. The maximum inside formula (2) is necessary to assure an invariance with respect to local unitary operations and shall be taken over the set of N locally mutually orthogonal pure states $|\chi^{\text{sep}}\rangle = \{|\chi_1^{\text{sep}}\rangle, \dots, |\chi_N^{\text{sep}}\rangle\}$.

B. Entanglement criteria based on pure-state collectibility

In Ref. [10] we have shown that the pure-state collectibility can serve as a simple entanglement test. In particular, we proved the existence of two upper bounds for $Y^{\max}[|\Psi\rangle]$. The first one,

$$Y^{\max}[|\Psi\rangle] \leqslant N^{-N}, \tag{3}$$

is valid for all states $|\Psi\rangle \in \mathcal{H}$, while the second one,

$$Y^{\max}[|\Psi_{\text{sep}}\rangle] \leqslant N^{-NK}, \tag{4}$$

must be satisfied if the state $|\Psi\rangle$ is a separable state $|\Psi_{\text{sep}}\rangle = |\Psi_A\rangle \otimes \cdots \otimes |\Psi_K\rangle$. Since the second bound is much sharper than Eq. (3) we obtain the following separability criteria based on the pure-state collectibility [10]:

$$(Y^{\max}[|\Psi\rangle] > \alpha_{K,N}) \Rightarrow (|\Psi\rangle\text{—entangled}), \tag{5}$$

where $\alpha_{K,N} = N^{-NK}$ plays the role of a discrimination parameter.

C. Experimentally accessible criteria for pure state of many qubits

In the case of K qubits we are able to perform analytically a first step of the maximization procedure; that is, to maximize over a pair of two vectors $|a_1^A\rangle, |a_2^A\rangle \in \mathcal{H}^A$ belonging to the first Hilbert subspace. We obtain an expression for the *collectibility* of a pure state [10]:

$$Y_a[|\Psi\rangle] = \max_{|a_1^A\rangle, |a_2^A\rangle} \prod_{j=1}^{N} |\langle\Psi|\chi_j^{\text{sep}}\rangle|^2$$
$$= \frac{1}{4}(\sqrt{G_{11}G_{22}} + \sqrt{G_{11}G_{22} - |G_{12}|^2})^2, \tag{6}$$

where the coefficients $G_{jk} = \langle\varphi_j|\varphi_k\rangle$ are elements of the Gram matrix for two ($j = 1,2$) vectors $|\varphi_j\rangle = (\langle a_j^B|| \otimes \cdots \otimes \langle a_j^K|)|\Psi\rangle \in \mathcal{H}^A$. The vectors $|\varphi_j\rangle$ represent the state $|\Psi\rangle$ projected onto two orthogonal, separable states $|a_j^B\rangle \otimes \cdots \otimes |a_j^K\rangle$ of $K-1$ qubits.

A particular case of the separability criterion (5) for the K-qubit system reads

$$(Y_a[|\Psi\rangle] > \alpha_{K,2}) \Rightarrow (|\Psi\rangle\text{— entangled}). \tag{7}$$

An important advantage of the above separability criterion based on collectibility is its possible experimental implementation. In Ref. [10] we proposed an experiment based on Hong-Ou-Mandel (H-O-M) [13] interferometry, in which all coefficients G_{jk} can be measured if two copies of a two-qubit pure state are given. If $Y_a[|\Psi\rangle_{AB}] > 1/16$ then the two-qubit state $|\Psi\rangle_{AB}$ is entangled. In the following sections of this work we generalize this approach for the mixed states of a multipartite system.
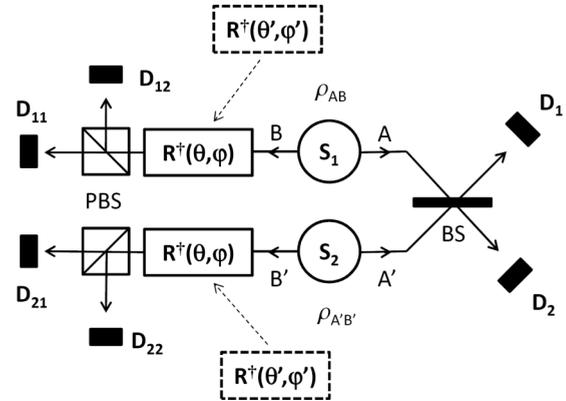

FIG. 1. Determination of Gram matrix via conditional overlapping in the case of two polarization-entangled photon pairs. Each source produces a pair of photons in a general mixed-polarization state ρ_{AB}. The statistics of pairs of clicks is collected on the left-hand side after two elements are measured. On the right side the H-O-M interference is performed. The square of the Gram matrix element equals the probability of the pair of clicks multiplied by that of the double click on the right. The extra boxes with the primed arguments of the rotations $R(\theta', \phi')$ represent the measurements in a basis complementary to the original basis associated with the rotations $R(\theta, \phi)$ (see main text). The Hermitian conjugate at the rotations correspond to the dualism of Schrödinger and Heisenberg pictures.

III. TWO QUBITS IN A MIXED STATE—PSEUDOPURE ENTANGLEMENT TEST BY CONTROLLING REMOTE PURITY

The starting point of our considerations shall be the original experimental setup [10] with the pure state $|\Psi\rangle_{AB}$ substituted by the mixed state ρ_{AB} shared by Alice and Bob. Apart from two sources of the same copy of the state it involves the $50:50$ beam splitter (BS), two polarization rotators $R^\dagger(\theta, \phi)$ in the same setting, and the polarized beam splitters (PBS) (see Fig. 1). Let us define $p_{ij}(+,+)$ as the probability of double click after the beam splitter. We also denote by $p_{1i} \equiv p_1((-1)^{i+1})$ [$p_{2i} \equiv p_2((-1)^{i+1})$] the probability of click in detector $D_{1,i}$ (detector $D_{2,i}$); that is, one of the detectors located after the upper PBS (lower PBS). The Gram matrix elements result in

$$|G_{ij}|^2 = p_{1i}p_{2j}[1 - 2p_{ij}(+,+)]. \tag{8}$$

Note now that whenever Alice gets the two results in the same index (which corresponds to the probabilities p_{1i}, p_{2i} for $i = 1,2$) then she remotely produces at Bob's site a pair of the same state; say σ_+—with probability $p_{11} = p_{21} \equiv p_+$—or σ_- with probability $p_{12} = p_{22} \equiv p_-$, respectively, which is further subject to H-O-M interference. However, when she gets the results with a different second index; namely, either $p_{11} \equiv p_+$, $p_{22} \equiv p_-$ or $p_{12} \equiv p_-$, $p_{21} \equiv p_+$ then she produces a pair of different states (i.e., σ_+, σ_- or σ_-, σ_+) which come into the H-O-M interferometer on the right-hand side. This means that finally we have the following relationship between the measurable quantities ($G_{\pm\pm}$ and $G_{\pm\mp}$) and the mathematical quantities (G_{ij}, $i,j = 1,2$): $G_{11} \equiv G_{++}$,

$|G_{12}| \equiv |G_{+-}|$, $|G_{21}| \equiv |G_{-+}|$, and $G_{22} \equiv G_{--}$, where

$$G_{++} = p_+\sqrt{\mathrm{Tr}(\sigma_+^2)}, \quad G_{--} = p_-\sqrt{\mathrm{Tr}(\sigma_-^2)}, \quad (9a)$$

and

$$|G_{+-}|^2 = p_+ p_- \mathrm{Tr}(\sigma_+\sigma_-) = |G_{-+}|^2. \quad (9b)$$

Some other quantities will also be used in our formulas:

$$\begin{aligned} \mathrm{Tr}(\sigma_+\sigma_-) &= 1 - 2p_{12}(+,+), \\ \mathrm{Tr}(\sigma_+^2) &= 1 - 2p_{11}(+,+), \\ \mathrm{Tr}(\sigma_-^2) &= 1 - 2p_{22}(+,+). \end{aligned} \quad (10)$$

We may expect that one has $p_{12}(+,+) = p_{21}(+,+)$ up to the measurement accuracy since the two copies of state ρ_{AB} are assumed to be the same.

A. Modified version of experiment involving two complementary observables

Consider the case in which Alice and Bob share a two-qubit state ρ_{AB} and Alice performs measurements of *two* complementary binary observables $\hat{n}\vec{\sigma}$ and $\hat{n}'\vec{\sigma}'$. Below we shall refer to the axes $\hat{x}$ and $\hat{z}$ but it is only to simplify the derivation, which is in fact invariant under the choice of the orthogonal pair $\hat{n}$ and $\hat{n}'$. Under each of the four results Alice produces remotely one of the four states on Bob's side: $\sigma_\pm$ (when she gets the result $\pm$ while measuring $\hat{n}$) or $\sigma'_\pm$ (when she gets the result $\pm$ while measuring $\hat{n}'$).

The experimental setup shown in Fig. 1 is almost the same as the one designed for the original inequality $Y_a[|\Psi\rangle_{AB}] > 1/16$. The only difference is that while, in the original scheme one performs a single measurement in both arms [determined by some specific choice of the unitary rotation $R(\theta,\phi)$], here we perform the same experiment for two selected rotations $R(\theta,\phi)$ and $R(\theta',\phi')$ labeled by a pair of angles defining two orthonormal versors $\hat{n}$ and $\hat{n}'$.

We measure the degree of purity of $\sigma_\pm$ and $\sigma'_\pm$ [which is directly measurable according to Eq. (10)] by four parameters $\epsilon_\pm$, $\epsilon'_\pm$ as follows:

$$\mathrm{Tr}(\sigma_\pm^2) \geqslant 1 - \epsilon_\pm, \quad (11a)$$

$$\mathrm{Tr}[(\sigma'_\pm)^2] \geqslant 1 - \epsilon'_\pm. \quad (11b)$$

Moreover, we have other parameters which are the probabilities that Alice produces remotely the states $\sigma_\pm$ ($\sigma'_\pm$) which we have denoted by $p_\pm$ ($p'_\pm$). The final parameters we need are the overlaps between the "σ" states, which are easily measurable as one can see from Eq. (10). The probabilities, purity parameters, and the overlaps are the only data from the experiment. While finally in practice we put equalities in the formulas (11a) and (11b) we keep inequalities to stress that this may take into account all experimental statistical errors resulting from data analysis. The test works well provided all four states produced remotely at Bob's site are characterized by a high purity, which resembles the ideal situation. If the initial state ρ_{AB} was a pure state, then all four states $\sigma_\pm$, $\sigma'_\pm$ would be completely pure. That is why we refer to the test as *pseudopure* but it is fully general (i.e., we do not need to have any additional assumption on the structure of the state). In order to generalize

the entanglement criteria $Y_a[|\Psi\rangle_{AB}] > 1/16$ to the case of the two-qubit mixed state ρ_{AB} we establish the following theorem:

Theorem 1. Any separable two-qubit state ρ_{AB} satisfies the following inequality:

$$G_{++}G_{--} - |G_{+-}|^2 \leqslant \frac{\eta + (G_{++} + G_{--})^2 - 1}{2}, \quad (12)$$

where

$$\eta \equiv \eta(\epsilon_+\epsilon_-; p_+p_-; \epsilon') = 8p_+ p_- \sqrt{\epsilon_+\epsilon_-} + 2\epsilon' \leqslant 2, \quad (13)$$

and $\epsilon' = \max[\epsilon'_+, \epsilon'_-]$. The proof of the above theorem together with an extended discussion of the inequality (12) can be found in Sec. VI. As a weaker consequence of Theorem 1 we obtain the separability criteria in terms of the collectibility Y_a and the parameter η given in Eq. (13):

$$\left[Y_a[\rho_{AB}] > \frac{1}{16} + \frac{1}{4}\left(\frac{\eta}{2} + \sqrt{\frac{\eta}{2}}\right) \right] \Rightarrow (\rho_{AB} \text{ —entangled}). \quad (14)$$

Note that we may rewrite the formula (12) as a nonlinear entanglement witness value

$$\mathcal{W}(\rho) = \tfrac{1}{2}(\eta + G_{++}^2 + G_{--}^2 + 2|G_{+-}|^2 - 1) \geqslant 0. \quad (15)$$

We have checked the above test for the Werner states with admixture of uniform noise with probability p:

$$\rho_{AB}(p) = (1 - p)|\Psi_+\rangle\langle\Psi_+| + p\frac{I \otimes I}{4}. \quad (16)$$

It is known that the threshold for separability is $p = 2/3 = 0.(6)$, while the scheme with two Hong-Ou-Mandel interferometers [14] (which realizes the purity-entropy separability test [15]) gives the threshold as $p = 1 - 1/\sqrt{3} \approx 0.4226$. The pseudomixture scheme proposed here provides a smaller value $p = 1 - \sqrt{3}/2 \approx 0.1180$. This implies that the pseudomixed test is essentially dedicated to high-quality sources (i.e., small perturbations of purity). It allows one to test them faster than in Ref. [14] (provided that the source is good enough) because it requires two settings on Alice's side and the interference on Bob's side as opposed to the usual tests where two Hong-Ou-Mandel interferences are used. In a sense it offers a balanced compromise between the number of observables measured and the number of interferences performed.

IV. MIXED-STATE COLLECTIBILITY FOR MULTIPARTITE SYSTEMS

A first natural effort to generalize the definition of the pure-state collectibility could be to rewrite the right-hand side of the definition (2) in the following form:

$$\max_{|\chi^{\mathrm{sep}}\rangle} \prod_{j=1}^{N} \langle\chi_j^{\mathrm{sep}}|\Psi\rangle\langle\Psi|\chi_j^{\mathrm{sep}}\rangle. \quad (17)$$

One could suspect that a simple modification $|\Psi\rangle\langle\Psi| \mapsto \rho$ would make the above formula suitable for an arbitrary mixed state ρ. However, we require the *mixed-state collectibility* to be capable of identifying entanglement. On the other hand, the separable maximally mixed state of rank N can be represented

as a sum of N mutually orthogonal separable states $|\chi^{\mathrm{sep}}\rangle$,

$$\rho_\chi = \frac{1}{N} \sum_{k=1}^N |\chi_k^{\mathrm{sep}}\rangle\langle\chi_k^{\mathrm{sep}}|. \tag{18}$$

This implies that formula (17) with $|\Psi\rangle\langle\Psi|$ replaced by ρ_χ immediately gives the value N^{-N}, which is the upper bound (3). The same bound is attained by any pure, maximally entangled state. The above observation shows that the quantity based on expression (17) does not allow us to distinguish between the maximally entangled pure state and the maximally mixed separable state.

In order to remove this ambiguity we introduce the following definition:

Definition 2. The mixed-state collectibility of a mixed state ρ is

$$Y^{\max}[\rho] = \left(\max_{|\chi^{\mathrm{sep}}\rangle} \prod_{j,k=1}^N \langle\chi_j^{\mathrm{sep}}|\rho|\chi_k^{\mathrm{sep}}\rangle \right)^{1/N}. \tag{19}$$

Since we also take into account the *off-diagonal* terms of the density matrix, we necessarily obtain for the maximally mixed state $Y^{\max}[\rho_\chi] = 0$, because the density matrix ρ_χ is diagonal.

Note that, for the mixed-state collectibility, we use the same symbol $Y^{\max}[\cdot]$ as for the pure-state collectibility, thus only the argument ($|\Psi\rangle$ or ρ) allows one to distinguish the difference. In fact, we can use the same symbol for both quantities, since the Nth root in the definition (19) assures that $Y^{\max}[|\Psi\rangle\langle\Psi|] = Y^{\max}[|\Psi\rangle]$. The mixed-state collectibility (19) calculated for a rank-one density operator $\rho = |\Psi\rangle\langle\Psi|$ is equal to the earlier-defined pure-state collectibility (2), which makes both definitions consistent.

A. Criteria based on mixed-state collectibility

After we have defined the mixed-state collectibility we would like to generalize the upper bounds (3) and (4). To this end let us consider a $D \times D$ Hermitian, positive semidefinite matrix $\rho \geqslant 0$, $\rho = \rho^\dagger$ with fixed trace $\mathrm{Tr}\rho = \mathrm{const.}$ and purity $\mathrm{Tr}(\rho^2) = \mathrm{const.}$ Let us denote by ρ_{ij}, $i,j = 1,\dots,D$ the matrix elements of ρ. We shall establish the following lemma:

Lemma 1. For every $N \leqslant D$ we have

$$\left(\prod_{i,j=1}^N \rho_{ij} \right)^{1/N} \leqslant r_N(D, \mathrm{Tr}\rho, \mathrm{Tr}(\rho^2)), \tag{20a}$$

where

$$r_N = \left(\frac{\mathrm{Tr}\rho}{N} \right)^N (1 - D\Phi)^{\frac{N-1}{2}} (1 - \tilde{D}\Phi)^{\frac{N+1}{2}}, \tag{20b}$$

$$\Phi = \frac{D + \tilde{D} - 1 - \sqrt{(D + \tilde{D} - 1)^2 + 4D\tilde{D}(\xi - 1)}}{2D\tilde{D}}, \tag{20c}$$

$$\xi = \frac{\mathrm{Tr}(\rho^2)}{(\mathrm{Tr}\rho)^2} \in [D^{-1}, 1], \quad \tilde{D} = D - N. \tag{20d}$$

Proof of Lemma 1 can be found in Appendix A.

A mixed-states analog of the general upper bound (3) reads

$$Y^{\max}[\rho] \leqslant r_N(N^K, 1, \mathrm{Tr}(\rho^2)), \tag{21}$$

because in our particular case we have $D = N^K$ and $\mathrm{Tr}\rho = 1$. One can check that for pure states we obtain $r_N(D, 1, 1) =$

N^{-N} independently of the dimension D of a composite Hilbert space, so that the bound (3) is recovered. Moreover, the bound (21) is an increasing function of purity $\mathrm{Tr}(\rho^2)$ with the minimal value equal to 0 for the maximally mixed state of rank N^K and the maximal value reached for pure states.

The task to generalize the upper bound (4) and the entanglement criteria (5) is more difficult, since the conditions for separability of mixed states are more complex than for pure states. In order to deal with the bipartite case we prove the following statement:

Theorem 2. Assume that $K = 2$, and $\mathcal{H}_2 = \mathcal{H}^A \otimes \mathcal{H}^B$. If ρ acting on $\mathcal{H}_2$ is PPT (i.e., has positive partial transpose) so that $\rho^{T_A} \geqslant 0$ where $T_A = T \otimes \mathbb{1}$, then

$$Y^{\max}[\rho] \leqslant N^{-2N}. \tag{22}$$

Proof. Denote $\rho_{\substack{jj \\ kk}} = \langle\chi_j^{\mathrm{sep}}|\rho|\chi_k^{\mathrm{sep}}\rangle$. The partial transposition $(\cdot)^{T_A}$ transforms indices as

$$(\cdot)^{T_A} : \substack{m\mu \\ n\nu} \mapsto \substack{n\mu \\ m\nu}. \tag{23}$$

After the partial transposition the state is nonnegative, which implies that [16]

$$\left| \rho_{\substack{jj \\ kk}} \right|^2 \leqslant \rho_{\substack{jk \\ jk}} \rho_{\substack{kj \\ kj}}. \tag{24}$$

The above inequalities are strongly related with the separability criteria derived in Refs. [17,18]. Making use of the property (24) we get

$$Y^{\max}[\rho] \leqslant \left(\prod_{j,k=1}^N \rho_{\substack{jk \\ jk}} \right)^{1/N}. \tag{25}$$

All factors appearing in the last product are diagonal elements of ρ. Moreover, ρ has exactly N^2 diagonal elements, thus all of them occur in the product inside (25). As the last step we shall use the inequality between arithmetic and geometric means and derive the result (22)

$$Y^{\max}[\rho] \leqslant \left(\prod_{j,k=1}^N \rho_{\substack{jk \\ jk}} \right)^{1/N} \leqslant \left(\frac{1}{N^2} \sum_{j,k=1}^N \rho_{\substack{jk \\ jk}} \right)^N = N^{-2N}, \tag{26}$$

where in the last equality we used the property that $\mathrm{Tr}\rho = 1$. $\blacksquare$

Consider now the case of K qubits (i.e., $N = 2$). Using the same method as before the following can be proven:

Fact 1. If $N = 2$ and ρ is PPT for partial transpositions with respect to all possible bipartite splittings of the K-partite system, then

$$Y^{\max}[\rho] \leqslant \frac{1}{16(2^{K-1} - 1)}. \tag{27}$$

Inequality (27) is sharp (can be saturated) for all values of K, and in the case of two qubits coincides with Eq. (22) for $N = 2$.

Using the inequalities (22) and (27) we are able to derive counterparts of the criteria (5) for two cases of mixed quantum states: (a) $K = 2$ and arbitrary N and (b) $N = 2$ and arbitrary

K, with a restriction that we detect whether the state is not PPT (NPPT) instead of being not separable.

B. Efficiency of NPPT-state detection

An answer to the question concerning the efficiency of the entanglement criteria considered depends strongly on the investigated state. In particular, there are states which are too close to the set of separable (or PPT) states to be detected by the given criteria. In this paragraph, we characterize the set of NPPT states which is covered by the entanglement test based on mixed-state collectibility.

At first we note that the purity $\text{Tr}(\rho^2)$ is bounded from below by the value $\mathcal{P}_{\min} = N^{-K}$ since $\dim \mathcal{H} = N^K$. The second observation is that, if $\text{Tr}(\rho^2) \leqslant \mathcal{P}_{\text{PPT}} = (N^K - 1)^{-1}$, then ρ is PPT with respect to all possible partial transpositions [19,20]. This result is based on Mehta's theorem [21].

To describe the ability to detect NPPT states by the mixed-state collectibility we introduce parameters $\mathcal{P}_N^{\text{crit}}$ and $\mathcal{P}_K^{\text{crit}}$, which are related to inequalities (22) and (27), respectively, and are defined as follows: If $\text{Tr}(\rho^2) \leqslant \mathcal{P}_N^{\text{crit}}$ or $\text{Tr}(\rho^2) \leqslant \mathcal{P}_K^{\text{crit}}$, then the corresponding inequalities (22) or (27) classify the state as PPT (even if it is NPPT). The critical values of purity are the solutions to the equations

$$r_N\left(N^2, 1, \mathcal{P}_N^{\text{crit}}\right) = N^{-2N}, \tag{28a}$$

$$r_2\left(2^K, 1, \mathcal{P}_K^{\text{crit}}\right) = \frac{1}{16(2^{K-1} - 1)}. \tag{28b}$$

This means that we are looking for the values of purity for which the general upper bound (21)—which increases with purity and tends to 0 for the maximally mixed state—goes below two fixed values (depending on N or K) which appear on the right-hand sides of (22) and (27).

In Table I we compared the critical values $\mathcal{P}_N^{\text{crit}}$ and $\mathcal{P}_K^{\text{crit}}$ with the general limitations given by $\mathcal{P}_{\min}$ and $\mathcal{P}_{\text{PPT}}$. As expected, both $\mathcal{P}_N^{\text{crit}}$ and $\mathcal{P}_K^{\text{crit}}$ are greater than $\mathcal{P}_{\text{PPT}}$, so not all NPPT states are detected by the criteria based on inequalities (22) and (27). However, in all cases the width of the range of purities for which the NPPT states are not detected is less than 0.05 and seems to be slightly smaller for many qubits than for two quNits. In both tables the second columns are the same because they refer to the case of two qubits.

V. BIPARTITE CASE

A. The generalized Werner state

We define a generalized Werner state on an $N \times N$ system as

$$\rho_w = \alpha(U \otimes V)|\psi_\lambda\rangle\langle\psi_\lambda|(U \otimes V)^\dagger + \frac{1-\alpha}{N^2}\mathbb{1}, \tag{29}$$

Here $|\psi_\lambda\rangle$ represents a normalized pure state with the following Schmidt decomposition:

$$|\psi_\lambda\rangle = \sum_{i=1}^{N} \sqrt{\lambda_i}|ii\rangle, \quad \sum_{i=1}^{N} \lambda_i = 1, \tag{30}$$

while $U \otimes V$ denotes a local unitary matrix. In Appendix B one can find a detailed derivation of the following expression for the mixed-state collectibility of Eq. (29):

$$Y^{\max}[\rho] = \alpha^{N-1} y(\lambda)\left(\alpha + \frac{1-\alpha}{N^2}[y(\lambda)]^{-1/N}\right), \tag{31a}$$

$$y(\lambda) \equiv Y^{\max}[|\psi_\lambda\rangle] = \left(\frac{\sum_i \sqrt{\lambda_i}}{N}\right)^{2N}. \tag{31b}$$

Since for $\alpha = 1$ we have $Y^{\max}[\rho_w] = y(\lambda)$, the function $y(\lambda)$ coincides with the pure-state collectibility of $|\psi_\lambda\rangle$ given by Eq. (30), as emphasized in Eq. (31b). Note that $y(\lambda)$ is a function of the Rényi entropy $H_{1/2}(\lambda)$ of the Schmidt vector $\lambda = \{\lambda_1, \ldots, \lambda_N\}$, as $Y^{\max}[|\psi_\lambda\rangle] = N^{-2N}\exp[NH_{1/2}(\lambda)]$.

In order to discuss the quality of the entanglement (PPT) criteria (22) we examine an example of the generalized Werner state (29) of two qubits. In this case we have only one Schmidt number λ; namely,

$$|\psi_\lambda\rangle = \sqrt{\lambda}|00\rangle + \sqrt{1-\lambda}|11\rangle. \tag{32}$$

If we apply the usual PPT criteria (in this case the PPT property is equivalent to separability) we find that the state (29) is separable if $\alpha \leqslant \alpha_T = 1/(1 + 4\omega)$, where $\omega = \sqrt{\lambda(1-\lambda)}$. When the state $|\psi_\lambda\rangle$ is separable, then ρ_w is also always separable (for $\alpha \leqslant 1$). In the opposite case, when $|\psi_\lambda\rangle$ is maximally entangled, then $\lambda = 1 - \lambda = 1/2$ and the state ρ_w is separable for $\alpha \leqslant 1/3$, as it shall be in the case of the original Werner state [1].

We would like to specify in which cases the mixed-state collectibility given by the formula (31a) is able to detect entanglement of a two-qubit generalized Werner state. This
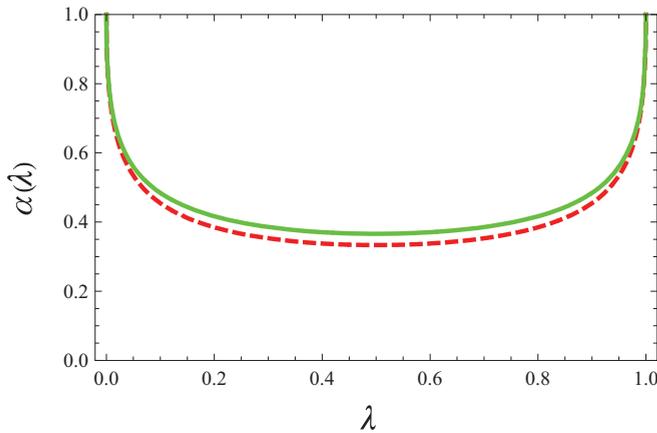
TABLE I. Minimal purities for which the NPPT property is detected. The parameter $\mathcal{P}_{\min}$ denotes the minimal possible purity of the system. The parameter $\mathcal{P}_{\text{PPT}}$ gives the bound for the purity below which all states are PPT. The critical parameters $\mathcal{P}_N^{\text{crit}}$ (left table; given as a function of N, for $K = 2$) and $\mathcal{P}_K^{\text{crit}}$ (right table; a function of K, for $N = 2$) provide the bounds for purities above which the NPPT property is detected by the criteria (22) and (27), respectively.

	Bipartite systems: $K = 2$				K qubits: $N = 2$		
N	2	3	4	K	2	3	4
$\mathcal{P}_{\min}$	0.2500	0.1111	0.0625	$\mathcal{P}_{\min}$	0.2500	0.1250	0.0625
$\mathcal{P}_{\text{PPT}}$	0.3333	0.1250	0.0667	$\mathcal{P}_{\text{PPT}}$	0.3333	0.1429	0.0667
$\mathcal{P}_N^{\text{crit}}$	0.3456	0.1728	0.1033	$\mathcal{P}_K^{\text{crit}}$	0.3456	0.1599	0.0808

FIG. 2. (Color online) Separability of the generalized Werner state of two qubits. Below the red dashed line (representing α_T) this state is separable. Above the green line (representing α_C) our criteria (22) based on the mixed-state collectibility detect its entanglement.

happens when $Y^{\max}[\rho_w] > 1/16$; thus, for

$$\alpha > \alpha_C = \frac{2}{1 + 2\omega + \sqrt{(1 + 2\omega)(1 + 10\omega)}}. \tag{33}$$

In Fig. 2 we compare two parameters α_T and α_C as functions of λ. Both curves lay close to each other, which shows that the mixed-state collectibility provides very good efficiency of entanglement detection in the case of the generalized Werner state of two qubits.

B. Pure-state collectibility as a bipartite entanglement measure

In the previous section we calculated the mixed-state collectibility (31a) for the generalized Werner state (29). We showed that the formula (31a) for the value $\alpha = 1$ is the pure-state collectibility of the bipartite state $|\psi_\lambda\rangle$ [i.e., $y(\lambda) \equiv Y^{\max}[|\psi_\lambda\rangle]$]. From that result we establish the following observation:

Fact 2. The pure-state collectibility of a bipartite state $|\Psi\rangle \in \mathcal{H}_2$ is a function of the negativity

$$Y^{\max}[|\Psi\rangle] = \frac{\{1 + (N - 1)\mathcal{N}[|\Psi\rangle]\}^N}{N^{2N}}. \tag{34}$$

Negativity $\mathcal{N}$ is a simple entanglement measure [20,22] which, for a pure state, reads

$$\mathcal{N}[|\Psi\rangle] = \frac{\|(|\Psi\rangle\langle\Psi|)^{T_B}\|_1 - 1}{N - 1}. \tag{35}$$

The above fact implies that the pure-state collectibility in the bipartite case can be alternatively defined in terms of the maximal fidelity [23,24]

$$Y^{\max}[|\Psi\rangle] = N^{-N} \max_{U,V} |\langle\Psi|\Phi^+\rangle|^{2N}, \tag{36}$$

with respect to the maximally entangled state

$$|\Phi^+\rangle = (U \otimes V)\frac{1}{\sqrt{N}} \sum_{i=1}^{N} |ii\rangle. \tag{37}$$

Maximization over the local unitaries $U \otimes V$ assures that $Y^{\max}$ depends on the overlap of the analyzed state $|\Psi\rangle$ with

the closest maximally entangled state. Thus, the pure-state collectibility can be considered as a quantity complementary to the geometric measure of entanglement [25,26]. The latter describes the minimal distance of the state analyzed to the closest separable state, while the former is related to its distance to the closest maximally entangled state (i.e., the Bell state).

VI. MIXED-STATES ENTANGLEMENT CRITERIA BASED ON PURE-STATE COLLECTIBILITY AND REMOTE PURITIES

The mixed-state entanglement criteria (14) are provided by inequality (12). In order to prove this inequality we shall derive, using the purity parameters and the separability assumption, the lower bound on the purity of the reduced (unconditional) Bob state ρ_B.

A. Bound on total Bob purity via his conditional purities

Consider a mixed state of two qubits ρ_{AB} in a usual block shape:

$$\rho_{AB} = \begin{bmatrix} A_+ & C \\ C^\dagger & A_- \end{bmatrix}. \tag{38}$$

We may locally rotate the state to the form in which the block A_+ is diagonal. Such an operation is performed on Bob's side, so it commutes with the von Neumann measurements on the side of Alice. We also put the following notation:

$$A_\pm = p_\pm \sigma_\pm, \tag{39}$$

where, as mentioned before, $p_\pm$ are probabilities of the results "$\pm$" of the von Neumann measurement of $\sigma_{\hat{z}}$ on system "A" and $\sigma_\pm$ are the states created remotely by Alice on Bob's side after getting the results $\pm$.

Note that the axis $\hat{n}$ may be in general arbitrarily chosen (here $\hat{n} = \hat{z}$). Let us write Eq. (38) in the form of the following 4×4 matrix:

$$\rho_{AB} = \begin{bmatrix} p_+ p_1 & w & c_{11} & c_{12} \\ w^* & p_+ p_2 & c_{21} & c_{22} \\ c_{11}^* & c_{21}^* & p_- q_1 & z \\ c_{12}^* & c_{22}^* & z^* & p_- q_2 \end{bmatrix}. \tag{40}$$

An analysis of the above structure leads immediately to the following lemma:

Lemma 2. Suppose that the state (40) is separable and in addition it satisfies the remote-purity condition (11a) under the measurement performed on system "A" in basis $\hat{n}$. Then we have

(1) $p_+ p_- p_i q_j \geqslant |c_{ij}|^2$ for $i, j = 1, 2$.

(2) $p_+ p_- p_2 q_1 \geqslant |c_{12}|^2$, $\quad p_+ p_- p_1 q_2 \geqslant |c_{21}|^2$.

(3) $p_1 p_2 \leqslant \epsilon_+/2$, $\quad q_1 q_2 \leqslant \epsilon_-/2$.

Proof. The first statement follows from positivity of ρ_{AB}, the second one comes from the PPT condition equivalent to separability, while the last one is implied by Eq. (11a).

Now we provide a key bound on the purity of the reduced state of Bob. ∎

Lemma 3. Suppose that the matrix from Lemma 2 fulfills the remote-purity condition (11b) under the measurement along

the perpendicular basis $\hat{n}'$, $(\hat{n} \cdot \hat{n}' = 0)$. Then its reduced Bob density matrix, $\rho_B \equiv A_+ + A_-$, satisfies the purity condition:

$$\mathrm{Tr}(\rho_B^2) \geqslant 1 - \eta(\epsilon_+ \epsilon_-; p_+ p_-; \epsilon'), \qquad (41)$$

with ϵ' and η introduced in Theorem 1.

A proof of Lemma 3 can be found in Appendix C. Clearly the dual inequality obtained by swapping the roles of $\hat{n}$ and $\hat{n}'$ is also satisfied:

$$\mathrm{Tr}[(\rho_B')^2] \geqslant 1 - \eta(\epsilon_+' \epsilon_-'; p_+' p_-'; \epsilon). \qquad (42)$$

An intriguing property of formula (41) is that, while in the first basis $\hat{n}$, it requires only one conditional purity to be small (since the product $\epsilon_+ \epsilon_-$ is involved), in the second basis $\hat{n}'$ it requires both purities to be small since $\epsilon' = \max[\epsilon_+', \epsilon_-']$.

B. Experimentally suitable uncertainty relation for mixed two-qubit states

Consider again the state ρ_B, obtained as a reduction of the original bipartite state (40). We have the following immediate fact:

Fact 3. If the purity of the state $\rho_B \equiv A_+ + A_-$ with $A_\pm = p_\pm \sigma_\pm$ defined for some probabilities $\{p_\pm\}$ and states $\{\sigma_\pm\}$ satisfies $\mathrm{Tr}(\rho_B^2) \geqslant 1 - \eta$, then

$$\mathrm{Tr}(A_+^2) + \mathrm{Tr}(A_-^2) + 2\mathrm{Tr}(A_+ A_-) \geqslant 1 - \eta. \qquad (43)$$

Recall that the quantities in our interferometric scheme (see Fig. 1) are *directly measurable* under measuring both Alice qubits along the direction $\hat{z}$. Equations (9a) and (9b) together with the definition of $A_\pm$ imply the following extensions for the expressions defined for the pure-state case:

$$G_{\pm\pm} := \sqrt{\mathrm{Tr}(A_\pm^2)}, \quad |G_{\pm\mp}|^2 = \mathrm{Tr}(A_\pm A_\mp). \qquad (44)$$

In a full analogy we may define the quantities $G_{\pm\pm}'$ and $G_{\pm\mp}'$ as putting the primes on the right-hand side of the above equations which means that all quantities were measured in the complementary basis (say $\hat{x}$). Combining Lemma 3 and Fact 3 we are immediately able to derive the central result:

Proof of Theorem 1. Directly from Eq. (44) we have

$$G_{++}^2 + G_{--}^2 + 2|G_{+-}|^2$$
$$= \mathrm{Tr}(A_+^2) + \mathrm{Tr}(A_-^2) + 2\mathrm{Tr}(A_+ A_-). \qquad (45)$$

Since we have assumed that the state ρ_{AB} is separable, according to Lemma 3 we obtain that $\mathrm{Tr}(\rho_B^2) \geqslant 1 - \eta$. Thus, the bound (43) applied to Eq. (45) boils down after short rearrangement to the desired inequality (12).

Inequality (12) implies a bound for the mixed-state generalization of the collectibility (6):

$$Y_a[\rho_{AB}] \leqslant \frac{(\sqrt{2 G_{++} G_{--}} + \sqrt{\eta + (G_{++} + G_{--})^2 - 1})^2}{8}. \qquad (46)$$

In order to derive the modified entanglement criteria (14) we finally bound from above the terms $\sqrt{G_{++} G_{--}}$ and $G_{++} + G_{--}$ by $1/2$ and 1, respectively.

The separable two-qubit state ρ_{AB} also satisfies the dual inequalities generated automatically by interchanging the roles of the two complementary bases. Such an operation is equivalent to putting primes in Eq. (46) for all the quantities

(particularly in the arguments of the η function) other than ϵ' and turning the latter into $\epsilon = \max[\epsilon_+, \epsilon_-]$.

C. Discussion and simplifications

Let us discuss the inequality (46) for three elementary examples. First of all, if the state is pure and separable we have $\eta = 0$ and $G_{++} + G_{--} = 1$ as any separable pure state is a product state. This implies that all elements of the Gram matrix G are equal so the inequality (46) is saturated. Secondly, if a separable mixed state has a product structure then the coefficients "G" are all equal, so that the left-hand side of this inequality reads $Y_a[\rho_{AB}] = G_{++} G_{--}/4$. Since the state is mixed there is a contribution of a positive parameter $\eta > 0$ on the right-hand side so in this case the inequality holds and it is strict. Finally, if the state is pure but maximally entangled then $\eta = 0$ so the right-hand side of Eq. (46) is equal to $1/16$ while the collectibility on the left side is equal to $1/4$, as shown earlier in Ref. [10], so the inequality is violated by a factor of four.

Let us emphasize that the performed analysis leading to Eq. (46) is independent of the choice of the two complementary observables, so we may consider an *optimization of the two dual inequalities* over two measurements of the two complementary observables $\{\hat{n}\vec{\sigma}, \hat{n}'\vec{\sigma}\}$ with the constraint $\hat{n} \cdot \hat{n}' = 0$.

Furthermore, let us observe that inequality (46) is a slightly stronger variant of the original result $Y_a[|\Psi\rangle_{AB}] \leqslant 1/16$ valid for separable pure states. Since the state is pure we shall put $\eta = 0$ and $G_{++} + G_{--} = 1$ in Eq. (46) and obtain

$$Y_a[|\Psi\rangle_{AB}] \leqslant \frac{G_{++} G_{--}}{4} \leqslant \frac{1}{16}. \qquad (47)$$

The above inequality is useful for entanglement detection in the case of pure states that are not maximally entangled.

Let us make a remark on statistical properties of these inequalities. Inequality (46), as well as the separability criteria based on $Y_a[\rho_{AB}]$, involves the square root calculated from the difference of the measurable quantities in the form $\sqrt{G_{++} G_{--} - |G_{+-}|^2}$. While in theory the argument inside the square root is always positive, in practice, due to experimental errors, a problems with the sign under the square root might appear. We shall keep in mind that the violation of the inequality is equivalent to "passing" the entanglement test. The most natural approach is to apply the term "passed" to all cases where the quantity under the square root in left-hand side of Eq. (46) is strictly positive up to the error bars. This is actually the test of the quantity of the entanglement produced since in all entangled states the term $G_{++} G_{--} - |G_{+-}|^2$ becomes positive, sometimes in a very drastic way. For instance, for maximally entangled states the second term $|G_{+-}|$ vanishes, and $G_{++} G_{--} = 1/4$. Therefore, testing the degree of entanglement of the state analyzed, we look for the case in which the number $G_{++} G_{--} - |G_{+-}|^2$ is significantly positive. This justifies the term "violated" or the term "passed" from the perspective of testing the entanglement property.

Finally, let us point out that other separability tests, based on a pair of inequalities (41) and (42) or (12) and its dual, do not involve the problem discussed above. In fact, these inequalities seem to be more suitable for an experimental

application. In this context the previous paragraph dealing with the issue of the square root may seem to be of purely academic character. We keep it since the reasoning as it is has an element of universality—in all cases when the error may spoil the mathematics of the test formula one should look at the region of parameters in which the test really matters.

VII. CONCLUSIONS

In this work we generalized the collectibility—a quantity initially designed to characterize entanglement of pure quantum states—for mixed states. On one hand we improved existing expressions for pure-state collectibility by taking into account contributions due to nonmaximal purity of the state investigated. This approach can be useful in practice to analyze experimentally states intended to be pure, which are characterized by a high purity. For the simplest case of two-qubit system the proposed measurement scheme requires two copies of the state analyzed, so a possible optical setup involves four-photon experiments.

For such pseudomixed states the essential element of the scheme is that one performs two measurements represented by the orthonormal Bloch vectors $\hat{n}$ and $\hat{n}'$ corresponding to the two polarization rotations $R^\dagger(\theta,\phi)$, and $R^\dagger(\theta',\phi')$. This allows us to avoid the assumption which was necessary in Ref. [10] that the state is pure.

On the other hand we introduced here a new notion of the mixed-state collectibility—a quantity defined for an arbitrary mixed state of a composite system which contains K subsystems, each describing an N-level system. Presence of the Nth root in the definition (19) assures that it is consistent with the former notion (2) of pure-state collectibility. In the particular case of a pure state of a bipartite system the collectibility is shown to be a function of the negativity. Moreover, in this case the collectibility depends on the minimal distance of the pure state analyzed to the closest maximally entangled state.

Explicit bounds for the mixed-state collectibility obtained for arbitrary quantum states, separable states, and states with positive partial transpose belong to the key results of the present paper. They allow us to design practical tests for entanglement of a given state. Such experimental schemes look realistic at least for the mixed state of a two-qubit system, for which a four-photon experiment is necessary. We are thus tempted to believe that such experiments, useful to demonstrate quantum entanglement of a given state without performing its quantum tomography, will be realized in the near future.

There are still few more questions for future research. One should explore possible structural connections of the present quantity to the symmetric measurement-state reconstruction of the type proposed in Ref. [27] and to the experimental discrimination of classes invariant to stochastic local operation and classical communication (SLOCC-invariant classes) [28]. The directly related problem is the issue of lower bounds for typical entanglement measures (like concurrence or the geometric measure) based on the measurement reproducing collectibility in analogy to the standard approach (see Refs. [29–32]). One of the interesting practical questions that will be considered elsewhere is what are the best possible lower

bounds on entanglement measures at the constraints given by full output data from the presented Hong-Ou-Mandel setup. While the issue of finding some lower bounds for the latter (two-qubit) case is relatively easily tractable, the analogous questions for higher dimensions and/or subsystems seem to require much more effort. Finally, there is an intriguing question of the relation of the test designed for pseudopure entangled states to the entanglement criteria based on mutually unbiased bases (MUBs) provided in Ref. [33]. In fact, the H-O-M–based test for pseudopure qubit entanglement uses incomplete MUBs in one laboratory being two Pauli matrices. However, one should be careful with the way the MUBs are used here in order to keep the parametric efficiency of the scheme in the context of the direct tomography method.

APPENDIX A: MAXIMUM OF PRODUCT OF MATRIX ELEMENTS

Proof of Lemma 1. We maximize $\prod_{i,j=1}^{N} \rho_{ij}$ over all possible choices of the matrix entries ρ_{ij} of a Hermitian $D \times D$ matrix ρ, assuming two constraints: $\mathrm{Tr}\rho = $ const. and $\mathrm{Tr}(\rho^2) = $ const. This means solving the following system of equations (for all $k,m = 1, \ldots, D$):

$$\frac{\partial}{\partial \rho_{km}} \left[\prod_{i,j=1}^{N} \rho_{ij} - \sum_{i,j=1}^{D} \rho_{ij} \left(\nu \delta_{ij} + \frac{\mu}{2} \rho_{ji} \right) \right] = 0, \qquad (A1)$$

where ν and $\mu/2$ are the Lagrange multipliers related to the constraints on the trace and purity, respectively. After differentiation with respect to ρ_{km} we obtain

$$C\rho_{km}^{-1} = \nu \delta_{km} + \mu \rho_{mk}, \quad k,m \leqslant N, \qquad (A2a)$$

$$0 = \nu \delta_{km} + \mu \rho_{mk} \quad \text{elsewhere}, \qquad (A2b)$$

where $0 \leqslant C \equiv \prod_{i,j=1}^{N} \rho_{ij}$ shall be from now on treated as a constant. Note that $\rho_{km}^{-1} \equiv 1/\rho_{km}$. In order to prove Lemma 1 we have to find the maximal value of C.

The structure of the above equations provides that the maximizing matrix ρ will have the form

$$\rho_{km} = \mathrm{Tr}\rho \begin{cases} \rho_a, & k,m \leqslant N \ \wedge \ k = m \\ \rho_b, e^{i\varphi_{km}}, & k,m \leqslant N \ \wedge \ k \neq m \\ \rho_c, & k,m > N \ \wedge \ k = m \\ 0, & k,m > N \ \wedge \ k \neq m, \end{cases} \qquad (A3)$$

determined by three real coefficients ρ_a, $\rho_b \geqslant 0$, ρ_c and all phases φ_{km} being arbitrary. For further convenience we have separated the normalization factor $\mathrm{Tr}\rho$.

The equations (A2a) and (A2b) together with both constraints give five conditions:

$$C(\rho_a \mathrm{Tr}\rho)^{-1} = \nu + \mu\rho_a \mathrm{Tr}\rho, \quad \text{(A4a)}$$

$$C(\rho_b \mathrm{Tr}\rho)^{-1} = \mu\rho_b \mathrm{Tr}\rho, \quad \text{(A4b)}$$

$$\nu + \mu\rho_c \mathrm{Tr}\rho = 0, \quad \text{(A4c)}$$

$$N\rho_a + (D-N)\rho_c = 1, \quad \text{(A4d)}$$

$$N\rho_a^2 + N(N-1)\rho_b^2 + (D-N)\rho_c^2 = \xi, \quad \text{(A4e)}$$

where $\xi = \mathrm{Tr}(\rho^2)/(\mathrm{Tr}\rho)^2$, so that $\xi \in [D^{-1}, 1]$. In order to solve the above system of equations we shall rewrite Eqs. (A4b) and (A4c) to the form $(\rho_b \mathrm{Tr}\rho)^2 = C/\mu$ and $\rho_c \mathrm{Tr}\rho = -\nu/\mu$ and eliminate both Lagrange multipliers from (A4a) to obtain

$$\rho_b^2 \rho_a^{-1} + \rho_c - \rho_a = 0. \quad \text{(A4f)}$$

From Eqs. (A4d)–(A4f) we find (we introduce the dual dimension $\tilde{D} = D - N$)

$$\rho_a = \frac{1 - \tilde{D}\rho_c}{N}, \quad \rho_b = \frac{\sqrt{(1-D\rho_c)(1-\tilde{D}\rho_c)}}{N}, \quad \text{(A5a)}$$

and the quadratic equation for ρ_c

$$D\tilde{D}\rho_c^2 - (D + \tilde{D} - 1)\rho_c + 1 - \xi = 0. \quad \text{(A5b)}$$

The above equation possesses two nonnegative solutions of the form

$$\rho_c^\pm = \frac{D + \tilde{D} - 1 \pm \sqrt{(D + \tilde{D} - 1)^2 + 4D\tilde{D}(\xi-1)}}{2D\tilde{D}} \quad \text{(A6)}$$

that lead to two independent solutions of the main problem:

$$C_\pm = \left(\frac{\mathrm{Tr}\rho}{N}\right)^{N^2} (1 - D\rho_c^\pm)^{\frac{N(N-1)}{2}} (1 - \tilde{D}\rho_c^\pm)^{\frac{N(N+1)}{2}}. \quad \text{(A7)}$$

Since $\rho_c^- \leqslant \rho_c^+$, the larger value is always given by C_- (this statement holds also when $1 - D\rho_c^+$ and/or $1 - \tilde{D}\rho_c^+$ become negative). The above conclusion finalizes the proof of Lemma 1—we shall only rename $[C_-]^{1/N}$ by $r_N(D, \mathrm{Tr}\rho, \mathrm{Tr}(\rho^2))$. It is important to point out that the solution presented above provides a global maximum, since the set of $D \times D$ hermitian matrices with given trace and Hilbert-Schmidt norm (which coincides with purity in the case of positive semidefinite matrices) has topology of $D^2 - 2$ sphere $\mathcal{S}^{D^2-2}$, so that it has no boundary.

APPENDIX B: COLLECTIBILITY OF GENERALIZED WERNER STATE

Before we start calculating the mixed-state collectibility for the generalized Werner state (29) we need the following lemma:

Lemma 4. For $b \geqslant 0$ and $q \geqslant 1$ the function $h : [0, \infty)^N \to [0, \infty)$

$$h(\boldsymbol{x}) = \prod_{i=1}^N (x_i^2 + b) x_i^q, \quad \text{(B1)}$$

is Schur concave.

Proof. Since $x_i \geqslant 0$, the Theorem II.3.14 from [34] states that to prove the Schur concavity of the function $h(\boldsymbol{x})$, it is sufficient to show that $\Theta[h(\boldsymbol{x})] \leqslant 0$, where

$$\Theta[h(\boldsymbol{x})] \equiv (x_j - x_k)\left(\frac{\partial h}{\partial x_j}(\boldsymbol{x}) - \frac{\partial h}{\partial x_k}(\boldsymbol{x})\right). \quad \text{(B2)}$$

An explicit computation gives

$$\Theta[h(\boldsymbol{x})] = -(x_j - x_k)^2 (x_j x_k)^{q-1} \prod_{i \notin \{j,k\}} (x_i^2 + b) x_i^q$$
$$\times [b_1 x_j^2 x_k^2 + b_2(x_k^2 + x_j^2) + b(x_k - x_j)^2 + qb^2], \quad \text{(B3)}$$

where $b_1 = q + 2$ and $b_2 = b(q-1)$. Since $b_1 \geqslant 0$ and $b_2 \geqslant 0$ we obtain that $\Theta[h(\boldsymbol{x})] \leqslant 0$; thus $h(\boldsymbol{x})$ is Schur concave.

The maximally mixed part of the Werner state (29) is invariant under local unitary operations. Since the definition (30) of $|\psi_\lambda\rangle$ involves $U \otimes V$, we can choose the basis of N separable states to be $|\chi_j^{\mathrm{sep}}\rangle = |jj\rangle$ without losing generality. The optimization over $|\chi^{\mathrm{sep}}\rangle$ shall then be substituted by maximization over $U \otimes V$, where

$$U = \sum_{m,n} u_{mn} |m\rangle\langle n|, \quad V = \sum_{\mu,\nu} v_{\mu\nu} |\mu\rangle\langle\nu|. \quad \text{(B4)}$$

An explicit calculation yields

$$\langle\chi_j^{\mathrm{sep}}|\rho_w|\chi_k^{\mathrm{sep}}\rangle = \alpha g_j g_k^* + \frac{1-\alpha}{N^2}\delta_{jk}, \quad \text{(B5)}$$

where

$$g_m = \sum_{n=1}^N u_{mn} v_{mn} \sqrt{\lambda_n}. \quad \text{(B6)}$$

Now we shall use the result (B5) to calculate the product appearing inside the definition (19):

$$\prod_{j,k=1}^N \langle\chi_j^{\mathrm{sep}}|\rho|\chi_k^{\mathrm{sep}}\rangle$$
$$= \prod_{i=1}^N \left[(\sqrt{\alpha}|g_i|)^2 + \frac{1-\alpha}{N^2}\right](\sqrt{\alpha}|g_i|)^{2(N-1)}. \quad \text{(B7)}$$

The right-hand side of (B7) might be viewed as the $h(\boldsymbol{x})$ function from Lemma 4 with $x_i = \sqrt{\alpha}|g_i|$, $b = (1-\alpha)/N^2$, and $q = 2(N-1)$. $\blacksquare$

Every vector $\boldsymbol{x} \in [0, \infty)^N$ majorizes the vector of mean values $\bar{\boldsymbol{x}} = (\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N)$, such that $\bar{x}_1 = \bar{x}_2 = \cdots = \bar{x}_N \equiv N^{-1}\sum_{j=1}^N x_j$. According to Lemma 4, $h(\boldsymbol{x})$ is the Schur concave function, thus since $\boldsymbol{x} \succ \bar{\boldsymbol{x}}$ we obtain that $h(\boldsymbol{x}) \leqslant h(\bar{\boldsymbol{x}})$. This means that in each factor (for each i) inside the product (B7) we can substitute $\sqrt{\alpha}|g_i|$ by the number $\sqrt{\alpha}N^{-1}\sum_{j=1}^N |g_j|$. Since we obtain then the Nth power of the same factor, after some simplifications we find that

$$\prod_{j,k=1}^N \langle\chi_j^{\mathrm{sep}}|\rho|\chi_k^{\mathrm{sep}}\rangle \leqslant \left[\alpha^{N-1}p(\lambda)\left(\alpha + \frac{1-\alpha}{N^2}[p(\lambda)]^{-\frac{1}{N}}\right)\right]^N. \quad \text{(B8)}$$

We introduced here the function $p(\lambda)$,

$$
p(\lambda) = \left(\frac{1}{N} \sum_{i=1}^{N} |g_i| \right)^{2N} = \left(\frac{1}{N} \sum_{i=1}^{N} \left| \sum_{n=1}^{N} u_{in} v_{in} \sqrt{\lambda_n} \right| \right)^{2N}.
$$

(B9)

Since the right-hand side of (B8) is an increasing function of $p(\lambda)$ we will prove the result (31a) by showing that $p(\lambda) \leqslant y(\lambda)$. To this end we establish the chain of inequalities

$$
\begin{aligned}
p(\lambda) &\leqslant \left[\frac{1}{N} \sum_{i=1}^{N} \sum_{n=1}^{N} |u_{in}||v_{in}| \sqrt{\lambda_n} \right]^{2N} \\
&= \left[\frac{1}{N} \sum_{n=1}^{N} \sqrt{\lambda_n} \left(\sum_{i=1}^{N} |u_{in}||v_{in}| \right) \right]^{2N} \\
&\leqslant \left[\frac{1}{N} \sum_{n=1}^{N} \sqrt{\lambda_n} \sqrt{\sum_{i=1}^{N} |u_{in}|^2} \sqrt{\sum_{i=1}^{N} |v_{im}|^2} \right]^{2N} = y(\lambda).
\end{aligned}
$$

(B10)

The first one is the inequality for the modulus of a sum which in the simplest form reads $|a + b| \leqslant |a| + |b|$. Then we change the order of summation and apply the Cauchy-Schwarz inequality for the sum over i. In the last step we use the fact that the matrices U and V are unitary, so that $\sum_{i=1}^{N} |u_{in}|^2 = 1$ and $\sum_{i=1}^{N} |v_{im}|^2 = 1$ for any index n and m.

Taking the Nth root of the expression (B8) and applying the inequality $p(\lambda) \leqslant y(\lambda)$ we find that

$$
Y^{\max}[\rho_w] \leqslant \alpha^{N-1} y(\lambda) \left(\alpha + \frac{1-\alpha}{N^2} [y(\lambda)]^{-1/N} \right).
$$

(B11)

The last step is to show that the above inequality can be saturated. Comparing (B7) with the inequality (B11) we observe that this can happen only if there exist unitary matrices U and V such that $|g_i| = [y(\lambda)]^{1/(2N)}$ for all i. In that case the matrix elements u_{in} and v_{in} need to fulfill N conditions of the form

$$
\forall_i \left| \sum_{n=1}^{N} u_{in} v_{in} \sqrt{\lambda_n} \right| = \frac{1}{N} \sum_{n=1}^{N} \sqrt{\lambda_n}.
$$

(B12)

The solution of (B12) might be constructed as $|u_{in}| = 1/\sqrt{N}$ and $v_{in} = u_{in}^*$. Finally, since inequality (B11) can be saturated, the mixed-state collectibility for the generalized Werner state (29) is given by Eq. (31a). ∎

APPENDIX C: PROPERTIES OF BOB STATE PURITY IN PSEUDOPURE TWO-QUBIT STATE

Proof of Lemma 3. After the measurement performed on the subsystem "A" in a basis complementary to $\hat{z}$ (let us say $\hat{x}$) the matrices $A'_\pm$ (in the second basis) are

$$
A'_\pm \equiv p'_\pm \sigma'_\pm = \tfrac{1}{2}[\rho_B \pm X],
$$

(C1)

where $\rho_B = A_+ + A_-$ and $X = C + C^\dagger$. Related normalization constants read

$$
p'_\pm \equiv \text{Tr}(A'_\pm) = \frac{1 \pm \text{Tr} X}{2}.
$$

(C2)

Note that the last term in Eq. (C1) is the most important since we are looking for a restriction on the purity of ρ_B.

We shall now consider the remote-purity condition (11b) together with the Eqs. (C1) and (C2) to obtain two inequalities (for both signs):

$$
\text{Tr}[(\rho_B \pm X)^2] \geqslant (1 \pm \text{Tr} X)^2 (1 - \epsilon'_\pm).
$$

(C3)

We extract the purity of the state ρ_B as follows:

$$
\begin{aligned}
\text{Tr}(\rho_B^2) &\geqslant (1 - \epsilon'_\pm)[1 + (\text{Tr} X)^2] - \text{Tr}(X^2) \\
&\quad \pm 2(1 - \epsilon'_\pm)\text{Tr} X \mp 2\text{Tr}(\rho_B X).
\end{aligned}
$$

(C4)

In order to eliminate the term $2\text{Tr}(\rho_B X)$ we shall add the inequalities of both signs and divide the result by two:

$$
\text{Tr}(\rho_B^2) \geqslant (1 - \bar{\epsilon}')[1 + (\text{Tr} X)^2] - \Delta' \text{Tr} X - \text{Tr}(X^2),
$$

(C5)

where $\bar{\epsilon}' = (\epsilon'_+ + \epsilon'_-)/2$ and $\Delta' = \epsilon'_+ - \epsilon'_-$. Next, we use the identity $2 \det X = (\text{Tr} X)^2 - \text{Tr}(X^2)$ valid for 2×2 matrices to rewrite

$$
\text{Tr}(\rho_B^2) \geqslant 1 - \bar{\epsilon}' + 2 \det X - \bar{\epsilon}'(\text{Tr} X)^2 - \Delta' \text{Tr} X.
$$

(C6)

Since the sign of $\det X$ is not fixed we shall involve an estimation $\det X \geqslant -|\det X|$ which becomes an equality whenever the sign of $\det X$ is negative. We obtain

$$
\text{Tr}(\rho_B^2) \geqslant 1 - [\bar{\epsilon}' + 2|\det X| + \bar{\epsilon}'(\text{Tr} X)^2 + \Delta' \text{Tr} X];
$$

(C7)

thus, in the next step we maximize the following expression:

$$
\bar{\epsilon}' + 2|\det X| + \bar{\epsilon}'(\text{Tr} X)^2 + \Delta' \text{Tr} X.
$$

(C8)

Since

$$
X = \begin{bmatrix} 2\text{Re} c_{11} & c_{12} + c_{21}^* \\ c_{12}^* + c_{21} & 2\text{Re} c_{22} \end{bmatrix},
$$

(C9)

we estimate the modulus of the determinant as

$$
\begin{aligned}
|\det X| &= |4\text{Re} c_{11} \text{Re} c_{22} - |c_{12} + c_{21}^*|^2| \\
&\leqslant 4|c_{11}||c_{22}| + (|c_{12}| + |c_{21}|)^2.
\end{aligned}
$$

(C10)

Multiplying different conditions from Lemma 2 we find the following estimates (only in this place we use the assumption that ρ_{AB} is separable):

$$
|c_{ij}|^2 \leqslant p_+ p_- \frac{\sqrt{\epsilon_+ \epsilon_-}}{2}, \quad i \neq j,
$$

(C11)

$$
|c_{11}||c_{22}| \leqslant p_+ p_- \frac{\sqrt{\epsilon_+ \epsilon_-}}{2},
$$

(C12)

which imply

$$
|\det X| \leqslant 4 p_+ p_- \sqrt{\epsilon_+ \epsilon_-}.
$$

(C13)

In order to estimate the quadratic expression we observe that, according to Eq. (C2), we have $\text{Tr} X \in [-1, 1]$. Moreover,

since $\bar{\epsilon}' \geqslant 0$ and $|\Delta'| \leqslant 1/2$, there are two cases. For $\Delta' \geqslant 0$ the maximum is attained for $\text{Tr}X = 1$, while for $\Delta' \leqslant 0$ the maximal value is provided by $\text{Tr}X = -1$. This implies the following estimate:

$$\bar{\epsilon}'(\text{Tr}X)^2 + \Delta'\text{Tr}X \leqslant \bar{\epsilon}' + |\Delta'|. \tag{C14}$$

Eventually, inequalities (C13) and (C14) give the result

$$\bar{\epsilon}' + 2|\det X| + \bar{\epsilon}'(\text{Tr}X)^2 + \Delta'\text{Tr}X$$
$$\leqslant 8p_+p_-\sqrt{\epsilon_+\epsilon_-} + 2\epsilon' \equiv \eta, \tag{C15}$$

where $\epsilon' = \max[\epsilon'_+, \epsilon'_-]$. In the last step we used the identity $\epsilon'_+ + \epsilon'_- + |\epsilon'_+ - \epsilon'_-| = 2\epsilon'$.

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