Distinguishability of generic quantum states

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Properties of random mixed states of dimension N distributed uniformly with respect to the Hilbert-Schmidt measure are investigated. We show that for large N, due to the concentration of measure, the trace distance between two random states tends to a fixed number $\tilde{D} = 1/4 + 1/\pi$, which yields the Helstrom bound on their distinguishability. To arrive at this result, we apply free random calculus and derive the symmetrized Marchenko-Pastur distribution, which is shown to describe numerical data for the model of coupled quantum kicked tops. Asymptotic value for the root fidelity between two random states, $\sqrt{F} = \frac{3}{4}$, can serve as a universal reference value for further theoretical and experimental studies. Analogous results for quantum relative entropy and Chernoff quantity provide other bounds on the distinguishablity of both states in a multiple measurement setup due to the quantum Sanov theorem. We study also mean entropy of coherence of random pure and mixed states and entanglement of a generic mixed state of a bipartite system.

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I. INTRODUCTION

Processing of quantum information takes place not in a Hilbert space, but in a laboratory [1]. However, the standard formalism of density operators acting on a finite-dimensional Hilbert space $\mathcal{H}_N$ proves to be extremely efficient in describing physical experiments. It is therefore important to investigate properties of the set Ω_N of density operators of a given size N, as it forms a scene for which screenplays for quantum pieces are written.

In the one-qubit case, the set Ω_2 forms the familiar Bloch ball, but the geometry of the set $\Omega_N \subset \mathbb{R}^{N^2-1}$ for $N > 3$ becomes much more complex [2]. Interestingly, for large N certain calculations become easier due to the measure concentration phenomenon: a slowly varying function of a random state typically takes values close to the mean value [3,4]. Extending the approach of [5,6], we focus our attention on the distinguishability between generic quantum states.

Two states ρ and σ can be distinguished by a suitable experiment with probability one if they are orthogonal, i.e., $\mathrm{Tr}\rho\sigma = 0$. The celebrated result of Helstrom [7] provides a bound for the probability p of the correct distinction between arbitrary two quantum states in a single experiment,

$$p \leqslant \tfrac{1}{2}[1 + D_{\mathrm{Tr}}(\rho,\sigma)], \tag{1}$$

where the definition of the *trace distance* reads as

$$D_{\mathrm{Tr}}(\rho,\sigma) = \tfrac{1}{2}\mathrm{Tr}|\rho - \sigma|. \tag{2}$$

For two orthogonal states this distance is maximal, $D_{\mathrm{Tr}} = 1$, so $p = 1$ as expected. The above bound can be achieved by a projective measurement onto positive and negative part of the *Helstrom matrix* $\Gamma = \rho - \sigma$.

In the case of measurement on m copies of both states, the probability of an error decreases exponentially with the number of copies. To characterize the optimal distinguishability rate, one can apply the quantum Stein lemma [8], the Sanov theorem [9], and the relative entropy between both states, or the quantum Chernoff bound [10].

The aim of this work is to analyze properties of generic quantum states in high dimensions and their distinguishability. For two random states distributed according to the flat, Lebesgue measure in the set Ω_N of quantum states of dimension $N \gg 1$, we derive the limiting values of fidelity, trace distance, relative entropy, and Chernoff quantity, which allow us to obtain universal bounds for their distinguishability. Furthermore, we study average coherence of a random state with respect to a given basis and average entanglement of random mixed state of a bipartite quantum system.

II. RANDOM QUANTUM STATES AND THEIR LEVEL DENSITY

Consider a Haar random unitary matrix U of dimension NK acting on a fixed bipartite state $|\phi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_K$. Then, the state $|\psi\rangle = U|\phi\rangle$ is distributed according to the Haar measure on the space of pure states, while its partial trace $\sigma = \mathrm{Tr}_K|\psi\rangle\langle\psi|$ is distributed in the set Ω_N according to the induced measure ν_K [11]. To be specific, the integral with respect to the induced measure ν_K on a set of quantum mixed states is equal to the integral of the partial trace over a group of unitary matrices of size $N \times K$ equipped with the normalized Haar measure

$$\int_{\Omega_N} f(\rho)d\nu_K(\rho) = \int f(\mathrm{Tr}_K(U|00\rangle\langle 00|U^\dagger))d\mu_H(U). \tag{3}$$

In the special case of $K = N$, we get the flat Hilbert-Schmidt measure $\nu_{\mathrm{HS}} = \nu_N$. In particular, setting $K = N = 2$ one generates the flat Lebesgue measure on the set Ω_2 of one-qubit mixed states equivalent to the Bloch ball. Alternatively, such random states can be constructed as $\rho = GG^\dagger/\mathrm{Tr}GG^\dagger$, where G stands for a random rectangular matrix of dimension $N \times K$ with independent complex normal entries. In the case where $K = N$ large, the flat measure in the set Ω_N leads asymptotically to the Marchenko-Pastur distribution $\mathcal{MP}(x)$

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[12]. In the general case, this distribution depends on the rectangularity parameter $c = K/N$. For $c \geqslant 1$ the density is continuous,

$$\mathcal{MP}_c(x) = \frac{1}{2\pi x}\sqrt{[x-(1-\sqrt{c})^2][(1+\sqrt{c})^2-x]}, \quad (4)$$

while for $c < 1$ the measure also contains a singular component and reads as $(1-c)\delta_0 + \mathcal{MP}_c(x)$. Here, $x = K\lambda$ is a rescaled eigenvalue of ρ. In the case where $c = 1$ the $\mathcal{MP}$ measure $\mu_{\mathcal{MP}}$ has a continuous density, $\mathcal{MP}(x) = (\sqrt{4/x-1})/2\pi$, supported on [0,4].

Consider a random quantum state ρ generated according to the flat measure $\nu_{\rm HS}$ in Ω_N with eigenvalues λ_i described by the measure $\mu_N^{(\rho)} = \frac{1}{N}\sum_i \delta_{N\lambda_i(\rho)}$. For a large N with probability one, the measure μ_N converges weakly to $\mu_{\mathcal{MP}}$. Thus, for any bounded and continuous matrix function g of a random state ρ, the trace of $g(\rho)$ converges almost surely (a.s.) to the mean value

$$\mathrm{Tr}(g(\rho)) = N\int g\left(\frac{t}{N}\right)d\mu_N^{(\rho)}(t) \xrightarrow[N\to\infty]{\text{a.s.}} \int \tilde{g}(t)d\mu_{\mathcal{MP}}(t), \quad (5)$$

where $\tilde{g}(t) = \lim_{N\to\infty} N g(\frac{t}{N})$.

III. TRACE DISTANCE AND A SINGLE SHOT EXPERIMENT

To show a simple application of Eq. (5), consider the trace distance of a random state ρ to the maximally mixed state $\rho_* = \mathbb{1}/N$. For a large dimension N, the value

$$\mathrm{Tr}|\rho-\rho_*| = \int |t-1|d\mu_N^{(\rho)}(t) \quad (6)$$

converges to the integral over the $\mathcal{MP}$ measure, so that the trace distance behaves almost surely as

$$D_{\rm Tr}(\rho,\rho_*) \xrightarrow[N\to\infty]{\text{a.s.}} \frac{1}{2}\int |t-1|d\mu_{\mathcal{MP}}(t) = \frac{3\sqrt{3}}{4\pi} \simeq 0.4135. \quad (7)$$

We wish to evaluate the trace distance between two random quantum states ρ,σ sampled from the set Ω_N according to an induced measure ν_K. To this end, we need to derive the spectral density of the Helstrom matrix $\Gamma = \rho - \sigma$, for which we use tools of free probability. In particular, we are going to apply the free additive convolution of two distributions [13], written $P_1 \boxplus P_2$, which describes the asymptotic spectrum of the sum of two random matrices X_1 and X_2, each described by the densities P_1 and P_2, under the assumption that both variables are independent and invariant with respect to unitary transformations $X_i \to UX_iU^\dagger$. As both random states ρ and σ are independent and free, the limiting spectral density of their difference is given by the free additive convolution of the Marchenko-Pastur distribution and its dilation $\mathcal{D}$ by factor minus one, $\mathcal{SMP} = \mathcal{MP} \boxplus \mathcal{D}_{-1}(\mathcal{MP})$. In order to derive $\mathcal{SMP}$, it is convenient to use the R transform of the $\mathcal{MP}$ distribution, given by

$$R(G(z)) + \frac{1}{G(z)} = z, \quad (8)$$

where $G(z)$ is the Cauchy transform of the distribution [14]. Hence, we get that the R transform of the $\mathcal{MP}$ distribution reads as $R_{\mathcal{MP}}(z) = c/(1-z)$ so the R transform of the symmetrized distribution is given by the sum

$$R_{\mathcal{SMP}}(z) = R_{\mathcal{MP}}(z) - R_{\mathcal{MP}}(-z) = \frac{2cz}{1-z^2}. \quad (9)$$

Inverting this transform, we obtain the desired *symmetrized Marchenko-Pastur* ($\mathcal{SMP}$) distribution

$$\mathcal{SMP}_c(y) = \frac{-1-4(c-1)c-3y^2+[Y_c(y)]^{2/3}}{2\sqrt{3}\pi y[Y_c(y)]^{1/3}}, \quad (10)$$

where

$$Y_c(y) = (2c-1)^3 + 9(c+1)y^2 + 3\sqrt{3}y \times \sqrt{(2c-1)^3 - y^4 + [2-(c-10)c]y^2}. \quad (11)$$

This measure is supported between $y_\pm = \pm\frac{1}{\sqrt{2}}\sqrt{-c^2+10c+(c+4)^{3/2}\sqrt{c}+2}$. If $c < 1$ the measure has a singularity at zero.

In the special case of $c = 1$, corresponding to the Hilbert-Schmidt measure $\nu_{\rm HS}$ in Ω_N, we obtain

$$\mathcal{SMP}(y) = \frac{-1-3y^2+[1+3y(\sqrt{3+33y^2-3y^4}+6y)]^{2/3}}{2\sqrt{3}\pi y[1+3y(\sqrt{3+33y^2-3y^4}+6y)]^{1/3}}. \quad (12)$$

This measure was earlier identified as a free commutator between two semicircular distributions [14] and appeared in the literature under the name of the tetilla law [15]. As a simple application of the $\mathcal{SMP}$ distribution we will show that the Hilbert-Schmidt distance tends to zero as N tends to infinity.

To obtain the required result, consider two independent random states ρ and σ, distributed according to the Hilbert-Schmidt measure and rewrite Eq. (5) as

$$\mathrm{Tr}(g(\rho-\sigma)) = N\int g\left(\frac{t}{N}\right)d\mu_N^{(\rho-\sigma)}(t). \quad (13)$$

From this follows

$$\|\rho-\sigma\|_2^2 = \mathrm{Tr}(\rho-\sigma)^2 = \frac{1}{N}\int t^2 d\mu_N^{(\rho-\sigma)}(t). \quad (14)$$

Now, taking the limit $N \to \infty$, the integral with probability one has a finite value $\int t^2 d\mu_{\mathcal{SMP}}(t) = 2$, hence, due to the prefactor N^{-1} we get asymptotically the desired result

$$D_{\rm HS}(\rho,\sigma) = \|\rho-\sigma\|_2 \simeq \sqrt{\frac{2}{N}} \xrightarrow[N\to\infty]{\text{a.s.}} 0\,. \quad (15)$$

An analogous result implying that the average HS distance $D_{\rm HS}(\rho,\rho_*)$ between a random state ρ and the maximally mixed state ρ_* for large dimension N tends to zero was earlier obtained in [4]. Although this distance is not monotone [2], it admits a direct operational interpretation [16].

The asymptotic value of $\mathrm{Tr}|\rho-\sigma| = \mathrm{Tr}|\Gamma|$ will be almost surely given by an integral over the $\mathcal{SMP}$ distribution, which yields the generic trace distance between two random states

$$D_{\mathrm{Tr}}(\rho,\sigma) \xrightarrow[N\to\infty]{\text{a.s.}} \tilde{D} = \frac{1}{2}\int |y| d\mu_{\mathcal{SMP}}(y) = \frac{1}{4} + \frac{1}{\pi} \simeq 0.5683. \tag{16}$$

Formulas (7) and (16) combined with the Helstrom theorem imply one of the key results of this work.

Proposition 1. Consider two independent random states ρ and σ drawn according to the HS measure in the space of states of dimension N. For large N the probability p_2 of correct distinction between both random states in a single measurement is bounded by $p_2 \leqslant \frac{1}{2} + \frac{1}{2}(\frac{1}{4} + \frac{1}{\pi}) \simeq 0.7842$, while the probability p_1 of distinguishing σ and the maximally mixed state $\rho_* = \mathbb{1}/N$ is bounded by $p_1 \leqslant \frac{1}{2} + \frac{1}{2}\frac{3\sqrt{3}}{4\pi} \simeq 0.7067$.

The above results improve our understanding [2] of the structure of the set Ω_N of mixed states of a large dimension N. The HS measure is concentrated in an ε neighborhood of the unitary orbit $U\rho U^\dagger$, where U is unitary and ρ is a random mixed quantum state with spectrum distributed according to $\mu_{\mathcal{MP}}$. The width of the orbit decreases as $\frac{1}{N}$ (this will be shown in the next section), while its diameter is given by the distance between two diagonal matrices with opposite order of the eigenvalues [17] $d = D_{\mathrm{Tr}}(p^\uparrow,p^\downarrow) = \int_0^4 x\,\mathrm{sign}(x-M)d\mu_{\mathcal{MP}}(x) \simeq 0.7875$, where M denotes the median $\int_0^M d\mu_{\mathcal{MP}} = 1/2$. A generic state ρ is located at the distance $a = \frac{3\sqrt{3}}{4\pi}$ from the center $\rho_* = \frac{\mathbb{1}}{N}$, its distance to the closest boundary state $\tilde{\rho}$ tends to zero, while the distances to the closest and to the most distant pure state, $|\phi_1\rangle$ and $|\phi_2\rangle$, tend to unity (see Fig. 1).

To study the case of a large environment $K \gg N$, we take the limit $c \to \infty$ of Eq. (10). This distribution, rescaled by a factor $1/\sqrt{2c}$, tends to the Wigner semicircle law $C(y)$ supported on $[-2,2]$, as its R transform converges to $R_W(z) = z$. The average $|y|$ over the semicircle is finite in analogy to (16), but due to the scaling factor $c^{-1/2}$ the typical trace distance with respect to the limiting measure ν_∞ tends to 0 (see Appendix B).

Note that the trace distance asymptotically tends to a fixed value (16), while the average HS distance tends to zero. This is a consequence of the known fact that if the taxi distance

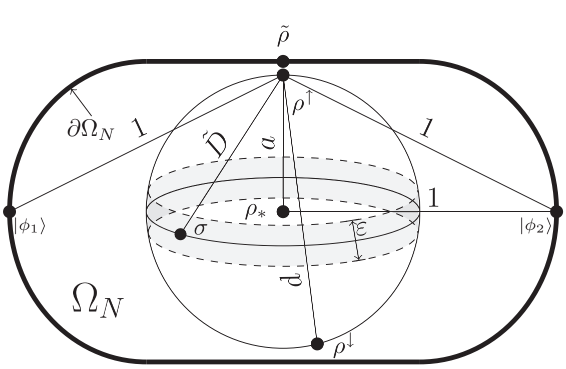

FIG. 1. Sketch of the set Ω_N of mixed states for large dimension: the measure is concentrated along the unitary orbit of a generic state $\rho = \rho^\uparrow$. Its trace distance to the center ρ_* of the set is $a \simeq 0.41$ and to another typical state σ is $\tilde{D} \simeq 0.57$.

TABLE I. Average distances of points in a unit ball of dimension n with respect to the norms L_1, L_2, and L_∞.

n	L_1	L_2	L_∞
1	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
2	$\frac{647+120\pi}{90\pi^2} \simeq 1.1528$	$\frac{128}{45\pi} \simeq 0.9054$	0.8151
3	$\frac{55\pi}{112} \simeq 1.15428$	$\frac{36}{35} \simeq 1.0286$	0.8549
...	...	...	...
∞	∞	$\sqrt{2}$	0

between two points in $\mathbb{R}^{n+1}$ is fixed, their Euclidean distance is typically much smaller since shortcuts across city blocks are allowed. More formally, one can show that a random point from the unit sphere S^n with respect to the taxi metric L_1 is located at the Euclidean distance $L_2 \sim 1/\sqrt{n}$ from the center of the sphere. A similar relation between the trace and the Hilbert-Schmidt distance holds in the space of quantum states and explains result (15).

To visualize this, we present here, for comparison, the average distances between two random points in a unit ball in $\mathbb{R}^n$ distributed according to the Euclidean measure. The distances with respect to the norms L_1, L_2, and L_∞ are provided in Table I. Note that the definition of the trace distance contains the prefactor $\frac{1}{2}$, so the L_1 distance has to be compared with $2D_{\mathrm{Tr}}$. For large dimensions, the full mass of the n ball is concentrated close to its surface, so the L_1 distance between two typical points diverges, while in the case of the set of mixed states, the layer of generic states remains distant from the subset of pure states and the average asymptotic distance becomes finite and deterministic.

IV. APPLICATION OF LEVY'S LEMMA

In this section, we present the Levy's lemma, used to strengthen our results and to characterize the convergence rate of the trace distance between two random states to the mean value $\tilde{D}$. Note that the key results from the previous section, showing that various distances between two generic states converge to their deterministic values, are independent of the Levy's lemma.

Lemma 1 (Levy's lemma). Let $f : S^{m-1} \to \mathbb{R}$ be a Lipschitz-continuous function with Lipschitz constant η, i.e.,

$$|f(x) - f(y)| \leqslant \eta\|x - y\|. \tag{17}$$

For a random element of the sphere $x \in S^{m-1}$ distributed uniformly, for all $\varepsilon > 0$ we get

$$P[|f(x) - \mathbb{E}f(x)| \geqslant \varepsilon] \leqslant 2\exp\left(-\frac{m\varepsilon^2}{9\pi^3\eta^2}\right). \tag{18}$$

Following [4] we are going to use the Levy's lemma for a set of bipartite pure states of a high dimension. Consider two states $|\psi\rangle,|\phi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$ and the function

$$f(|\psi\rangle,|\phi\rangle) = D_{\mathrm{Tr}}(\rho,\sigma), \tag{19}$$

where $\rho = \mathrm{Tr}_B|\psi\rangle\langle\psi|$ and $\sigma = \mathrm{Tr}_B|\phi\rangle\langle\phi|$ represent reduced states. For a fixed state $|\phi\rangle$ the above function has the Lipschitz property with the Lipschitz constant equal to unity. To show this, we write

$$|f(|\psi\rangle,|\phi\rangle) - f(|\xi\rangle,|\phi\rangle)| = \left|\tfrac{1}{2}\|\rho-\sigma\|_1 - \tfrac{1}{2}\|\rho'-\sigma\|_1\right| \leqslant \tfrac{1}{2}\|\rho-\rho'\|_1, \quad (20)$$

where $\rho' = \mathrm{Tr}_B|\xi\rangle\langle\xi|$. Next, we rely on the inequality of Fuchs and van de Graaf [2,18]

$$D_{\mathrm{Tr}}(\rho,\rho') \leqslant \sqrt{1-F(\rho,\rho')}, \quad (21)$$

where the fidelity F between two states reads as

$$F(\rho,\sigma) = (\mathrm{Tr}|\rho^{1/2}\sigma^{1/2}|)^2. \quad (22)$$

Using an elementary inequality $\sqrt{1-x^2} \leqslant \sqrt{2-2x}$ for $x \in [0,1]$, we write

$$D_{\mathrm{Tr}}(\rho,\rho') \leqslant \sqrt{2-2\sqrt{F(\rho,\rho')}} \leqslant \|\,|\psi\rangle - |\xi\rangle\,\|. \quad (23)$$

The last inequality follows from the Uhlmann theorem [2] relating fidelity to the maximal overlap of purifications. This consideration gives us that for a fixed $|\phi\rangle$ the function $f(|\psi\rangle,|\phi\rangle)$ is 1-Lipschitz.

Using the Levy's lemma, for a fixed state $|\phi\rangle$ and $|\psi\rangle$ distributed uniformly on a set of pure states in $\mathcal{H}_N \otimes \mathcal{H}_N$ we obtain

$$\begin{aligned} &P[|f(|\psi\rangle,|\phi\rangle) - \mathbb{E}f(|\psi\rangle,|\phi\rangle)| > \varepsilon] \\ &= P[|D_{\mathrm{Tr}}(\rho,\sigma) - \mathbb{E}D_{\mathrm{Tr}}(\rho,\sigma)| > \varepsilon] \\ &\leqslant \exp\left(-\frac{N^2\varepsilon^2}{9\pi^3}\right). \end{aligned} \quad (24)$$

Let us now consider a sequence of fixed states $\{|\phi_N\rangle\}_N$, such that the empirical distributions of eigenvalues of the reduced states σ_N tend to the Marchenko-Pastur law. For a random state $|\psi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$ with $\rho = \mathrm{Tr}_B|\psi\rangle\langle\psi|$ we define an expectation value with respect to the distribution of $|\psi\rangle$:

$$\Delta_N = \mathbb{E}f(|\psi\rangle,|\phi_N\rangle) = \mathbb{E}D_{\mathrm{Tr}}(\rho,\sigma_N). \quad (25)$$

Unitary invariance of the trace distance and unitary invariance of the distribution of ρ implies that $\Delta_N \to \tilde{D} = \frac{1}{4} + \frac{1}{\pi}$. The triangle inequality gives us the following set of inclusion relations: $\{\rho : |D_{\mathrm{Tr}}(\rho,\sigma_N) - \Delta_N| + |\Delta_N - \tilde{D}| < \varepsilon\} \subset \{\rho : |D_{\mathrm{Tr}}(\rho,\sigma_N) - \tilde{D}| < \varepsilon\}$. For dimension N so large that $|\Delta_N - \tilde{D}| < \varepsilon$ this implies the following bounds:

$$\begin{aligned} &P[|D_{\mathrm{Tr}}(\rho,\sigma_N) - \tilde{D}| < \varepsilon] \\ &\geqslant P[|D_{\mathrm{Tr}}(\rho,\sigma_N) - \Delta_N| + |\Delta_N - \tilde{D}| < \varepsilon] \\ &= P[|D_{\mathrm{Tr}}(\rho,\sigma_N) - \Delta_N| < \varepsilon - |\Delta_N - \tilde{D}|] \\ &\geqslant 1 - 2\exp\left[-\frac{N^2(\varepsilon - |\Delta_N - \tilde{D}|)^2}{9\pi^3}\right]. \end{aligned} \quad (26)$$

Since $|\Delta_N - \tilde{D}| = o(1)$ we arrive at the following result.

Proposition 2. Consider two independent random states ρ and σ distributed according to the Hilbert-Schmidt measure in the set Ω_N of quantum states of dimension N. For sufficiently large N, when $|\mathbb{E}D_{\mathrm{Tr}}(\rho,\sigma) - \tilde{D}| < \frac{\varepsilon}{2}$, their trace distance is close to the number $\tilde{D} = \frac{1}{4} + \frac{1}{\pi}$, as the deviations become exponentially rare:

$$P[|D_{\mathrm{Tr}}(\rho,\sigma) - \tilde{D}| > \varepsilon] \leqslant 2\exp\left[-\frac{N^2(\varepsilon/2)^2}{9\pi^3}\right]. \quad (27)$$

The formal statement above can be interpreted that the orbit of generic quantum states containing the entire mass of the set Ω_N, represented in Fig. 1 by an ε strip, for large N becomes infinitesimally narrow as $\varepsilon \sim 1/N$. Since the exponent in Eq. (27) behaves as $N^2\varepsilon^2$, the width ε of the strip decays with the dimension faster than $N^{\delta-1}$ for any $\delta > 0$.

V. SEVERAL MEASUREMENTS, RELATIVE ENTROPY, AND CHERNOFF BOUND

Consider the Chernoff function [10,19] of two random states $Q_s(\rho,\sigma) = \mathrm{Tr}\rho^s\sigma^{1-s}$ and the Chernoff information $Q := \min_{s\in[0,1]} Q_s$. Based on the discussion in Appendix A, the Chernoff function can be asymptotically represented as a product of two integrals over the Marchenko-Pastur measure

$$\begin{aligned} Q_s &\xrightarrow[N\to\infty]{\text{a.s.}} \int t^s d\mu_{\mathcal{MP}}(t) \int t^{1-s} d\mu'_{\mathcal{MP}}(t) \\ &= \frac{4\Gamma\left(\frac{3}{2}-s\right)\Gamma\left(s+\frac{1}{2}\right)}{\pi\Gamma(3-s)\Gamma(s+2)}. \end{aligned} \quad (28)$$

The above expression, involving the gamma function $\Gamma(x)$, takes its minimal value at $s = \frac{1}{2}$, so that the generic value of the quantum Chernoff information Q reads as

$$Q = \min_s(\lim_{N\to\infty} Q_s) = \left(\frac{8}{3\pi}\right)^2 \simeq 0.7205. \quad (29)$$

The Kullback-Leibler relative entropy reads as $S(\rho\|\sigma) = \mathrm{Tr}\rho(\ln\rho - \ln\sigma)$. Applying again the reasoning presented in Appendix A we get

$$S(\rho\|\sigma) \xrightarrow[N\to\infty]{\text{a.s.}} \int (t\ln t - t\ln s)d\mu_{\mathcal{MP}}(t)d\mu_{\mathcal{MP}}(s) = \frac{3}{2}. \quad (30)$$

In general, the relative entropy is not symmetric, but for two generic states, belonging to the same unitary orbit one has $S(\rho\|\sigma) = S(\sigma\|\rho)$. In the case when one of the states is maximally mixed, the symmetry is broken and we get $S(\rho\|\rho_*) \xrightarrow[N\to\infty]{\text{a.s.}} \frac{1}{2}$, while $S(\rho_*\|\rho) \xrightarrow[N\to\infty]{\text{a.s.}} 1$. Results (29) and (30), compatible with [5], combined with the quantum Sanov [9] and Chernoff bounds [10,19] imply the following.

Proposition 3. Perform m measurements on a generic quantum state ρ of a large dimension. Probability of obtaining results compatible with measurements performed on another generic state σ scales as $\exp(-3m/2)$. In a symmetric setup, probability of erroneous discrimination between both states behaves as $\exp(-64m/9\pi^2)$.

VI. TRANSMISSION DISTANCE AND BURES DISTANCE

The Jensen-Shannon divergence (JSD) characterizes concavity of the entropy. Its quantum analog (QJSD) deals with the von Neumann entropy $S(\rho) = -\mathrm{Tr}\rho\ln\rho$ and is equal to the Holevo quantity χ:

$$\mathcal{J}(\rho,\sigma) := S[(\rho+\sigma)/2] - [S(\rho)+S(\sigma)]/2. \quad (31)$$

Level density of the sum of two generic random states $\rho+\sigma$ is asymptotically described by the $\mathcal{MP}_2$ distribution. As the average entropy for $\mathcal{MP}_c$ distribution for $c \geqslant 1$ reads as [20]

$$\langle S\rangle_c = -\int x \ln x d\mu_{\mathcal{MP}_c(x)} = -\frac{1}{2} - c\ln c, \tag{32}$$

using Eq. (5) it is easy to show that

$$\begin{aligned}
&\mathcal{J}(\rho,\sigma)\\
&= -\frac{1}{2}\int t[\ln t - \ln 2N]d\mu_N^{\rho+\sigma}(t)\\
&\quad+\frac{1}{2}\int t[\ln t - \ln N]\big[d\mu_N^{\rho}(t) + d\mu_N^{\sigma}(t)\big] \xrightarrow[N\to\infty]{\text{a.s.}}\\
&\quad-\frac{1}{2}\int t\ln t d\mu_{\mathcal{MP}_2}(t) + \ln 2\\
&\quad+\int t\ln t d\mu_{\mathcal{MP}}(t) = \frac{1}{4}.
\end{aligned} \tag{33}$$

JSD is a symmetric function of both arguments but it does not satisfy the triangle inequality. However, its square root is known [21] to yield a metric for classical states, which leads to the transmission distance [22] $D_T(\rho,\sigma) = \sqrt{\mathcal{J}(\rho,\sigma)} \to 1/2$.

Squared overlap of two pure states, $|\langle\psi|\phi\rangle|^2$, can be interpreted in terms of probability. A suitable generalization of this notion for mixed states, called *fidelity*, is defined in (22). The average fidelity and root fidelity between random states for small N was analyzed in [23]. To cope with the same problem for large dimensions we use the Fuss-Catalan distribution [24]

$$\mathcal{FC}(x) = \frac{\sqrt[3]{2}\sqrt{3}}{12\pi}\,\frac{[\sqrt[3]{2}(27+3\sqrt{81-12x})^{\frac{2}{3}} - 6\sqrt[3]{x}]}{x^{\frac{2}{3}}(27+3\sqrt{81-12x})^{\frac{1}{3}}}, \tag{34}$$

which is supported in $[0,27/4]$ and describes the asymptotic level density of a product $\rho\sigma$ of two random density matrices. Hence, the average root fidelity can be asymptotically expressed as an integral over the Fuss-Catalan measure

$$\sqrt{F(\rho,\sigma)} = \sum_i \sqrt{\lambda_i(\rho\sigma)} \to \int \sqrt{x}\,\mathcal{FC}(x)dx = \frac{3}{4}. \tag{35}$$

This important result, compatible with [23], determines the limiting behavior of the Bures distance [2]

$$D_B(\rho,\sigma) = \sqrt{2[1-\sqrt{F(\rho,\sigma)}]} \to \sqrt{2(1-3/4)} = \frac{\sqrt{2}}{2}. \tag{36}$$

Observe that the known bounds [19] between fidelity, Chernoff information, and trace distance $F \leqslant Q \leqslant \sqrt{F} \leqslant \sqrt{1-\tilde{D}^2}$ are generically satisfied with a healthy margin, $0.562 < 0.720 < 0.75 < 0.823$.

Making use of the Marchenko-Pastur distribution, we calculate the average root fidelity between a random state ρ and the maximally mixed state ρ_*:

$$\sqrt{F(\rho,\rho_*)} \to \int \sqrt{t}d\mu_{\mathcal{MP}}(t) = \frac{8}{3\pi} \approx 0.8488. \tag{37}$$

In a similar way, one can treat the related Hellinger distance [2] $D_H(\rho,\sigma) = \sqrt{2 - 2\mathrm{Tr}\rho^{\frac{1}{2}}\sigma^{\frac{1}{2}}}$. Making use of the average (29),

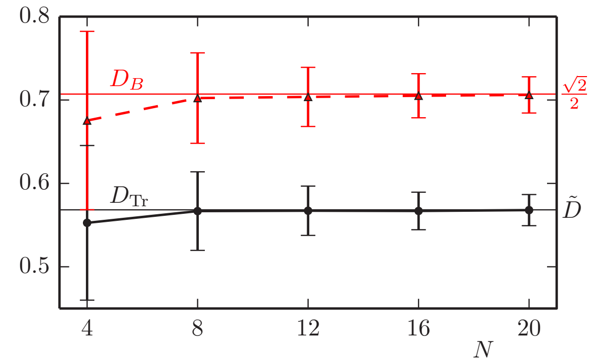

FIG. 2. Dependence of average distance between two generic states on the dimension N. Dashed (red) line shows the Bures distance and solid (black) line shows the trace distance. The horizontal lines mark the asymptotic values.

we find the asymptotic behavior

$$D_H(\rho,\sigma) \to \sqrt{2 - 2\left(\frac{8}{3\pi}\right)^2} \simeq 0.7476. \tag{38}$$

Dependencies of averaged values of selected distances between two generic states on the dimensionality shown in Fig. 2 are consistent with rumors that form the point of view of random matrix theory a dozen is already close to infinity.

VII. AVERAGE ENTANGLEMENT OF QUANTUM STATES

Generic pure quantum states are strongly entangled [4,11]. Integration over the $\mathcal{MP}$ measure may be applied to evaluate average entanglement of a random pure state $|\psi\rangle \in \mathcal{H}_N \otimes \mathcal{H}_N$. Its G concurrence is defined [25] as $G = N(\det\rho)^{1/N}$, where $\rho = \mathrm{Tr}_N|\psi\rangle\langle\psi|$. For large N the concurrence converges almost surely to $\langle G\rangle \to \exp[\int \ln(x)d\mu_{\mathcal{MP}}(x)] = 1/e$, in accordance with [26].

Results developed in the former sections allow us to study the overlap between a given pure bipartite state $|\psi\rangle \in \Omega_N \otimes \Omega_N$ and the nearest maximally entangled state. This overlap, also called maximal entanglement fidelity, is related to another measure $\mathcal{N}$ of entanglement, called negativity. The relation is as follows:

$$\begin{aligned}
\mathcal{N}(|\psi\rangle\langle\psi|) &= \frac{1}{2}\left[\left(\sum_i \sqrt{\lambda_i}\right)^2 - 1\right]\\
&= \frac{1}{2}\left(d \max_{|\psi_+\rangle\in\Omega_{N^2}^{\mathrm{ME}}} |\langle\psi|\psi_+\rangle|^2 - 1\right),
\end{aligned} \tag{39}$$

where $\Omega_{N^2}^{\mathrm{ME}}$ denotes the set of maximally entangled states and λ_i are the Schmidt coefficients of the state $|\psi\rangle$, which satisfy $\sum_i \lambda_i = 1$. For a partial trace, $\rho = \mathrm{Tr}_1|\psi\rangle\langle\psi|$, we have

$$\sum_i \sqrt{\lambda_i} = \mathrm{Tr}\sqrt{\rho}. \tag{40}$$

Now, we consider a random pure state distributed according to the unitarily invariant Haar measure. Integral (37) implies that

$$\frac{1}{\sqrt{N}}\mathrm{Tr}\sqrt{\rho} \to \frac{8}{3\pi}, \tag{41}$$

almost surely when $N \to \infty$. The above result gives us the asymptotic behavior of the negativity for pure states

$$\mathcal{N}(|\psi\rangle\langle\psi|) \simeq \frac{1}{2}\left[\left(\frac{8}{3\pi}\right)^2 N - 1\right]. \tag{42}$$

Formally, we have an almost sure convergence in the asymptotic limit $N \to \infty$:

$$\frac{1}{N}\mathcal{N}(|\psi\rangle\langle\psi|) \to \frac{1}{2}\left(\frac{8}{3\pi}\right)^2 \approx 0.3602. \tag{43}$$

Average entanglement of random mixed states can also be analyzed [27]. As shown by Aubrun [28], the level density of partially transposed random states, written ρ^{T_A}, of a bipartite system obeys asymptotically the shifted semicircular law of Wigner

$$\lambda(\rho^{T_A}) \sim \frac{1}{2\pi c}\sqrt{4c-(x-c)^2}. \tag{44}$$

Therefore, assuming $0 < c < 4$, the fraction $f_{\mathcal{N}}$ of negative eigenvalues depends on the measure

$$\begin{aligned} f_{\mathcal{N}} &= \int_{c-2\sqrt{c}}^{0} \frac{1}{2\pi c}\sqrt{4c-(x-c)^2}dx \\ &= \frac{4\arccos\left(\frac{\sqrt{c}}{2}\right) - \sqrt{4c-c^2}}{4\pi}, \end{aligned} \tag{45}$$

and tends to 0 for $c \to 4$ in agreement with [28]. Furthermore, the average negativity $\mathcal{N}(\rho) = \mathrm{Tr}|\rho^{T_A}| - 1$ of a random mixed state behaves as

$$\begin{aligned} \mathcal{N} &= \int_{c-2\sqrt{c}}^{0} \frac{|x|}{2\pi c}\sqrt{4c-(x-c)^2}dx \\ &= \frac{8\sqrt{4c-c^2} + \sqrt{4c^3-c^4} - 12c\arccos\left(\frac{\sqrt{c}}{2}\right)}{12\pi}. \end{aligned} \tag{46}$$

For the flat measure $c = 1$, we get $f_{\mathcal{N}} = \frac{1}{3} - \frac{\sqrt{3}}{4\pi} \simeq 0.1955$ and $\mathcal{N} = \frac{3\sqrt{3}}{4\pi} - \frac{1}{3} \simeq 0.080$, respectively, which implies that a generic bipartite state is weakly entangled. The above numbers can be compared with the values for the maximally entangled state $\rho_+ = \frac{1}{N}\sum_{ij}|ii\rangle\langle jj|$, which read as $f_{\mathcal{N}}(\rho_+) \to \frac{1}{2}$ and $\mathcal{N}(\rho_+) = N - 1$.

VIII. CLASSICAL PROBABILITY VECTORS AND AVERAGE COHERENCE

Let $q = (q_1, \ldots q_N)$ be a normalized probability vector, so that $q_i \geqslant 0$ and $\sum_i q_i = 1$. A vector q belongs to the probability simplex of dimension $N - 1$, in which one defines a family of *symmetric Dirichlet measures*, parametrized by a real index s:

$$P_s(q) = C_s\, \delta\left(\sum_{i=1}^{N} q_i - 1\right)(q_1 q_2 \ldots q_N)^{s-1}. \tag{47}$$

Here, C_s denotes a suitable normalization constant, which is dimension dependent. Note that the flat measure in the simplex is recovered for $s = 1$, while the case $s = \frac{1}{2}$ corresponds to the statistical measure [2].

Both of the above measures can be related to ensembles of random matrices. The flat measure $s = 1$ describes distribution of squared absolute values of a column (or a row) of a random matrix distributed according to the Haar measure on the unitary group, while the statistical measure $s = \frac{1}{2}$ corresponds to random orthogonal matrices [11]. It is known [29] that eigenstates of matrices pertaining to circular unitary and circular orthogonal ensembles are distributed according to the Haar measure on unitary and orthogonal groups, respectively. Therefore, these distributions characterize statistics of eigenvectors of random unitary matrices pertaining to circular unitary ensemble (CUE) and circular orthogonal ensemble (COE), respectively [30].

It is also possible to relate these measures with ensembles of random quantum states. Writing a projector on a random pure state $|\psi\rangle$ in the computational basis (or the basis state $|1\rangle\langle 1|$ in a random basis), we arrive at a random projector matrix $\omega = |\psi\rangle\langle\psi| = \omega^2$. The coarse-graining channel describes decoherence, as it sends any state ρ into a diagonal matrix

$$\Phi_{CG}(\rho) = \sum_i |i\rangle\langle i|\rho|i\rangle\langle i| = \mathrm{diag}(\rho). \tag{48}$$

Applying this channel to a projector ω we obtain a matrix with classical probability vector $q_i = \omega_{ii} = |\langle i|\psi\rangle|^2$, at the diagonal. If $|\psi\rangle$ represents a complex random vector, then the joint probability distribution $P(q)$ is given by the Dirichlet distribution (47) with $s = 1$, while random real vector leads to the case $s = \frac{1}{2}$.

Integrating out $N - 1$ variables from the Dirichlet distribution (47), one obtains the distribution of a single component $x = q_i$ of the probability vector. For a fixed dimension N these distributions read as, for $s = \frac{1}{2}$ and 1, respectively,

$$\begin{aligned} \tilde{P}_{1/2}(x) &= \frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)} \frac{(1-x)^{(N-3)/2}}{\sqrt{\pi x}}, \\ \tilde{P}_1(x) &= (N-1)(1-x)^{N-2}. \end{aligned} \tag{49}$$

In the asymptotic case $N \to \infty$, the above distributions converge, after an appropriate scaling, to the χ^2_ν distribution with the number ν of degrees of freedom equal to 1 (orthogonal case) and 2 (unitary case) [29]:

$$\begin{aligned} P_{1/2}(t) &= \chi_1^2(t) = \frac{1}{\sqrt{2\pi t}}e^{-t/2}, \\ P_1(t) &= 2\chi_2^2(2t) = e^{-t}. \end{aligned} \tag{50}$$

Here, $t = Nx$ denotes a rescaled variable, such that $\langle t\rangle = 1$, due to the fact that an element $\langle x\rangle$ is proportional to $1/N$. The former formula, also known as the Porter-Thomas distribution, describes statistics of a squared real Gaussian variable, while the latter one describes distribution of the squared modulus of a complex Gaussian variable.

Using the distributions (50) we calculate the asymptotic mean L_1 distance between two probability vectors p and q, distributed according to the statistical D^{s} and the flat measure D^{f}, respectively. Furthermore, we find their root fidelity

$$\sqrt{F(p,q)} = \sum_i \sqrt{p_i q_i}. \tag{51}$$

This quantity is also known as the Bhattacharya coefficient [2]. Finally, we compute the root fidelity and the Bures distance from the maximally mixed vector p_*. We get the following results for the root fidelity $\sqrt{F^{\rm s}}$ and $\sqrt{F^{\rm f}}$ averaged over statistical and flat measures, respectively:

$$\begin{aligned}
\langle\sqrt{F^{\rm s}(p,p_*)}\rangle &\to \int \sqrt{t}P_{1/2}(t)dt = \sqrt{\frac{2}{\pi}} \approx 0.7979,\\
\langle\sqrt{F^{\rm f}(p,p_*)}\rangle &\to \int \sqrt{t}P_{1}(t)dt = \frac{\sqrt{\pi}}{2} \approx 0.8862,\\
\langle\sqrt{F^{\rm s}(p,q)}\rangle &\to \int \sqrt{ts}P_{1/2}(t)P_{1/2}(s)dtds = \frac{2}{\pi} \approx 0.6366,\\
\langle\sqrt{F^{\rm f}(p,q)}\rangle &\to \int \sqrt{ts}P_{1}(t)P_{1}(s)dtds = \frac{\pi}{4} \approx 0.7853.
\end{aligned} \tag{52}$$

Observe that for both measures, the average root fidelity between two random probability vectors asymptotically equals the average fidelity with respect to the uniform vector $\langle\sqrt{F(p,q)}\rangle = \langle\sqrt{F(p,p_*)}\rangle^2 = \langle F(p,p_*)\rangle$.

The above numbers can be directly compared with results obtained in Sec. VI for the quantum case. For instance, the average root fidelity between two random classical states distributed according to the flat measure (0.78) is larger than the analogous result (0.75) for quantum states. Hence, the average Bures distance satisfies the reversed inequality $\langle D_B(p,q)\rangle < \langle D_B(\rho,\sigma)\rangle$.

Mean L_1 distances between classical states averaged over the probability simplex can be expressed as integrals over the measures (49). For large N one can use asymptotic measures (50), which yield the following results:

$$\begin{aligned}
\frac{1}{2}\langle D^{\rm s}(p,p_*)\rangle &\to \frac{1}{2}\int |t-1|P_{1/2}(t)dt = \sqrt{\frac{2}{\pi e}},\\
\frac{1}{2}\langle D^{\rm f}(p,p_*)\rangle &\to \frac{1}{2}\int |t-1|P_{1}(t)dt = \frac{1}{e},\\
\frac{1}{2}\langle D^{\rm s}(p,q)\rangle &\to \frac{1}{2}\int |t-s|P_{1/2}(t)P_{1/2}(s)dtds = \frac{2}{\pi},\\
\frac{1}{2}\langle D^{\rm f}(p,q)\rangle &\to \frac{1}{2}\int |t-s|P_{1}(t)P_{1}(s)dtds = \frac{1}{2}.
\end{aligned} \tag{53}$$

To make a direct comparison with the results obtained in the quantum case easier, each formula above contains the same prefactor $\frac{1}{2}$, present in the definition (2) of the trace distance.

In order to quantify coherence of a state ρ with respect to a given basis, one uses the *relative entropy of coherence* [31], defined as the difference between the entropy of the diagonal of the density matrix and its von Neumann entropy

$$C_{\rm rel.ent}(\rho) = S(\Phi_{CG}(\rho)) - S(\rho). \tag{54}$$

It belongs to $[0,\ln N]$ and is equal to the increase of the entropy of a state ρ under the action of the coarse-graining channel (48).

The maximal value $C_{\rm rel.ent} = \ln N$ is achieved for contradiagonal states [32], for which $\rho_{ii} = 1/N$. If $U_{\min}$ denotes the unitary matrix of eigenvectors of ρ, so that $\rho_D = U_{\min}^\dagger \rho U_{\min}$ is diagonal, then $\rho_C = U_{\max}^\dagger \rho U_{\max}$ is contradiagonal in the related basis $U_{\max} = U_{\min}F$ for any unitary complex Hadamard matrix F, which satisfies $FF^\dagger = \mathbb{1}$ and $|F_{ij}| = 1/N$ [33].

For any pure state the second term in (54) vanishes, so for a complex random pure state of a large dimension N its average entropy of coherence equals

$$\begin{aligned}
\langle C_{\rm rel.ent}(|\psi_{\mathbb{C}}\rangle)\rangle &= \langle S(p)\rangle_{\rm CUE}\\
&= -\int t\ln t P_1(t)dt = \ln N - (1-\gamma),
\end{aligned} \tag{55}$$

where $\gamma \approx 0.5772$ denotes the Euler constant. An analogous result for a random real pure state reads as

$$\begin{aligned}
\langle C_{\rm rel.ent}(|\psi_{\mathbb{R}}\rangle)\rangle &= \langle S(p)\rangle_{\rm COE}\\
&= -\int t\ln t P_{1/2}(t)dt\\
&= \ln N - (2-\gamma-\ln 2),
\end{aligned} \tag{56}$$

where the average entropies were studied first by Jones [34]. Thus, a random state is characterized with a high degree of coherence, close to the maximal one, $\ln N$. Interestingly, this statement holds almost surely for any choice of the basis, with respect to which the decoherence takes place. We also note that these results are compatible with the ones presented in [35] when we let $N \to \infty$.

For comparison, let us consider a random mixed state ρ distributed according the Hilbert-Schmidt measure. Its average entropy is close to the maximal, as it reads as [2] for large dimensions $\langle S(\rho)\rangle_{\rm HS} = \ln N - 1/2$. Therefore, its relative entropy of coherence, equal to the entropy gain induced by the decoherence channel (48), cannot be large. In the generic bases the diagonal elements of a complex Wishart matrix are asymptotically distributed according to the χ^2_ν distribution with $\nu = 2N$ degrees of freedom, the entropy of the diagonal behaves as $S(\mathrm{diag}(\rho)) \simeq \ln N - 1/2N$. Therefore, the relative entropy of coherence of a random mixed state ρ asymptotically tends to a constant

$$\langle C_{\rm rel.ent}(\rho)\rangle = \langle S(\mathrm{diag}(\rho))\rangle - \langle S(\rho)\rangle_{\rm HS} \to 1/2, \tag{57}$$

characteristic to random mixed states in a contradiagonal form, for which $C_{\rm rel.ent}$ is maximal. Thus, the average coherence of random states decreases with their purity, as expected: It behaves asymptotically as $\ln N$ for a random pure state and it is equal to a constant for a random mixed state. This result is compatible with the one presented in Table II in [36], when we multiply the values in the second column by $\ln N$.

Note that result (55) can be used to explain recent findings of [37]. The mean entropy $\langle S_{\rm diag}(\tau)\rangle$ of the diagonal of a quantum state averaged over a sufficiently long time τ tends to the average over the ensemble of random states, which in the case of complex pure states tends to the mean entropy (55) over the flat measure on the simplex. It is equal to $\ln N - (1-\gamma)$ and becomes larger for mixed states. As the entropy of the time-averaged state $S_{\bar\rho}$ is limited by $\ln N$, their difference $\Delta S = S_{\bar\rho} - \langle S_{\rm diag}(\tau)\rangle$ for a generic unitary evolution satisfies the bound $\Delta S \leqslant 1-\gamma$, in agreement with [38]. However, taking an initial real pure state ρ_R and restricting attention to purely imaginary generators, $H = iA$ with A real antisymmetric, $A = -A^T$, one obtains an orthogonal evolution matrix $O = e^{iH} = e^{-A}$, such that the state $\rho_R' = O\rho_R O^T$ remains real. In such a case, the average entropy of the diagonal is equal to the entropy averaged over the statistical measure (56),

TABLE II. Typical distances D_x between generic mixed states ρ and σ and pure states $|\psi\rangle$ and $|\phi\rangle$ of a large dimension N. Definitions of D_{Tr}, D_T, D_B, and D_H are given in the text, while $D_{\mathrm{HS}}(\rho,\sigma) = [\frac{1}{2}\mathrm{Tr}(\rho-\sigma)^2]^{1/2}$, and $D_\infty(\rho,\sigma) = \max_i |\lambda_i(\rho-\sigma)|$, and the entropic distance [45] is $D_E^2(\rho,\sigma) = H_2\{\frac{1}{2}[1-\sqrt{F(\rho,\sigma)}]\}$, with $H_2(x) = -x\ln x - (1-x)\ln(1-x)$. Analytical formulas for the numbers T_1 and E_1 are provided in Appendix C.

x	$D_x(\rho,\frac{1}{N}\mathbb{1})$	$D_x(\rho,\sigma)$	$D_x(\lvert\psi\rangle,\lvert\phi\rangle)$
Tr	$\frac{3\sqrt{3}}{4\pi} \approx 0.414$	$\frac{1}{4}+\frac{1}{\pi} \approx 0.568$	1
HS	0	0	1
∞	0	0	1
T	$T_1 \approx 0.368$	$\frac{1}{2}$	$\sqrt{\ln 2} \approx 0.833$
B	$\sqrt{2-\frac{16}{3\pi}} \approx 0.550$	$\frac{\sqrt{2}}{2} \approx 0.707$	$\sqrt{2} \approx 1.414$
E	$E_1 \approx 0.518$	$\frac{1}{2\sqrt{2}}\sqrt{\ln\frac{88}{77}} \approx 0.614$	$\sqrt{\ln 2} \approx 0.833$
H	$\sqrt{2-\frac{16}{3\pi}} \approx 0.550$	$\sqrt{2-2\left(\frac{8}{3\pi}\right)^2} \approx 0.748$	$\sqrt{2} \approx 1.414$

so that the bound for the entropy difference becomes weaker,

$$\Delta S \leqslant 2 - \gamma - \ln 2 \approx 0.7297. \quad (58)$$

Next, we wish to calculate the L_1 coherence of a generic quantum state

$$C_{L_1} = \sum_{i\neq j} |\rho_{i,j}|. \quad (59)$$

In order to achieve this, observe first that for any $N > 1$ and a random pure state $\rho = |\phi\rangle\langle\phi|$ distributed according to the Haar measure, we have

$$|\rho_{ij}| = \sqrt{|\phi_i\phi_j|^2}. \quad (60)$$

Next, we note that the vector $q = [|\phi_1|^2,\dots,|\phi_N|^2]$ pertains the N-dimensional symmetric Dirichlet distribution. For real states we get $q \sim P_{1/2}(q)$ and for complex states we get $q \sim P_1(q)$. Next, we note that $\mathbb{E}(|\rho_{ij}|) = \mathbb{E}(\sqrt{q_i}\sqrt{q_j})$. To calculate the above expectational value we will use a formula for mixed moments of the Dirichlet distribution. For a random vector q distributed according to the Dirichlet distribution with parameters given by the vector α we have [39]

$$\mathbb{E}\left(\prod_{i=1}^N q_i^{\beta_i}\right) = \frac{\Gamma\left(\sum_{i=1}^N \alpha_i\right)}{\Gamma\left(\sum_{i=1}^N \alpha_i + \beta_i\right)} \prod_{i=1}^N \frac{\Gamma(\alpha_i+\beta_i)}{\Gamma(\alpha_i)}. \quad (61)$$

To calculate the expected coherence of a pure state, we set $\alpha_i = 1$ for complex states and $\alpha_i = \frac{1}{2}$ for real vectors. In both cases, we set $\beta_1 = \beta_2 = \frac{1}{2}$ and $\beta_i = 0$ for $i \geqslant 3$. We also note that for large N, we have $C_{L_1} \simeq N(N-1)\langle|\rho_{ij}|\rangle$. We obtain

$$C_{L_1}{}^{\mathbb{R}}(\psi_{\mathbb{R}}) = (N-1)\frac{2}{\pi},$$
$$C_{L_1}{}^{\mathbb{C}}(\psi_{\mathbb{C}}) = (N-1)\frac{\pi}{4}. \quad (62)$$

Values obtained above are related to the limiting values of the root fidelity for classical probability vectors presented in Eq. (52). This happens because in the asymptotic scenario, the distribution of the probability components and nondiagonal elements of a pure state differ only by a scaling factor.

These values can be compared with the maximal value attained for pure state in a contradiagonal form $\rho_{\mathbb{C}}^{\psi}$ for which $|(\rho_{\mathbb{C}}^{\psi})_{ij}| = 1/N$, so that $C_1(\rho_{\mathbb{C}}^{\psi}) = N(N-1)\frac{1}{N} = N-1$. This result is greater than the mean value (62) for a complex random state in a generic basis, by a factor $\pi/4 \sim 0.785$.

The random mixed state $\rho = GG^\dagger/\mathrm{Tr}GG^\dagger$ can be considered as a normalized Gram matrix for an ensemble of N random complex vectors. In the asymptotic limit, the normalization prefactor $N/\mathrm{Tr}GG^\dagger$ tends to a constant. Thus, in the limiting case we may consider

$$\rho_{ij} = \frac{1}{N}\sum_{k=1}^N G_{ik}\overline{G}_{jk}. \quad (63)$$

From the central limit theorem we get that ρ_{ij} has a normal distribution for $N \gg 1$. This implies that the absolute value $|\rho_{ij}|$ is described, up to a scaling factor, by an appropriate χ_ν distribution. It is the distribution of the square root of the sum of squares of independent random variables having a standard normal distribution [40]. The χ_ν distribution has a single parameter, the number ν of degrees of freedom, equal to the number of summed normal variables. Hence, we get

$$P_{\mathbb{R}}(y) = \chi_1(y) = \sqrt{\frac{2}{\pi}}e^{-y^2/2},$$
$$P_{\mathbb{C}}(y) = \sqrt{2}\chi_2(\sqrt{2}y) = 2ye^{-y^2}, \quad (64)$$

where $y = \sqrt{N}|\rho_{ij}|$. Figure 3 shows that statistical distribution of modulus of off-diagonal elements of a random mixed state which can be asymptotically described by the χ distribution. This property allows us to describe asymptotic behavior of the coherence of real and complex random mixed states:

$$C_{L_1}^{\mathbb{R}}(\rho^{\mathbb{R}}) \simeq \sqrt{N}\int_0^\infty xP_{\mathbb{R}}dx = \sqrt{N}\sqrt{\frac{2}{\pi}}, \quad (65)$$
$$C_{L_1}^{\mathbb{C}}(\rho^{\mathbb{C}}) \simeq \sqrt{N}\int_0^\infty xP_{\mathbb{C}}dx = \sqrt{N}\frac{\sqrt{\pi}}{2}. \quad (66)$$

In the above, we have shown that the L_1 coherence, C_{L_1}, scales as the dimension N for generic pure states and as $\sqrt{N}$ for typical mixed states. Note that expression (66) asymptotically holds also for contradiagonal states, for which $\rho_{ii} = 1/N$.

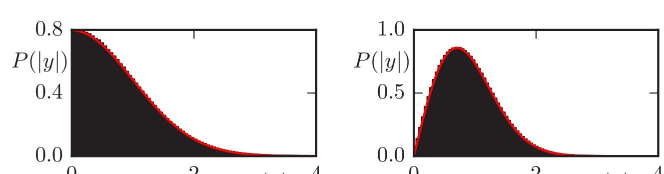

FIG. 3. Histogram of absolute values of rescaled off-diagonal elements $y = \rho_{ij}$ of random mixed state. Left real state and right complex state. The solid (red) curves represent the distributions $\chi_1(|y|)$ (left) and $\sqrt{2}\chi_2(\sqrt{2}|y|)$ (right).

IX. DYNAMICAL SYSTEM-COUPLED QUANTUM KICKED TOPS

To show a direct link with the theory of quantized chaotic systems, we analyze the model of quantum kicked top. It is described by the angular momentum operators J_x, J_y, J_z, satisfying standard commutation rules. The dynamics of a single system consists of a periodic unitary evolution in the Hilbert space of dimension $N = 2j + 1$, followed by an infinitesimal perturbation characterized by the kicking strength k. To analyze the effects of quantum entanglement, the model consisting of two coupled tops described by the Hamiltonian $H(t) = H_1(t) + H_2(t) + H_{12}(t)$, with $H_i(t) = p_i J_{y_i} + \frac{k}{2j_i} J_{z_i}^2 \sum_n \delta(t-n)$, $H_{12}(t) = \epsilon/\bar{j}(J_{z_1} \otimes J_{z_2}) \sum_n \delta(t-n)$, and $\bar{j} = \frac{j_1+j_2}{2}$. This model was studied in [41–43].

We set the standard values of the parameters $p_1 = p_2 = \pi/2$ and consider the unitary one-step time evolution operator $U = U_{12}(U_1 \otimes U_2)$, where

$$U_i = \exp\left(-i\frac{k}{2j_i}J_{z_i}^2\right)\exp\left(-i\frac{\pi}{2}J_{y_i}\right), \quad i = 1,2$$

$$U_{12} = \exp\left(-i\frac{\epsilon}{\bar{j}}J_{z_1} \otimes J_{z_2}\right). \tag{67}$$

Consider a pure separable state $|l\rangle \otimes |l\rangle$ evolved unitarily for t steps, which gives $|\psi_l^t\rangle = U^t(|l\rangle \otimes |l\rangle)$ for $l = 1,2$. After the unitary dynamics, we perform partial trace over the second subsystem, thus defining two states:

$$\sigma_1(t) = \mathrm{Tr}_2|\psi_1^t\rangle\langle\psi_1^t| \quad \text{and} \quad \sigma_2(t) = \mathrm{Tr}_2|\psi_2^t\rangle\langle\psi_2^t|, \tag{68}$$

and study the spectrum of their difference $\Gamma = \sigma_1 - \sigma_2$. Level density of Γ obtained for $k = 6$, corresponding to the chaotic regime, with Wigner level spacing statistics $\epsilon = 0.01$ and four values of the dimensionality ratio $c = (2j_2 + 1)/(2j_1 + 1)$, can be described by distribution (10) as shown in Fig. 4. This observation confirms the conjecture that a quantized deterministic chaotic system may lead to generic, random states. Consider the trace distance $D_{\mathrm{Tr}}(\sigma_1(t),\sigma_2(t))$, which at

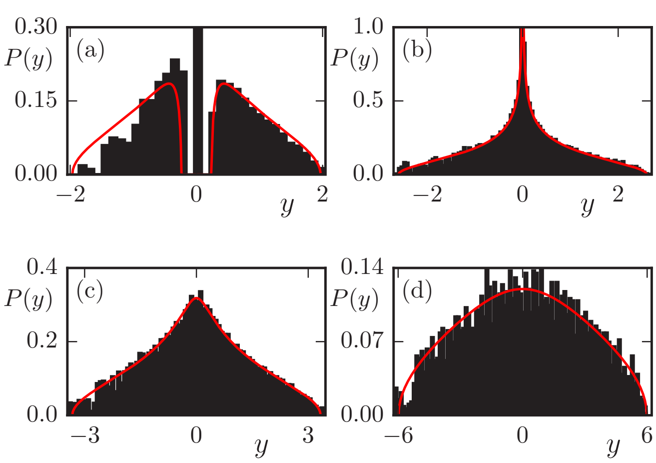

FIG. 4. Symmetrized Marchenko-Pastur distribution (10) for $c = 0.2$ (a), 0.5 (b), 1.0 (c), 4.0 (d) denoted by the solid (red) line. Histograms represent numerical results for the model of coupled kicked tops (67), (68) with $N = 100$ cumulated out of 100 realizations of the system.

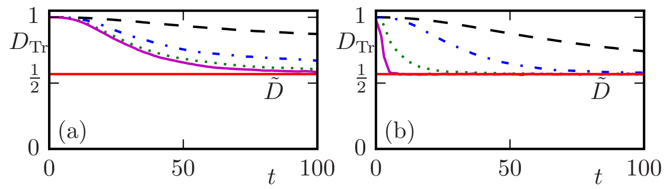

FIG. 5. Trace distance D_{Tr} between initially orthogonal reduced states of the kicked tops as a function of time t obtained for $j_1 = j_2 = 60$. Convergence to the asymptotic value $\tilde{D}$ represented by the horizontal line depends on (a) kicking strength $k = 3.2$ (upper dashed line), $k = 3.5, 3.8$ and $k = 4.0$ (solid line) and (b) the coupling constant $\epsilon = 0.005$ (upper dashed line), $\epsilon = 0.01$, 0.1 and $\epsilon = 1$ (solid line).

$t = 0$ is equal to unity as the initial states are orthogonal. For larger times t the trace distance tends to the number $\tilde{D}$ derived in (16) and the convergence rate is faster for strongly chaotic systems (large k) and large coupling strength ϵ (see Fig. 5).

The trace distance $D_{\mathrm{Tr}}(\sigma_1(t),\sigma_2(t))$ between two initially orthogonal states undergoing the dynamics of the coupled kicked top (67) decreases monotonously and tends to the universal number $\tilde{D}$. The rate of convergence to the asymptotic value depends on the parameters of the model, as shown in Fig. 5. However, if we repeat the procedure taking for initial states the tensor products of two coherent states localized nearby, the trace distance between reduced states initially grows in time. This feature is related to the fact that the dynamics taking place in the reduced system is not Markovian [44] and that there exist correlations between both systems.

Note that the investigated state of the first subsystem is obtained by a reduction over the second subsystem $\sigma(t) = \mathrm{Tr}_2|\psi(t)\rangle\langle\psi(t)|$. Thus, to obtain the next iteration $\sigma(t+1)$, one needs to purify $\sigma(t)$ in a specific way to get the bipartite pure state $|\psi(t)\rangle$, evolve it unitarily into $|\psi(t+1) = U|\psi\rangle$, and perform the partial trace $\sigma(t+1) = \mathrm{Tr}_2|\psi(t+1)\rangle\langle\psi(t+1)|$. The purification procedure is not unique, and the information encoded in σ is not sufficient to recover the desired pure state $|\psi(t)\rangle$. Hence, the dynamics of the reduced state $\sigma(t) \to \sigma(t+1)$ is not Markovian. Furthermore, the quantum dynamics is explicitly defined for a full system consisting of two coupled tops, but the reduced dynamics of a single subsystem is not well defined independently of the correlations.

X. CONCLUDING REMARKS

We evaluated common distances between two generic quantum states. Due to the concentration of measure in high dimensions, these distances converge to deterministic values (see Table II). Our results are directly applicable for quantum hypothesis testing [19] as they imply concrete bounds on the distinguishability between generic states. Asymptotic value of the root fidelity $\sqrt{F} = \frac{3}{4}$ can be used as a universal benchmark for future theoretical and experimental studies based on this quantity and Bures distance.

Although obtained expectation values are exact in the asymptotic limit, our numerical results presented in Fig. 2

show that they can be used for N of the order of 10. Furthermore, results obtained improve our intuition concerning the structure of the body Ω_N of quantum states and describe coherence with respect to a generic basis of a typical pure and mixed quantum state. Moreover, we demonstrated that a typical mixed state of a large bipartite system is weakly entangled.

Results presented in this paper are obtained with use of the free probability calculus and hold in the asymptotic limit $N \to \infty$. It would be therefore interesting to apply other methods to obtain finite-size corrections to the expressions derived in this work and to show analytically for what system sizes the correction terms can be neglected in practice. In the case of the $\mathcal{MP}$ distribution, this has been studied in [46], although these results do not directly apply in the case studied in this work. The free convolution works only in the asymptotic limit, thus obtaining a distribution similar to $\mathcal{SMP}$ for finite N requires careful treatment. The problem of estimating the average coherence for quantum systems of a finite size was recently studied in [35,36] and their results are consistent with our asymptotic expressions.

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APPENDIX A: LIMITING BEHAVIOR OF FUNCTIONS OF TWO INDEPENDENT STATES

In this appendix, we consider the limiting value of the trace of a product of two functions on random states. Let ρ and σ be random mixed states generated according to the flat measure ν_{HS} in Ω_N with eigenvalues described by the measures $\mu_N^{(\rho)} = \frac{1}{N}\sum_i \delta_{N\lambda_i(\rho)}$ and $\nu_N^{(\rho)} = \frac{1}{N}\sum_i \delta_{N\lambda_i(\sigma)}$. For large N with probability one, the measures μ_N and ν_N converge weakly to $\mu_{\mathcal{MP}}$. Because ρ and σ are asymptotically free random matrices, thus, for any analytic functions g and h the matrices $g(\rho)$ and $h(\sigma)$ are also asymptotically free. The product $g(\rho)h(\sigma)$ has a limiting, nonrandom distribution of eigenvalues, given by an appropriate multiplicative free convolution. As a result, the normalized trace of the product tends, almost surely, to a deterministic quantity, equal to the mean

$$\text{Tr} g(\rho)h(\sigma) \xrightarrow[N\to\infty]{\text{a.s.}} m, \tag{A1}$$

where $m = \lim_{N\to\infty} \mathbb{E}\text{Tr} g(\rho)h(\sigma)$.

Now, we will calculate the limiting value of the mean trace of the product. The eigendecomposition gives us

$$\mathbb{E}\text{Tr} g(\rho)h(\sigma) = \mathbb{E}\sum_{ij} g[\lambda_i(\rho)]h[\lambda_j(\sigma)]|U_{ij}|^2, \tag{A2}$$

where U is a unitary transition matrix from eigenbasis of ρ to eigenbasis of σ. The independence of eigenvectors distribution from eigenvalues distribution gives us

$$\mathbb{E}\text{Tr} g(\rho)h(\sigma) = \mathbb{E}\sum_{ij} g[\lambda_i(\rho)]h[\lambda_j(\sigma)](\mathbb{E}|U_{ij}|^2). \tag{A3}$$

Since the distribution of ρ and σ is invariant with respect to unitary rotations, U has a Haar distribution, for which we have, by the permutation symmetry, $\mathbb{E}|U_{ij}|^2 = \frac{1}{N}$. This gives us

$$\mathbb{E}\text{Tr} g(\rho)h(\sigma) = \frac{1}{N}\mathbb{E}\text{Tr} g(\rho)\mathbb{E}\text{Tr} h(\sigma). \tag{A4}$$

Now, we can make use of the weak convergence of the eigenvalues distributions similarly to (5) and get

$$\mathbb{E}\text{Tr} g(\rho)h(\sigma) \xrightarrow[N\to\infty]{\text{a.s.}} \int B_{g,h}(s,t)d\mu_{\mathcal{MP}}(t)\mu_{\mathcal{MP}}(s), \tag{A5}$$

where

$$B_{g,h}(s,t) = \lim_{N\to\infty} Ng\left(\frac{t}{N}\right)h\left(\frac{s}{N}\right). \tag{A6}$$

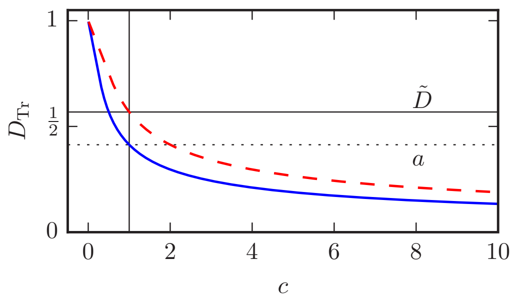
APPENDIX B: DISTANCES AS A FUNCTION OF THE RECTANGULARITY PARAMETER

In this appendix, we present the average trace and the rescaled Hilbert-Schmidt distance between random states distributed with respect to the induced measure ν_K. It is convenient to parametrize the measure by the rectangularity parameter $c = K/N$, where N stands for the dimension of the principal system, while K denotes the dimension of the environment.

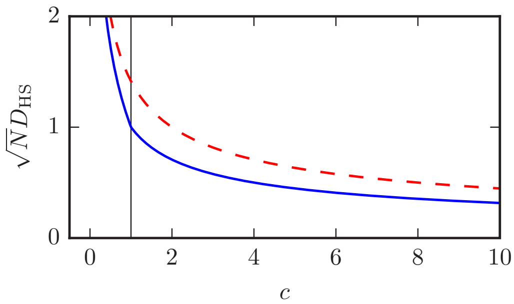
Following (5) we integrate the function $|t - c|$ over the Marchenko-Pastur distribution $\mathcal{MP}_c$ to get the average trace distance $D_{\text{Tr}}(\rho,\rho_*)$ between a random state ρ and the center of the set $\rho_* = \mathbb{1}/N$. In analogy to Eq. (16), computing the average $|x|$ with respect to the symmetrized MP distribution $\mathcal{SMP}_c$ one obtains the mean value of the trace distance $D_{\text{Tr}}(\rho,\sigma)$ between two random states.

These quantities for the trace distance are shown in Fig. 6(a) as a function of the rectangularity parameter c. In the limit $c \to \infty$, the distribution $\mathcal{SMP}_c$ tends to the circular law rescaled by $1/\sqrt{2c}$ so the distance $D_{\text{Tr}}(\rho,\sigma)$ tends to zero as $4\sqrt{2}/(3\pi\sqrt{c})$. Note that above average improves the bound recently derived [47] in order to show that exponential decay of correlations implies area law. Integrating the function $|t - c|$ over the $\mathcal{MP}_c$ distribution, we obtain that the average trace distance from the maximally mixed state ρ_*, $D_{\text{Tr}}(\rho,\rho_*)$, tends to zero as $4/(3\pi\sqrt{c})$.

In order to calculate the rescaled Hilbert-Schmidt distance between two generic states $\sqrt{N}D_{\text{HS}}(\rho,\sigma)$, we evaluate the



(a) D_{Tr}



(b) $\sqrt{N}D_{\mathrm{HS}}$

FIG. 6. Trace distance (a) and rescaled Hilbert-Schmidt distance (b) as a function of the rectangularity parameter c. Solid blue curve denotes the distance of a random state ρ from the maximally mixed state ρ_*. Dashed red curve shows the distance between two random states. The vertical lines mark $c = 1$. The solid horizontal line marks $\tilde{D}$, the distance between two generic states for $c = 1$. The dashed horizontal line marks a, the distance of a generic state form the maximally mixed state ρ_*.

second moment of the $\mathcal{SMP}_c$ distribution. In the limit $c \to \infty$, the distance tends to zero as $\sqrt{2}/\sqrt{c}$. To obtain average Hilbert-Schmidt distance of a generic state from the maximally mixed state $\sqrt{N}D_{\mathrm{HS}}(\rho,\rho_*)$, we integrate the function $(t-c)^2$ over the Marchenko-Pastur distribution $\mathcal{MP}_c$. In the limit $c \to \infty$, this distance behaves as $1/\sqrt{c}$ [see Fig. 6(b)]. Note that for both distances in the limit $c \to \infty$ we get $D_x(\rho,\sigma) = \sqrt{2}D_x(\rho,\rho_*)$, where x stands either for the trace or the Hilbert-Schmidt distance.

APPENDIX C: TRANSMISSION DISTANCE AND ENTROPIC DISTANCE

We provide here an analytical expression for two distances listed in Table II. Square root of quantum Jensen-Shannon divergence (QJSD) leads to the transmission distance [22] $D_T(\rho,\sigma) = \sqrt{\mathcal{J}(\rho,\sigma)}$. Taking one random state ρ and the maximally mixed state ρ_* we find that for large dimension N this distance converges to

$$T_1^2 = [D_T(\rho,\rho_*)]^2 \to \frac{1}{8} + \frac{\sqrt{5}}{16} + \frac{15}{16}\ln 2 + \ln \sqrt[16]{4870847 - 2178309\sqrt{5}}, \quad \text{(C1)}$$

so that $T_1 \simeq 0.368$. In a similar way, we find the entropic distance defined in the caption to the Table I:

$$E_1 = D_E(\rho,\rho_*) \to \sqrt{\frac{3\pi \ln\left(\frac{36\pi^2}{9\pi^2-64}\right) - 16\coth^{-1}\left(\frac{3\pi}{8}\right)}{6\pi}} = 0.518. \quad \text{(C2)}$$

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