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Product numerical range in a space with tensor product structure

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ABSTRACT

We study operators acting on a tensor product Hilbert space and investigate their product numerical range, product numerical radius and separable numerical range. Concrete bounds for the product numerical range for Hermitian operators are derived. Product numerical range of a non-Hermitian operator forms a subset of the standard numerical range containing the barycenter of the spectrum. While the latter set is convex, the product range needs not to be convex nor simply connected. The product numerical range of a tensor product is equal to the Minkowski product of numerical ranges of individual factors.

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1. Introduction

Let X be an operator acting on an N -dimensional Hilbert space \mathcal{H}_N . Let $\Lambda(X)$ denote its *numerical range*, i.e. the set of all λ such that there exists a normalized state $|\psi\rangle \in \mathcal{H}_N$, $\|\psi\| = 1$, which satisfies $\langle \psi | X | \psi \rangle = \lambda$.

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In this work we study an analogous notion defined for operators acting on a composite Hilbert space with a tensor product structure. Consider first a bi-partite Hilbert space,

$$\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M, \quad (1)$$

of a composite dimension $N = KM$.

Definition 1 (*Product numerical range*). Let X be an operator acting on the composite Hilbert space (1). We define the *product numerical range* $\Lambda^\otimes(X)$ of X , with respect to the tensor product structure of \mathcal{H}_N , as

$$\Lambda^\otimes(X) = \{ \langle \psi_A \otimes \psi_B | X | \psi_A \otimes \psi_B \rangle : |\psi_A\rangle \in \mathcal{H}_K, |\psi_B\rangle \in \mathcal{H}_M \}, \quad (2)$$

where $|\psi_A\rangle \in \mathcal{H}_K$ and $|\psi_B\rangle \in \mathcal{H}_M$ are normalized.

Definition 2 (*Product numerical radius*). Let $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$ be a tensor product Hilbert space. We define the *product numerical radius* $r^\otimes(X)$ of X , with respect to this tensor product structure, as

$$r^\otimes(X) = \max\{|z| : z \in \Lambda^\otimes(X)\}. \quad (3)$$

The notion of numerical range of a given operator, also called “field of values” [1, Chapter 1], has been extensively studied during the last few decades [2–4] and its usefulness in quantum theory has been emphasized [5]. Several generalizations of numerical range are known – see e.g. [1, Section 1.8]. In particular, Marcus introduced the notion of *decomposable numerical range* [6,7], the properties of which are a subject of considerable interest [8,9].

The product numerical range, which forms the central point of this work, can be considered as a particular case of the decomposable numerical range defined for operators acting on a tensor product Hilbert space. This notion may also be considered as a numerical range *relative* to the proper subgroup $U(K) \times U(M)$ of the full unitary group $U(KM)$.

In papers [10–13] the same object was called *local numerical range* in view of notation common in quantum mechanics. This name seems to be natural for the physicists audience, but to be more consistent with the mathematical terminology we will use in this paper the name *product numerical range*, although a longer version “local product numerical range” would be even more accurate.

In a recent paper of Dirr et al. [10] some geometric properties of the product numerical range and product C -numerical range were investigated. Another paper of the same group [12] demonstrates the possible application of these concepts in the theory of quantum information and the theory of quantum control. Product numerical range of unitary operators was very recently used by Duan et al. to tackle the problem of local distinguishability of unitary operators [11]. Knowing product numerical range of a Hermitian operator one could solve other important problems in the theory of quantum information, as establishing whether a given quantum map is positive, or obtaining bounds for the minimum output entropy of a quantum channel [14].

The main goal of this paper is to stimulate research on product numerical ranges. We derive several bounds for the product numerical range of a Hermitian operator defined on a space with a twofold tensor structure, which corresponds to a bi-partite physical system. In the non-Hermitian case we use the relation between the product numerical range of a tensor product and the Minkowski product of numerical ranges to establish a general bound for the product numerical range based on the operator Schmidt decomposition. We show that a tensor product of two operators, acting on a Hilbert space with a twofold tensor structure, has a simply connected product numerical range. A similar property does not hold for operators acting on a space with a larger number of factors. We introduce a class of product diagonalizable operators, for which a convenient method to parameterize product numerical range is proposed.

Although this work leaves several problems related to product numerical range unsolved, we believe it could point out directions for further mathematical research, which will find direct applications in the theory of quantum information. In order to invite reader to contribute to this field we conclude the paper with a list of exemplary open problems.

2. Properties of product numerical range

In this section we are going to consider arbitrary operators acting on a bipartite Hilbert space (1).

2.1. General case

It is not difficult to establish the basic properties of the product numerical range which are independent of the partition of the Hilbert space and of the structure of the operator. We list them below leaving some simple items without a proof.

2.1.1. Basic properties

We begin this section with some simple topological facts concerning product numerical range for general operators.

Property 1. *Product numerical range forms a connected set in the complex plane.*

Proof. The above is true because product numerical range is a continuous image of a connected set. \square

Property 2 (Subadditivity). *Product numerical range is subadditive. For all $A, B \in \mathbb{M}_n$*

$$\Lambda^\otimes(A + B) \subset \Lambda^\otimes(A) + \Lambda^\otimes(B). \tag{4}$$

Property 3 (Translation). *For all $A \in \mathbb{M}_n$ and $\alpha \in \mathbb{C}$*

$$\Lambda^\otimes(A + \alpha \mathbb{1}) = \Lambda^\otimes(A) + \alpha. \tag{5}$$

Property 4 (Scalar multiplication). *For all $A \in \mathbb{M}_n$ and $\alpha \in \mathbb{C}$*

$$\Lambda^\otimes(\alpha A) = \alpha \Lambda^\otimes(A). \tag{6}$$

Property 5 (Product unitary invariance). *For all $A \in \mathbb{M}_{mn}$*

$$\Lambda^\otimes\left((U \otimes V)A(U \otimes V)^\dagger\right) = \Lambda^\otimes(A), \tag{7}$$

for unitary $U \in \mathbb{M}_m$ and $V \in \mathbb{M}_n$.

Property 6. *Let $A \in \mathbb{M}_m$ and $B \in \mathbb{M}_n$*

1. *If one of them is normal then the numerical range of their tensor product coincides with the convex hull of the product numerical range,*

$$\Lambda(A \otimes B) = \text{Co}(\Lambda^\otimes(A \otimes B)). \tag{8}$$

2. *If $e^{i\theta}A$ is positive semidefinite for some $\theta \in [0, 2\pi)$, then*

$$\Lambda(A \otimes B) = \Lambda^\otimes(A \otimes B). \tag{9}$$

Proof. This property can be proven using Lemma 1 stated in the following subsection and Theorem 4.2.16 in [1]. \square

$$\text{Let } H(A) = \frac{1}{2}(A + A^\dagger) \text{ and } S(A) = \frac{1}{2}(A - A^\dagger).$$

Property 7 (Projection). *For all $A \in \mathbb{M}_n$*

$$\Lambda^\otimes(H(A)) = \text{Re } \Lambda^\otimes(A) \tag{10}$$

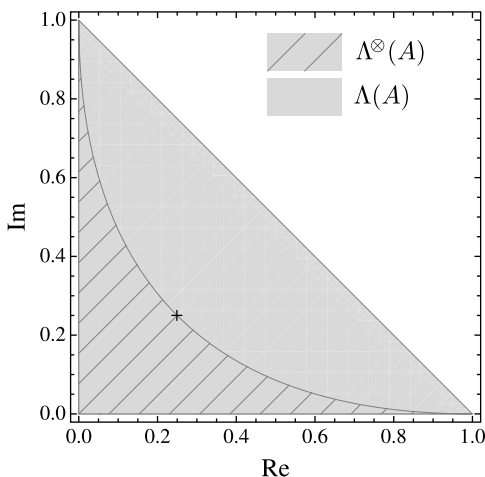


Fig. 1. The comparison of the numerical range (gray triangle) and the product numerical range (dashed set) for matrix A defined in Eq. (12).

and

$$\Lambda^\otimes(S(A)) = i \operatorname{Im} \Lambda^\otimes(A). \tag{11}$$

Property 8. The product numerical range does not need to be convex.

Proof. Consider the following simple example. \square

Example 1. Let

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + i \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{12}$$

Then A is normal matrix with eigenvalues $0, 1, i$. It is easy to see that $1 \in \Lambda^\otimes(A)$ and $i \in \Lambda^\otimes(A)$, but $(1 + i)/2 \notin \Lambda^\otimes(A)$. Actually, by direct computation we have

$$\Lambda^\otimes(A) = \{x + yi : 0 \leq x, 0 \leq y, \sqrt{x} + \sqrt{y} \leq 1\}. \tag{13}$$

Product numerical range of matrix A is presented in Fig. 1. \square

Product numerical range forms a nonempty set for a general operator. In particular it contains the barycenter of the spectrum.

Property 9. Product numerical range of $A \in \mathbb{M}_{KM}$ includes the barycenter of the spectrum,

$$\frac{1}{KM} \operatorname{tr} A \in \Lambda^\otimes(A). \tag{14}$$

Proof. Let A be an operator acting on a tensor product Hilbert space $\mathcal{H}_K \otimes \mathcal{H}_M$. Let us write

$$\frac{1}{KM} \operatorname{tr} A = \frac{1}{KM} \operatorname{tr} (A(\mathbb{1} \otimes \mathbb{1})) = \frac{1}{K} \sum_{i=1}^M \frac{1}{M} \operatorname{tr} (A(\mathbb{1} \otimes |\psi_i\rangle\langle\psi_i|)), \tag{15}$$

where $\{\psi_i\}_{i=1}^M$ is an arbitrary orthonormal basis in \mathcal{H}_M . The last sum in (15) is a convex combination of elements in the numerical range of $\operatorname{tr}_1 A$, where tr_1 denotes the partial trace with respect to \mathcal{H}_K . Remember that $\Lambda(\operatorname{tr}_1 A)$ is convex. Hence there exists an element $\psi \in \mathcal{H}_M$ of norm one such that

$$\sum_{i=1}^M \frac{1}{M} \text{tr} (A (\mathbb{1} \otimes |\psi_i\rangle\langle\psi_i|)) = \langle\psi|\text{tr}_1 A|\psi\rangle = \text{tr} (A (\mathbb{1} \otimes |\psi\rangle\langle\psi|)). \tag{16}$$

By repeating the same trick, we can replace the remaining identity with a single one-dimensional projector and obtain

$$\begin{aligned} \frac{1}{KM} \text{tr} A &= \frac{1}{K} \text{tr} (A (\mathbb{1} \otimes |\psi\rangle\langle\psi|)) \\ &= \text{tr} (A (|\phi\rangle\langle\phi| \otimes |\psi\rangle\langle\psi|)) = \langle\phi \otimes \psi|A|\phi \otimes \psi\rangle, \end{aligned} \tag{17}$$

for some $|\phi\rangle \in \mathcal{H}_K$ and $|\psi\rangle \in \mathcal{H}_M$ of norm one. The last equality in (17) means the same as $(\text{tr} A)/KM \in A^{\otimes}(A)$, which we wanted to prove. \square

In the particular case $\text{tr} A = 0$, Property 9 has already been used in [11]. The above reasoning can be generalized to the multipartite case (cf. Section 3.1).

Note that the barycenter does not have to lie in the interior of the product numerical range. For example, point $(\frac{1}{4}, \frac{1}{4})$, denoted by the black cross in Fig. 1.

Property 10. *Product numerical radius is a vector norm on matrices, but it is not a matrix norm. Product numerical radius is invariant with respect to local unitaries, which have the tensor product structure.*

Proof. The only hard part of the proof is positivity. In the Hermitian case this follows from Proposition 3, the non-Hermitian case can be reduced to Hermitian by the projection property. \square

Property 11. *If $A \in \mathbb{M}_{KM}$ can be diagonalized to Σ using product unitary matrices $U \in \mathbb{M}_K, V \in \mathbb{M}_M$ (i.e. there exist unitary U, V such that $(U \otimes V)A(U^\dagger \otimes V^\dagger) = \Sigma$) then*

$$A^{\otimes}(A) = \{z : z = ((p_1, p_2, \dots, p_K) \otimes (q_1, q_2, \dots, q_M)) \cdot (\Sigma_{1,1}, \Sigma_{2,2}, \dots, \Sigma_{KM,KM})\}, \tag{18}$$

where $\sum_k p_k = 1, \sum_m q_m = 1$ and $p_k, q_m \geq 0$.

Proof. This follows from more general Proposition 6 in Section 3.1. \square

Example 2. We can apply the last property to Example 1. The matrix in this example is given by

$$A = \text{diag}(1, 0, 0, i). \tag{19}$$

By Property 11 we have the following parametrization of the product numerical range of A :

$$pq + i(1 - p)(1 - q), \quad p, q \in [0, 1]. \tag{20}$$

2.1.2. Relation to Minkowski geometric algebra

We shall start this section by recalling the Minkowski geometric algebra of complex sets as developed by Farouki et al. [15]. For any sets Z_1 and Z_2 in the complex plane one defines their *Minkowski sum*,

$$Z_1 \boxplus Z_2 = \{z : z = z_1 + z_2, z_1 \in Z_1, z_2 \in Z_2\} \tag{21}$$

and *Minkowski product*,

$$Z_1 \boxtimes Z_2 = \{z : z = z_1 z_2, z_1 \in Z_1, z_2 \in Z_2\}. \tag{22}$$

Note that the above operations are not denoted by \oplus and \otimes as in the original paper [15], in order to avoid the risk of confusion with the direct sum of operators or their tensor product.

A simple lemma concerning the Minkowski sum and product has interestingly deep consequences. Let us define the Kronecker sum of two operators as

$$A \boxplus B = A \otimes \mathbb{1} + \mathbb{1} \otimes B. \tag{23}$$

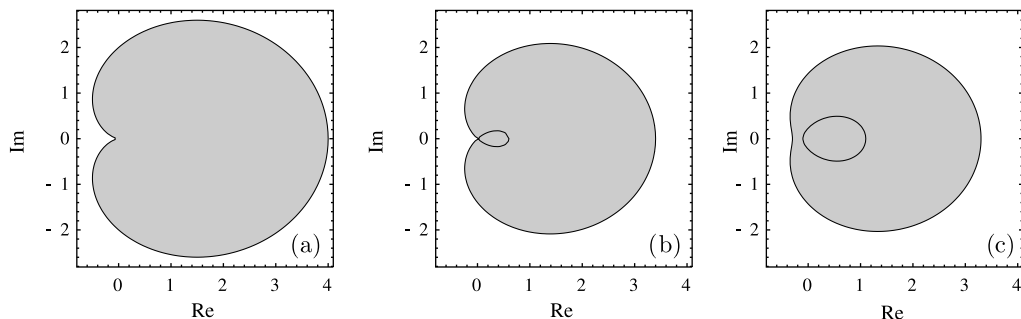


Fig. 2. Product numerical range for the operator Y defined in (26) with (r_1, r_2) equal to (a) $(1, 1)$ (cardioid), (b) $(0.7, 1)$ (limaçon of Pascal), and (c) $(0.5, 1.2)$ (Cartesian oval).

Lemma 1. Product numerical range of the Kronecker product of arbitrary operators is equal to the Minkowski product of the numerical ranges of both factors,

$$\Lambda^{\otimes} (A \otimes B) = \Lambda(A) \boxtimes \Lambda(B), \tag{24}$$

while product numerical range of the Kronecker sum of arbitrary operators is equal to the Minkowski sum of the numerical ranges of both factors,

$$\Lambda^{\otimes} (A \oplus B) = \Lambda(A) \boxplus \Lambda(B). \tag{25}$$

Proof follows directly from the definition of the product numerical range. First part of the above lemma has already been used in [10]. Observe that the definition (22) and the property (24) can be naturally generalized to an arbitrary number of factors.

Thus the problem of finding the product numerical range of a tensor product can be analyzed by checking what sorts of subsets of the complex plane one can obtain by multiplying two or more numerical ranges. This very problem has recently been investigated in a series of papers by Farouki et al. – see [15,16] and references therein. For instance, the structure of the Minkowski product of several discs in the complex plane was analyzed in detail by Farouki and Pottmann [16].

The above results concerning Minkowski product can be used directly to find the product numerical range of a tensor product of an arbitrary number of factors acting on two-dimensional subspaces. The numerical range of any matrix of order two forms an ellipse [17], which may degenerate to an interval or a point. For instance it is known that the numerical range of the matrix $X = \begin{pmatrix} c & 2r \\ 0 & c \end{pmatrix}$, forms a disk of radius $|r|$ centered at c . Consider a family of operators with a tensor product structure

$$Y(r_1, r_2) = X_1 \otimes X_2 = \begin{pmatrix} 1 & 2r_1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 2r_2 \\ 0 & 1 \end{pmatrix}. \tag{26}$$

The product numerical range of Y takes different shapes depending on the values of the radii r_1 and r_2 of both discs. According to the results of [16], one may find the values of these parameters for which the boundary of the product numerical range is a cardioid, a limaçon of Pascal, or the outer loop of a Cartesian oval – see Fig. 2.

Analysis of the Minkowski product of two sets becomes easier if none of them contains 0. In such a case one may use the log-polar coordinates in the complex plane and reduce the Minkowski product to a Minkowski sum in the new coordinates.

Let us now consider the opposite case.

Lemma 2. If the numerical range of one of the factors A_1 or A_2 contains 0, then the product numerical range of the tensor product $\Lambda^{\otimes} (A_1 \otimes A_2)$ is star-shaped.

Proof. We have

$$\Lambda^\otimes(A_1 \otimes A_2) = \bigcup_{z \in \Lambda(A_2)} z\Lambda(A_1). \tag{27}$$

Without loss of generality, we may assume that $0 \in \Lambda(A_1)$. It is known that the numerical range $\Lambda(A_1)$ is convex. Hence $z\Lambda(A_1)$ is star-shaped with respect to 0 for arbitrary $z \in \mathbb{C}$. We get that $\Lambda^\otimes(A_1 \otimes A_2)$ is star-shaped with respect to 0 and therefore simply connected. \square

It is conceivable that the assertion of Lemma 2 may also hold without the assumption $0 \in \Lambda(A_1) \cup \Lambda(A_2)$, but so far, we were not able to prove or disprove this.

We are now in a position to formulate the main result of this section.

Proposition 1. *Let A_1, A_2 be arbitrary operators. The product numerical range of $A_1 \otimes A_2$ is simply connected.*

The above proposition, proved in Appendix A, does not hold for a threefold tensor product. In this way we confirm the conjecture formulated in [10] that one needs to work with at least tripartite systems to construct a tensor product operator whose product numerical range is not simply connected.

2.1.3. Inclusion properties

Generically, operators acting on a bipartite Hilbert space (1) do not exhibit the tensor product form. However, treating operators as vectors in the Hilbert–Schmidt space endowed with the Hilbert–Schmidt scalar product, $\langle A|B \rangle = \text{tr } A^\dagger B$, we may use the operator Schmidt decomposition. In close analogy to regular Schmidt decomposition, any operator X acting on $\mathcal{H}_K \otimes \mathcal{H}_K$ can be decomposed as a sum of not more than K^2 terms,

$$X = \sqrt{\mu_1}A_1 \otimes B_1 + \dots + \sqrt{\mu_{K^2}}A_{K^2} \otimes B_{K^2}. \tag{28}$$

To find the explicit form of this decomposition, it is convenient to use the reshuffled matrix, $Y = X^R$, such that in the product basis it consists of the same entries as the original matrix X , but ordered differently, $\langle i, j|Y|i', j' \rangle = \langle i, i'|X|j, j' \rangle$. Then the non-negative components μ_i of the Schmidt vector are equal to the singular values of the non-negative matrix YY^\dagger of order K^2 , while operators A_i and B_i with $i = 1, \dots, K^2$ are obtained by reshaping eigenvectors of Hermitian matrices, YY^\dagger and $Y^\dagger Y$, respectively [18].

Making use of the Schmidt decomposition (28) of an arbitrary bipartite operator X we can formulate a proposition concerning its product numerical range.

Proposition 2

$$\begin{aligned} \Lambda^\otimes(X) &\subset \sqrt{\mu_1}(\Lambda(A_1) \boxtimes \Lambda(B_1)) \boxplus \dots \boxplus \sqrt{\mu_{K^2}}(\Lambda(A_{K^2}) \boxtimes \Lambda(B_{K^2})) \\ &= \Lambda^\otimes(\sqrt{\mu_1}A_1 \otimes B_1) \boxplus \dots \boxplus \Lambda^\otimes(\sqrt{\mu_{K^2}}A_{K^2} \otimes B_{K^2}). \end{aligned} \tag{29}$$

Proof. It follows directly from Property 2 and from the fact that the product numerical range of a Kronecker product of operators equals the Minkowski product of their numerical ranges (Lemma 1). \square

2.2. Hermitian case

2.2.1. Basic properties

Consider a Hermitian operator $X = X^\dagger$ with ordered spectrum $\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N$. A closed interval $[\lambda_k, \lambda_{k+1}]$ will be called a *segment* of the spectrum. Thus the spectrum consists of $N - 1$ segments, some of which reduce to a single point, if the spectrum is degenerated. The numerical range of a Hermitian operator reads $\Lambda(X) = [\lambda_1, \lambda_N]$ [1]. In this case also product numerical range is given by an interval, $\Lambda^\otimes(X) = [\lambda_{\min}^\otimes, \lambda_{\max}^\otimes]$, bounded by the points λ_{\min}^\otimes and λ_{\max}^\otimes – the extremal points of product

numerical range under the set of all product states. Hence, these points determine the maximal (the minimal) expectation values of an observable X among all product pure states.

Observe that these extremal values determine the product numerical radius,

$$r^\otimes(X) = \max\{|\lambda_{\min}^\otimes(X)|, |\lambda_{\max}^\otimes(X)|\}. \tag{30}$$

This relation simplifies for quantum states. Their positivity implies that $r^\otimes(\rho) = \lambda_{\max}^\otimes(\rho)$.

Note that the product numerical range by definition depends on the concrete form of the tensor product Hilbert space $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$. For instance, if X also possesses the similar product structure, $X = X_A \otimes X_B$, then all its eigenstates, $|\phi_i\rangle, i = 1, \dots, N$, are of the product form and are called *separable states*. Any pure state which is not of the product form is called *entangled*. Thus in this particular case both ranges are equal, $\Lambda^\otimes(X) = \Lambda(X) = [\lambda_1, \lambda_N]$. This is also the case if the eigenstates $|\phi_1\rangle$ and $|\phi_N\rangle$, corresponding to the extremal eigenvalues λ_1 and λ_N , are of the product form. Hence the product structure of X is not necessary to assure that both ranges coincide.

Let us consider an arbitrary orthonormal product basis, $|i, j\rangle = |i\rangle \otimes |j\rangle$ of \mathcal{H}_N . The states $|i\rangle \in \mathcal{H}_K, i = 1, \dots, K$ and $|j\rangle \in \mathcal{H}_M, j = 1, \dots, M$ satisfy the orthogonality relation $\langle i, j | i', j' \rangle = \delta_{ii'} \delta_{jj'}$. Let us also use the composed indices denoted by Greek letters, $\mu = (i, j), \nu = (i', j')$, to represent X in the product basis

$$X_{\mu\nu} = \langle \mu | X | \nu \rangle = \langle i, j | X | i', j' \rangle. \tag{31}$$

In such a product representation any diagonal element $X_{\mu\mu}$ denotes the expectation value of X in a product state $|\mu\rangle = |i, j\rangle$, so it belongs to the product numerical range, $X_{\mu\mu} \in \Lambda^\otimes(X)$.

Interestingly, some information about the product numerical range can be obtained even without specifying a concrete tensor product structure in \mathcal{H}_N and a product basis.

2.2.2. Invariant features

In this section we investigate some basic properties of the product numerical range of a Hermitian operator, $X = X^\dagger$, which hold independently from the splitting of the Hilbert space $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$. In the general case of an operator X acting on a $K \times M$ space, not so much can be said about its product range.

Proposition 3. *Product numerical range forms, by definition, a non-empty subset of the numerical range, $\emptyset \neq \Lambda^\otimes(X) \subset \Lambda(X)$. Furthermore, $\Lambda^\otimes(X)$ reduces to a single point λ if and only if the operator X is proportional to the identity, $X = \lambda \mathbb{1}$. Speaking in terms of the standard Lebesgue volume of an interval we arrive at the following statement. If $\text{Vol}[\Lambda(X)] > 0$, then $\text{Vol}[\Lambda^\otimes(X)] > 0$.*

The assertion that $\Lambda^\otimes(X)$ is a single point if and only if $X = \lambda \mathbb{1}$ holds also when we do not assume Hermiticity of X .

Proposition 4. *For any Hermitian operator X , acting on an N -dimensional Hilbert space, its product numerical range $\Lambda^\otimes(X)$ is convex and forms an interval of the real line.*

Proof. Let us assume that a_1 and a_2 belong to $\Lambda^\otimes(X)$. Hence there exist two pairs of product vectors, such that $a_1 = \langle x_1, y_1 | X | x_1, y_1 \rangle$ and $a_2 = \langle x_2, y_2 | X | x_2, y_2 \rangle$. Since it is possible to find a continuous family of vectors $|\phi_A(s)\rangle \in \mathcal{H}_K$, which interpolates between $|x_1\rangle$ and $|x_2\rangle$ and another family $|\phi_B(s)\rangle \in \mathcal{H}_M$, which interpolates between $|y_1\rangle$ and $|y_2\rangle$, the expectation values of X between these product states interpolate between a_1 and a_2 . Thus the entire interval belongs to the product numerical range, $[a_1, a_2] \subset \Lambda^\otimes(X)$. \square

Note that the above reasoning, applied to an arbitrary (non-Hermitian) operator X , shows that the product numerical range forms a connected set in the complex plane. However, as shown in further sections of this paper, this set does not need to be convex nor simply connected.

Before we turn to more specific theorems, let us invoke a useful result.

Lemma 3. Consider a tensor product complex Hilbert space $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$. Then we have the following:

- (a) any subspace $S_d \subset \mathcal{H}_N$ of dimension $d = (K - 1)(M - 1) + 1$ contains at least one separable state,
- (b) there exists a subspace of dimension $d - 1 = (K - 1)(M - 1)$ which does not contain any separable state.

Proof. Part (a) of the above Lemma follows directly from Proposition 6 in a paper by Cubitt et al. [19], but this fact was earlier proven in [20,21].

Concerning part (b), for given integers K, M we give an example of family of $(K - 1) \times (M - 1)$ matrices $A^{(ij)}$ $i = 1, \dots, K - 1$ and $j = 1, \dots, M - 1$ such that no linear combination of $\{A^{(ij)} : 1 \leq i < K, 1 \leq j < M\}$ is of rank one.

Let $A^{(ij)} = E^{(ij)} + E^{(i+1,j+1)}$ for $1 \leq i < K$ and $1 \leq j < M$. Explicitly, let us denote by $E^{(ij)}$ the $K \times M$ matrix containing 1 at the position (i, j) and zeros elsewhere. Suppose that $T = \sum_{i=1}^{K-1} \sum_{j=1}^{M-1} \alpha_{ij} A^{(ij)}$ is of rank one. As the top-right entry of T is zero, then the first row or the M -th column of T is zero. Thus we have $\alpha_{1,j} = 0$ for all j or $\alpha_{i,M} = 0$ for all i . Deleting the first row or the M -th column of T we can proceed by induction on K and M to get the desired result.

Thus the subspace spanned by the vectorizations $\sum_{k,l} A_{lk}^{(ij)} |k, l\rangle$ of the matrices $A^{(ij)}$ does not contain any product state. \square

A subspace containing no product states is called *completely entangled* and it was shown in [22] that a generic subspace of dimension $(K - 1)(M - 1)$ possesses this property.

In the proof of part (b) of Lemma 3, we used the fact that any pure state in a $N = KM$ dimensional bipartite Hilbert space can be represented in terms its *Schmidt decomposition*,

$$|\psi\rangle = \sum_{i=1}^K \sum_{j=1}^M A_{ij} |i\rangle \otimes |j\rangle = \sum_{i=1}^K \sqrt{\mu_i} |i'\rangle \otimes |i''\rangle. \tag{32}$$

We have assumed here that $K \leq M$ and denoted a suitably rotated product basis by $|i'\rangle \otimes |i''\rangle$. The eigenvalues μ_i of a positive matrix AA^\dagger are called the *Schmidt coefficients* of the bipartite state $|\psi\rangle$. The normalization condition $|\psi|^2 = \langle \psi | \psi \rangle = 1$ implies that $\|A\|_{HS}^2 = \text{tr} AA^\dagger = 1$, so the Schmidt coefficients μ_i form a probability vector.

The state $|\psi\rangle$ is separable iff the $K \times M$ matrix of coefficients A is of rank one, so the corresponding vector of the Schmidt coefficients is pure. Hence Lemma 3 is equivalent to the following.

Lemma 4. Consider a set of k complex rectangular matrices A_i of size $K \times M$, which are orthogonal with respect to the Hilbert–Schmidt scalar product, $\langle A_i | A_j \rangle = \text{tr} A_i^\dagger A_j = \delta_{ij}$.

- (a) If $k \geq d = (K - 1)(M - 1) + 1$ there exists a complex vector \vec{c} determining a linear combination of these matrices, $A_{av} = \sum_{i=1}^k c_i A_i$, such that A_{av} is of rank one.
- (b) Moreover, simple dimension counting arguments presented in [19] imply that, for a generic set of $(d - 1)$ such matrices A_i , a rank one linear combination does not exist.

Proposition 5 (General $K \times M$ problem). The following lower bound for the product numerical radius is true,

$$\lambda_{\max}^\otimes \geq \lambda_{K+M-1}$$

and a symmetric upper bound for the product minimum holds,

$$\lambda_{\min}^\otimes \leq \lambda_{(K-1)(M-1)+1}$$

Furthermore, there exist $X_1 = X_1^\dagger$ and $X_2 = X_2^\dagger$ acting on \mathcal{H}_{KM} such that $\lambda_{\max}^\otimes(X_1) < \lambda_{K+M}(X_1)$ and $\lambda_{\min}^\otimes(X_2) > \lambda_{(K-1)(M-1)}(X_2)$.

Before proving this proposition let us state some of its implications. For any Hermitian operator X acting on the 2×2 Hilbert space its product numerical range contains the central segment of the spectrum, $[\lambda_2, \lambda_3] \subset \Lambda^\otimes(X)$. A similar statement holds for the $2 \times m$ problem, $[\lambda_m, \lambda_{m+1}] \subset \Lambda^\otimes(X)$. Proposition 5 implies that for any X acting on a 3×3 system $\lambda_5 \in \Lambda^\otimes(X)$.

Proof. Lemma 3 implies in this case that any $d = (M - 1)(K - 1) + 1$ dimensional subspace contains a product state. Applying this to the subspace spanned by the $(M - 1)(K - 1) + 1$ eigenstates corresponding to the $(M - 1)(K - 1) + 1$ smallest eigenvalues we obtain a bound $\lambda_{\min}^\otimes \leq \lambda_{(M-1)(K-1)+1}$. The $(M - 1)(K - 1) + 1$ -dimensional subspace spanned by the eigenstates corresponding to largest eigenvalues also contains at least one product state, which implies that $\lambda_{\max}^\otimes \geq \lambda_{K+M-1}$.

To demonstrate the optimality, it is enough to find a single operator X of size KM such that $\lambda_{(K-1)(M-1)} \notin \Lambda^\otimes(X)$. By Lemma 3, there exists a subspace \mathcal{K} of dimension $(K - 1)(M - 1)$ which does not contain a separable state. Let $|\psi_1\rangle, \dots, |\psi_{(M-1)(K-1)}\rangle$ be some orthonormal basis of this subspace, and let vectors $|\psi_{(M-1)(K-1)+1}\rangle, \dots, |\psi_{KM}\rangle$ enlarge the previous system to the orthonormal basis of \mathcal{H}_{KM} . We define matrix X as

$$X = \sum_{i=(M-1)(K-1)+1}^{KM} |\psi_i\rangle\langle\psi_i|. \tag{33}$$

Each product unit length vector can be written in the above basis as

$$|x, y\rangle = \sum_{i=1}^{(M-1)(K-1)} \alpha_i |\psi_i\rangle + \sum_{i=(M-1)(K-1)+1}^{KM} \beta_i |\psi_i\rangle. \tag{34}$$

Because the subspace \mathcal{K} does not contain any separable state, in the above decomposition of a product state there exists $i \in \{(M + 1)(K - 1) + 1, \dots, KM\}$ such that $\beta_i \neq 0$. Thus for any product state we have

$$\langle x, y | X | x, y \rangle > 0 = \lambda_{(M-1)(K-1)}. \quad \square \tag{35}$$

Note that it is not possible to obtain complete information about the position of the product numerical range relying only on unitarily invariant properties. In general, one has to consider a concrete splitting of the composite Hilbert space.

3. Generalizations

3.1. Multipartite operators

3.1.1. Definition

It is straightforward to generalize the notions of product numerical range to spaces with a multipartite structure. In this section we study Hilbert spaces formed by a tensor product of an arbitrary number of factors,

$$\mathcal{H}_N = \mathcal{H}_{m_1} \otimes \dots \otimes \mathcal{H}_{m_k}, \tag{36}$$

with $N = m_1 \dots m_k$.

For instance, in the definition of product numerical radius one has to perform the maximization over the direct product group of local unitary transformations,

$$U_{\text{loc}} := U(m_1) \times U(m_2) \times \dots \times U(m_k), \tag{37}$$

embedded in the group of global unitaries, $U(N) = U(m_1 \dots m_k)$.

3.1.2. Parametrization

The following proposition gives a parametrization of the product numerical range for operators which can be diagonalized by a product of unitary transformations.

Definition 3 (Product diagonalizable operator). We call operator $A \in \mathbb{M}_{m_1 m_2 \dots m_k}$ product diagonalizable iff there exists unitary operators $U_{m_1} \in \mathbb{M}_{m_1}, U_{m_2} \in \mathbb{M}_{m_2}, \dots, U_{m_k} \in \mathbb{M}_{m_k}$ such that

$$(U_{m_1} \otimes U_{m_1} \otimes \dots \otimes U_{m_k})A(U_{m_1} \otimes U_{m_1} \otimes \dots \otimes U_{m_k})^\dagger = \Sigma \tag{38}$$

and Σ is a diagonal matrix.

In the special case where the operator A is Hermitian and positive, such states are called *classically correlated* [23] or *locally diagonalizable* [18].

Proposition 6. Let A be a product diagonalizable operator on $\mathcal{H}_{m_1 m_2 \dots m_k} = \mathcal{H}_{m_1} \otimes \mathcal{H}_{m_2} \otimes \dots \otimes \mathcal{H}_{m_k}$. By

$$\lambda_{l_1, l_2, \dots, l_k} = \langle l_1, l_2, \dots, l_k | A | l_1, l_2, \dots, l_k \rangle \tag{39}$$

we denote the vector of diagonal elements of A after diagonalization using product unitary matrices. With the above notation, we have the parametrization of the product numerical range of A ,

$$A^\otimes(A) = \left\{ z : z = \sum_{l_1=0}^{m_1-1} \sum_{l_2=0}^{m_2-1} \dots \sum_{l_k=0}^{m_k-1} p_{l_1}^{(1)} p_{l_2}^{(2)} \dots p_{l_k}^{(k)} \lambda_{l_1, l_2, \dots, l_k} \right\}, \tag{40}$$

where $p_0^{(r)}, \dots, p_{m_r-1}^{(r)} \geq 0$ and $p_0^{(r)} + \dots + p_{m_r-1}^{(r)} = 1$ for $r = 1, \dots, k$.

Remark 1. Note that if A is a product diagonalizable operator on $\mathcal{H}_{2^k} = \mathcal{H}_2 \otimes \mathcal{H}_2 \otimes \dots \otimes \mathcal{H}_2$, then the above parametrization simplifies and we have

$$A^\otimes(A) = \left\{ z : z = \left(\left\{ p^{(1)}, 1 - p^{(1)} \right\} \otimes \dots \otimes \left\{ p^{(k)}, 1 - p^{(k)} \right\} \right) \cdot \lambda, \right. \\ \left. p^{(1)}, \dots, p^{(k)} \in [0, 1] \right\}, \tag{41}$$

where ‘ \cdot ’ denotes the scalar product.

A proof of the above proposition is given in [Appendix B](#). Later we will use this proposition to study some concrete examples.

3.1.3. Properties

Proposition 7. Product numerical range of an operator X acting on a tensor product Hilbert space $\mathcal{H}_N = \mathcal{H}_{m_1} \otimes \dots \otimes \mathcal{H}_{m_k}$ includes the barycenter of the spectrum i.e.,

$$\frac{1}{N} \text{tr} X \in A^\otimes(X). \tag{42}$$

Proof. We can follow the same line of reasoning as in the proof of Property 9. \square

The next example, adopted from [12], shows that, in the case of a three-partite system, product numerical range is not necessarily simply connected.

Example 3. Consider two unitary matrices U_1 and U_2 written in the standard computational basis $\{|000\rangle, |001\rangle, \dots, |111\rangle\}$,

$$U_1 = \text{diag} \left(1, e^{\frac{2i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, 1 \right) \tag{43}$$

and

$$U_2 = \text{diag} \left(1, e^{\frac{2i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, e^{-\frac{2i\pi}{3}}, 1 \right). \tag{44}$$

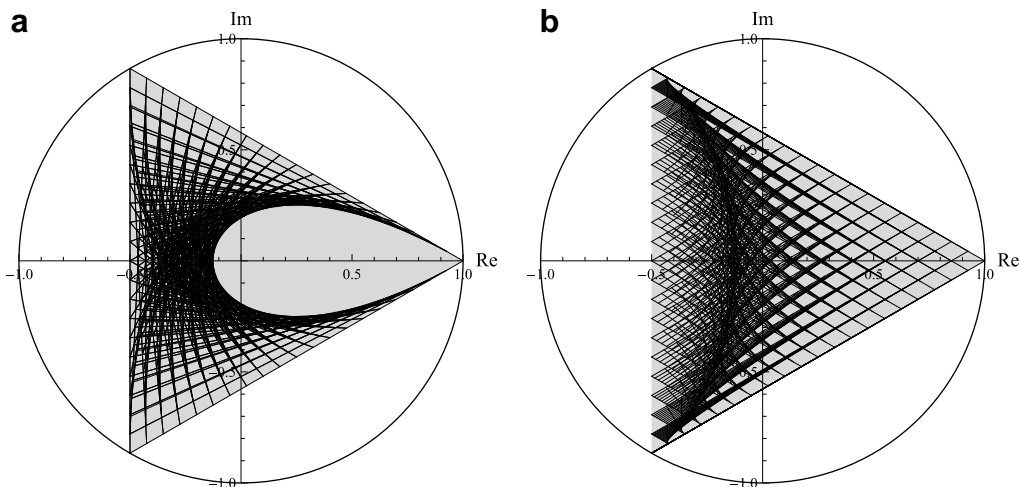


Fig. 3. (a) Λ^\otimes of matrix given by (43). (b) Λ^\otimes of matrix given by (44). Figures obtained from parametrizations (45) and (46). In both cases the numerical range forms an equilateral triangle plotted in grey.

Both U_1 and U_2 have identical eigenvalues, however they have different eigenvectors associated with these values.

Product numerical range of the operator U_1 acting on a three-partite system is not simply connected, as shown in Fig. 3(a). However, exchanging the position of two middle eigenvalues one obtains a unitary operator U_2 with the same spectrum, for which the product numerical range is convex and coincides with the standard numerical range, $\Lambda^\otimes(U_2) = \Lambda(U_2)$, see: Fig. 3(b).

Making use of Proposition 6, we can explicitly parameterize the product numerical range of these matrices,

$$\Lambda^\otimes(U_1) = \left\{ z : z = (\{p, 1 - p\} \otimes \{q, 1 - q\} \otimes \{r, 1 - r\}) \cdot (\{U_1\}_{ii})_{i=1}^8 \right\} \tag{45}$$

and

$$\Lambda^\otimes(U_2) = \left\{ z : z = (\{p, 1 - p\} \otimes \{q, 1 - q\} \otimes \{r, 1 - r\}) \cdot (\{U_2\}_{ii})_{i=1}^8 \right\} \tag{46}$$

for $p, q, r \in [0, 1]$.

Example 4 (Genus 2). Consider a unitary operator of order 16, acting on a four-qubit system. Assume that the corresponding matrix A is diagonal in the product basis $\{|0000\rangle, \dots, |1111\rangle\}$,

$$A = \text{diag} \left(e^{\frac{i\pi}{4}}, i, i, e^{\frac{3i\pi}{4}}, -1, e^{-\frac{3i\pi}{4}}, e^{-\frac{i\pi}{4}}, 1, -1, e^{-\frac{i\pi}{4}}, e^{-\frac{3i\pi}{4}}, 1, e^{\frac{3i\pi}{4}}, -i, -i, e^{\frac{i\pi}{4}} \right). \tag{47}$$

Applying Proposition 6, we parameterize the product numerical range of this matrix in the following manner

$$\Lambda^\otimes(A) = \{ z : z = (\{p, 1 - p\} \otimes \{q, 1 - q\} \otimes \{r, 1 - r\} \otimes \{s, 1 - s\}) \cdot \lambda \} \tag{48}$$

for $p, q, r, s \in [0, 1]$, where ‘ \cdot ’ denotes the scalar product. Product numerical range of the matrix A acting on the four-partite system forms a set of genus 2, as demonstrated in Fig. 4.

4. Concluding remarks

In this work we established basic properties of product numerical range (PNR) of operators acting on the Hilbert space with a tensor product structure. Even though the definition of this extension of the standard numerical range is rather straightforward, characterizing this set for a general operator

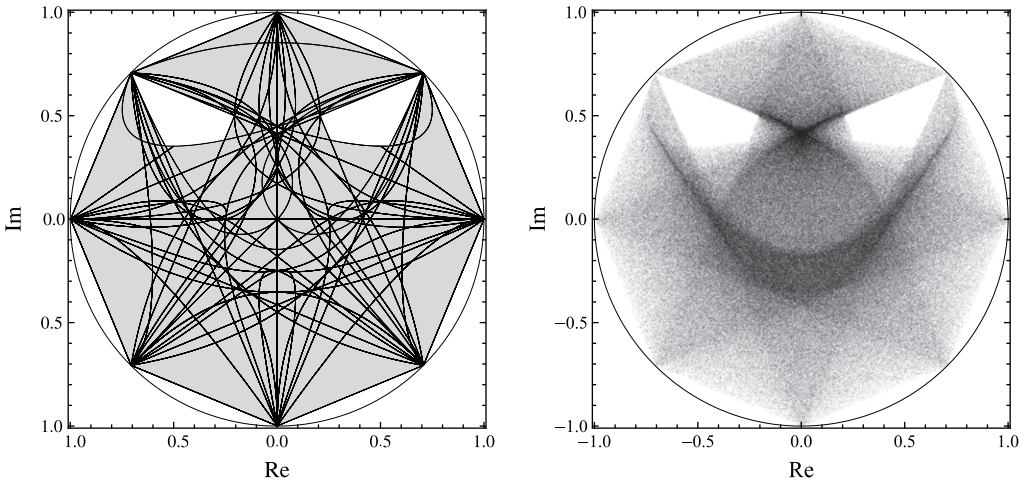


Fig. 4. Product numerical range of matrix A defined in (47). Panel (a) shows an image of the product numerical range (plotted in gray) obtained by parametrization (48), while panel (b) shows its image obtained by random sampling. The standard numerical range forms a regular octagon.

occurs to be a difficult problem. We have not managed therefore to present its general solution, but we obtained concrete bounds for this set (from inside and from outside) useful in various cases. Several special cases of this general problem are relevant in view of practical applications of product numerical range in the theory of quantum information [14]. To stimulate interest of the mathematical community in this subject we conclude the work by providing a list of open problems and sketching their motivation stemming from physics.

1. For a given Hermitian operator H acting on a bi-partite Hilbert space, $\mathcal{H}_N = \mathcal{H}_K \otimes \mathcal{H}_M$ find its local numerical range $[\lambda_{\min}^{\otimes}, \lambda_{\max}^{\otimes}]$.

Also precise bounds for PNR in this case would be useful. Of particular importance would be methods of establishing whether for a given H the inequality $\lambda_{\min}^{\otimes} \geq 0$ is satisfied. This statement is equivalent of saying that the operator H is *block positive* (with respect to the given decomposition of the space \mathcal{H}_N), hence the corresponding quantum map is positive [24,25].

2. Find the local product range $\Lambda^{\otimes}(U)$ for a possibly large class of unitary operators acting on a Hilbert space with the tensor product structure.

It is equally important to establish bounds applicable for PNR of an arbitrary unitary U , suitable to answer the question if $0 \in \Lambda^{\otimes}(U)$. This very issue occurs to be crucial in solving physical problems related to local distinguishability of unitary quantum gates [11,14].

3. For a given operator X acting on a Hilbert space with the tensor product structure find bounds for minimum of the modulus $|z|$, such that $z \in \Lambda^{\otimes}(X)$.

This question is related to finding the optimal gate fidelity [26] or optimizing local fidelity between two arbitrary quantum states [14].

4. Analyze the genus of PNR in the general case of operators acting on an k -partite Hilbert space, $\mathcal{H}_1 \otimes \dots \otimes \mathcal{H}_k$.

In particular verify the conjecture that the genus g is not larger than $k - 2$. If this conjecture does not hold in general one may still try to check whether it is true for operators that are k -fold tensor products. This issue is then related to the question what the genus of the Minkowski product of k convex sets on the complex plane is.

As a more specific question, one may ask the following:

5. Is $\Lambda^{\otimes}(A \otimes B)$ star-shaped for arbitrary operators A, B acting on $\mathcal{H}_K, \mathcal{H}_M$, respectively?

We have already mentioned that problem in Section 2.1.2.

Furthermore, one may also consider similar problems in a more general set-up by studying corresponding product analogues of other generalizations of the notion of numerical range. For instance,

mathematical results on ‘local (product) C-numerical range’ and ‘local (product) C-numerical radius’ [10,12,13], separable numerical range [14] and ‘Liouville numerical range’ [26] will find direct applications in several problems in theoretical physics.

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Appendix A. Proof of Proposition 1

As a consequence of Lemma 2, we can limit our proof to the case $0 \notin \Lambda(A_1) \cup \Lambda(A_2)$. Rotations of $\Lambda(A_1)$ or $\Lambda(A_2)$ in the complex plane cannot affect the genus of $\Lambda(A_1 \otimes A_2) = \Lambda(A_1) \boxtimes \Lambda(A_2)$. Thus we may assume for the beginning that $\mathbb{R}_+ \cap (\Lambda(A_1) \cup \Lambda(A_2)) = \emptyset$. We can achieve this by suitably rotating $\Lambda(A_1)$ and $\Lambda(A_2)$, because $\mathbb{R}_+ \cap \Lambda(A_j) = \emptyset$ or $-\mathbb{R}_+ \cap \Lambda(A_j) = \emptyset$ for either j , as a consequence of the convexity of A_j and the assumption that $0 \notin A_j$. Now we can rotate $\Lambda(A_1)$ and $\Lambda(A_2)$ clockwise, so that $\phi = 0$ becomes the minimal number ϕ in $[0, 2\pi)$ for which $\mathbb{R}_+ e^{i\phi} \cap \Lambda(A_1) \neq \emptyset$, as well as the minimal number ϕ for which $\mathbb{R}_+ e^{i\phi} \cap \Lambda(A_2) \neq \emptyset$. By our construction, keeping in mind that the sets A_j are closed, there exist $\varepsilon_1, \varepsilon_2 > 0$ such that $\mathbb{R}_+ e^{-i\varepsilon} \cap \Lambda(A_j) = \emptyset \forall \varepsilon \in (0, \varepsilon_j)$ for $j = 1, 2$. Using the convexity of $\Lambda(A_j)$ for $j = 1, 2$ again, we see that both $\Lambda(A_j)$ must be contained in the closed upper half-plane. Given $\mathbb{R}_+ \cap \Lambda(A_j) \neq \emptyset$, by the same argument as above we get $-\mathbb{R}_+ \cap \Lambda(A_j) = \mathbb{R}_+ e^{i\pi} \cap \Lambda(A_j) = \emptyset$. Closeness of A_j implies that

$$\phi_+^j = \max \left\{ \phi \in [0, 2\pi) \mid \mathbb{R}_+ e^{i\phi} \cap \Lambda(A_j) \neq \emptyset \right\} < \pi, \tag{A.1}$$

for $j = 1, 2$. Since $0 \notin \Lambda(A_1) \cup \Lambda(A_2)$ and $\phi_+^j < 2\pi$, we can describe the sets A_j in the log-polar coordinates $\mathcal{E}(z) = (\log |z|, \text{Arg}(z))$. Because the sets A_j are convex, $\mathbb{R}_+ e^{i\phi} \cap \Lambda(A_j)$ must be a closed nonempty interval or a point for arbitrary ϕ and j . Thus in the new coordinates we can write $\Lambda(A_j)$ as

$$\mathcal{E}(\Lambda(A_j)) = \bigcup_{\xi \in [0, \phi_+^j]} [r_-^j(\xi), r_+^j(\xi)] \times \{\xi\}. \tag{A.2}$$

Using the convexity and closeness of A_j , it is easy to prove that the functions r_{\pm}^j are continuous, which implies that (A.2) is a simply connected subset of \mathbb{R}^2 . Because the multiplication of complex numbers $z_1 = |z_1|e^{i\phi_1}, z_2 = |z_2|e^{i\phi_2}$ corresponds to the addition of their arguments ϕ_1, ϕ_2 as well as the addition of the logarithms of their modules, $\log |z_1|$ and $\log |z_2|$, it is straightforward to write an explicit formula for $\Lambda(A_1)\Lambda(A_2)$ in the coordinates \mathcal{E} ,

$$\bigcup_{\xi \in [0, \phi_+^1 + \phi_+^2]} \bigcup_{\varphi \in [\max(0, \xi - \phi_+^2), \min(\phi_+^1, \xi)]} [r_+^1(\varphi) r_+^2(\xi - \varphi), r_-^1(\varphi) r_-^2(\xi - \varphi)] \times \{\xi\} \tag{A.3}$$

or

$$\bigcup_{\xi \in [0, \phi_+^1 + \phi_+^2]} [R_+(\xi), R_-(\xi)] \times \{\xi\}, \tag{A.4}$$

where the functions R^\pm are defined in the following way,

$$R^+(\xi) = \max_{[\max(0, \phi_+^1 - \phi_+^2), \min(\phi_+^1, \xi)]} (r_+^1(\varphi) r_+^2(\xi - \varphi)), \tag{A.5}$$

$$R^-(\xi) = \min_{[\max(0, \xi - \phi_+^2), \min(\phi_+^1, \xi)]} (r_-^1(\varphi) r_-^2(\xi - \varphi)). \tag{A.6}$$

It turns out that R_+ and R_- are continuous and the interval $[R_-(\xi), R_+(\xi)]$ is nonempty for arbitrary $\xi \in [0, \phi_+^1 + \phi_+^2]$. Therefore (A.3) is a simply connected subset of $\mathbb{R} \times [0, \phi_+^1 + \phi_+^2]$. Since $\phi_+^1 + \phi_+^2 < 2\pi$, \mathcal{E} is a homomorphism between $\mathbb{R} \times [0, \phi_+^1 + \phi_+^2]$ and $\mathcal{E}^{-1}(\mathbb{R} \times [0, \phi_+^1 + \phi_+^2])$. Therefore $\Lambda(A_1) \boxtimes \Lambda(A_2)$ is simply connected, as a preimage of a simply connected set by the isomorphism \mathcal{E} . This is what we wanted to prove.

Appendix B. Proof of Proposition 6

Each product vector $|x\rangle = |x_1\rangle \otimes |x_2\rangle \cdots \otimes |x_k\rangle$ of unit norm can be rewritten in its computational basis,

$$(x_{1,0}|0\rangle + \cdots + x_{1,m_1-1}|m_1 - 1\rangle) \otimes \cdots \otimes (x_{k,0}|0\rangle + \cdots + x_{k,m_k-1}|m_k - 1\rangle), \tag{B.1}$$

where

$$|x_{r,0}|^2 + \cdots + |x_{r,m_r-1}|^2 = 1 \quad \text{for } r = 1, \dots, k. \tag{B.2}$$

Thus we have

$$\begin{aligned} \langle x|A|x\rangle &= \sum_{l_1=0}^{m_1-1} \sum_{l_2=0}^{m_2-1} \cdots \sum_{l_k=0}^{m_k-1} \sum_{s_1=0}^{m_1-1} \sum_{s_2=0}^{m_2-1} \cdots \sum_{s_k=0}^{m_k-1} \\ & x_{1,l_1}^* x_{2,l_2}^* \cdots x_{k,l_k}^* x_{1,s_1} x_{2,s_2} \cdots x_{k,s_k} \langle l_1 l_2 \cdots l_k | A | s_1 s_2 \cdots s_k \rangle. \end{aligned} \tag{B.3}$$

Now we must note that, since A is diagonal with respect to the product computational basis, we have

$$\langle l_1 l_2 \cdots l_k | A | s_1 s_2 \cdots s_k \rangle = \lambda_{l_1 l_2 \cdots l_k} \delta_{\{l_1 l_2 \cdots l_k, s_1 s_2 \cdots s_k\}}. \tag{B.4}$$

Thus we get

$$\langle x|A|x\rangle = \sum_{l_1=0}^{m_1-1} \sum_{l_2=0}^{m_2-1} \cdots \sum_{l_k=0}^{m_k-1} |x_{1,l_1}|^2 |x_{2,l_2}|^2 \cdots |x_{k,l_k}|^2 \lambda_{l_1 l_2 \cdots l_k}. \tag{B.5}$$

If we denote $p_i^{(r)} := |x_{r,i}|^2$, then we have $p_0^{(r)} + p_1^{(r)} + \cdots + p_{m_r-1}^{(r)} = 1$ and

$$\langle x|A|x\rangle = \sum_{l_1=0}^{m_1-1} \sum_{l_2=0}^{m_2-1} \cdots \sum_{l_k=0}^{m_k-1} p_{l_1}^{(1)} p_{l_2}^{(2)} \cdots p_{l_k}^{(k)} \lambda_{l_1 l_2 \cdots l_k}. \tag{B.6}$$

Now it is easy to notice that if we take all possible product states, we will obtain all possible decompositions

$$p_0^{(r)}, p_1^{(r)}, \dots, p_{m_r-1}^{(r)} \geq 0, \quad p_0^{(r)} + p_1^{(r)} + \cdots + p_{m_r-1}^{(r)} = 1 \tag{B.7}$$

for $r = 1, \dots, k$. Thus the proof is complete.

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