Spectrum of the Frobenius–Perron operator for systems with stochastic perturbation

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Abstract

We investigate dynamical systems with stochastic perturbation and study to what extend analytical properties of the noise present influence the spectrum of the associated Frobenius–Perron operator. We suggest to distinguish a "physical" part of the spectrum of the deterministic system, as this robust with respect to the perturbation. For exemplary system studied such eigenvalues of the FP-operator are located outside the *essential spectrum* and have direct physical meaning: they determine the rate of the exponential decay of correlations in the system. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Consider a discrete dynamical system f acting on a phase space Ω. Time evolution of a classical density ρ (a probability distribution function on Ω) is governed by the Frobenius–Perron operator M [1,2]:

$$\rho'(y) = (M\rho)(y) = \int_{\Omega} \delta\big(f(x) - y\big)\rho(x)\,dx,$$

where $\int_{\Omega} \rho(x)\,dx = 1.$ (1)

Its spectral properties are of considerable interest, since they influence properties of the correlation func-

tion [3], which may be measured in a physical system. Spectrum of the FP-operator was calculated for classical dynamical systems including tent map, Bernoulli shift, baker map [4–6] and the so-called "four legs" map [7]. However, results obtained may depend on the choice of the function space FP-operator operates in. Such a case was explicitly demonstrated by Antoniou et al. [8], in which two different spectra for a certain random dynamical system were found. One might ask, therefore, which of those constructions lead to physically meaningful results.

In all physical situations the system under consideration is inevitably subjected to a stochastic perturbation which we usually want to reduce as much as we can. The idea we want to present here is that the presence of a perturbation of a small amplitude may choose one particular space of functions and thus define a "physical" decomposition of the FP-operator, without ambiguities related to the choice of the space

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of the eigenstates. Note that the natural invariant measure (so-called SRB measure [9]) of a classical dynamical system may be defined in a slightly analogous way, as the unique invariant measure stable with respect to stochastic perturbations. Moreover, it is instructive to recall the quantum mechanical problem of calculating corrections to degenerated energy levels via perturbation theory. A priori none of the basis in the subspace of degenerated states is distinguished. However, one may distinguish a certain basis by applying an arbitrary perturbation and later decreasing its amplitude to zero.

2. Model system

We analyze a model system introduced in [10] which allows one for an *exact* representation of the FP-operator describing dynamical system with noise as a finite-dimensional matrix. The construction proceeds as follows.

For simplicity we consider a discrete dynamical system f acting on the interval $\Omega = [0, 1)$ and subjected to an additive noise (with periodic boundary conditions)

$$x_{n+1} = f(x_n) + \xi_n \quad (\text{mod } 1), \qquad (2)$$

where $\xi_1, \xi_2, \ldots$ are independent random variables fulfilling

stationarity: $\quad \mathcal{P}(\xi_n) = \mathcal{P}(\xi),$

zero mean: $\quad \langle \xi_n \rangle = 0,$

finite variance: $\quad \langle \xi_n \xi_m \rangle = \sigma^2 \delta_{mn}.$

We choose a noise, for which the probability of transition from x to y $(\mathcal{P}(x, y) = \mathcal{P}(\xi))$ is

homogeneous:

$$\mathcal{P}(x, y) \equiv \mathcal{P}(x - y), \qquad (3a)$$

periodic:

$$\mathcal{P}(x, y) \equiv \mathcal{P}(x + 1, y) \equiv \mathcal{P}(x, y + 1), \qquad (3b)$$

decomposable:

$$\mathcal{P}(x, y) = \sum_{l,r=0}^{N} A_{lr} u_r(x) v_l(y), \qquad (3c)$$

where $A = (A_{lr})_{l,r=0,\ldots,N}$ is a real matrix of finite size $(N + 1)$ of expansion coefficients and $(u_r)_{r=0,\ldots,N}$, $(v_l)_{l=0,\ldots,N}$ are two sets of linearly independent real valued functions.

All these conditions are satisfied by the trigonometric noise

$$\mathcal{P}_N(x, y) = C_N \cos^N(\pi[x - y]), \qquad (4)$$

where N is even and the normalization constant $C_N = \sqrt{\pi}\,\Gamma(N/2 + 1)/\Gamma(N/2 + 1/2)$ assures $\int_0^1 \mathcal{P}_N(x)\,dx = 1$. The noise strength may be parametrized either by N or by its variance [10]

$$\sigma_N^2 = \frac{1}{2\pi^2} \Psi'(N/2 + 1) = \frac{1}{2\pi^2}\left(\sum_{k=N/2+1}^{\infty} \frac{1}{k^2}\right)$$

$$= \frac{1}{12} - \frac{1}{2\pi^2}\sum_{k=1}^{N/2} \frac{1}{k^2}, \qquad (5)$$

where Ψ' stands for the derivative of the digamma function [11]. Asymptotically,

$$\sigma_N^2 \sim 1/N \underset{(N\to\infty)}{\to} 0.$$

In this case $A_{lr} = \binom{N}{l}\delta_{lr}$, while $u_l(x) = v_l(x) = \sin^l(\pi x)\cos^{N-l}(\pi x)$.

For a deterministic system, the action of the FP-operator is described by (1). In the presence of a stochastic perturbation this equation needs to be modified,

$$\rho'(y) = \int \mathcal{P}(f(x), y)\rho(x)\,dx. \qquad (6)$$

Due to the decomposition property (3c) we may write

$$\rho'(y) = \sum_{l,r=0}^{N} A_{lr} \int_0^1 u_r(f(x))v_l(y)\rho(x)\,dx$$

$$= \sum_{r=0}^{N}\left[\int_0^1 u_r(f(x))\rho(x)\,dx\right]\tilde{v}_r(y), \qquad (7)$$

where $\tilde{v}_r = \sum_{l=0}^{N} A_{lr} v_l$. Thus any initial density is projected by the FP-operator into the vector space spanned by the functions $\tilde{v}_r$. Eventually, we can represent the action of this operator by a matrix of

size $N + 1$ acting on the vector of expansion coefficients α_k. Writing $\rho(x) = \sum_l \alpha_l \tilde{v}_l(x)$ we obtain

$$\rho'(y) = \sum_k \tilde{v}_k(y)\alpha_k' = \sum_{kl} \tilde{v}_k(y)\mathbf{D}_{kl}\alpha_l, \qquad (8)$$

where $\mathbf{D}_{kl} = \int dx\, u_k(f(x))\tilde{v}_l(x)$. It is worth emphasizing that the matrix representation of the FP-operator for the system with noise is *finite-dimensional*, while the FP-operator connected with the deterministic case acts in an infinite-dimensional space.

3. Examples

In [12] it was conjectured that for continuous dynamics the spectrum of the system with noise should tend to the spectrum of the operator describing the deterministic system, if the amplitude of noise tends to zero. This correspondence is based on solving eigenvalue problem for the Fokker–Planck equation—which a priori is not a simpler task than calculating the spectrum of the Frobenius–Perron operator. In our approach we can find approximation of the exact spectrum just by matrix diagonalization. The larger dimension of the matrix $\mathbf{D}$, the smaller variance σ^2 of the noise, and the better the approximation of the spectrum of the deterministic system. In order to show how the spectrum of the Frobenius–Perron operator could be approximated by our procedure we calculate it for a simple one-dimensional map.

First of all we have to consider stability of the spectrum if the system is subjected to a stochastic perturbation. To address this problem it is useful to introduce a notion of the *essential spectrum* [13,14]. It is a part of the spectrum contained in the smallest disc of radius r such that all eigenvalues outside it are isolated and of finite multiplicity. [1] Blank and Keller have shown [13] that for piecewise expanding maps (without periodic points, for which the derivative f' is not defined) subjected to any local perturbation, eigenvalues outside essential spectrum are close to those of the deterministic system.

We analyze a family of piecewise expanding maps which have isolated eigenvalues outside the essential

spectrum,

$$f(x) = \begin{cases} 2(x + \epsilon) + \frac{1}{2}, & x < \frac{1}{4} - \epsilon, \\ \frac{3}{2} - 2(x + \epsilon), & x \in \left[\frac{1}{4} - \epsilon, \frac{3}{4} - \epsilon\right), \\ 2(x + \epsilon) - \frac{3}{2}, & x \geqslant \frac{3}{4} - \epsilon, \end{cases} \qquad (9)$$

and subject it to perturbation (4) (non-local in sense of [13]). Since the absolute value of the slope $|f'(x)|$ is constant and equal to 2, the radius of the essential spectrum equals $1/2$ independently of the parameter ϵ. For $\epsilon = 0$ there are two eigenvalues with modulus 1 (± 1)—the system separates into two subsystems $\Omega_1 = (0, 1/2)$ and $\Omega_2 = (1/2, 1)$ which exchange positions with each other under every iteration of the map (Fig. 1). This fact is related to the presence of $\lambda_2 = -1$ in the spectrum. If we increase ϵ then the modulus of the negative eigenvalue will decrease. However, for sufficiently small ϵ the subleading (with second largest modulus) eigenvalue is still located outside the essential spectrum, so we expect to approximate it with our procedure. For certain values of ϵ we may construct Markov partitions of Ω, write down the corresponding stochastic transition matrix $\mathbf{T}$ and find analytically its subleading eigenvalue λ_2 which determines the rate of convergence to the equilibrium. Fig. 2 presents map (9) for $\epsilon = 1/20$ together with its
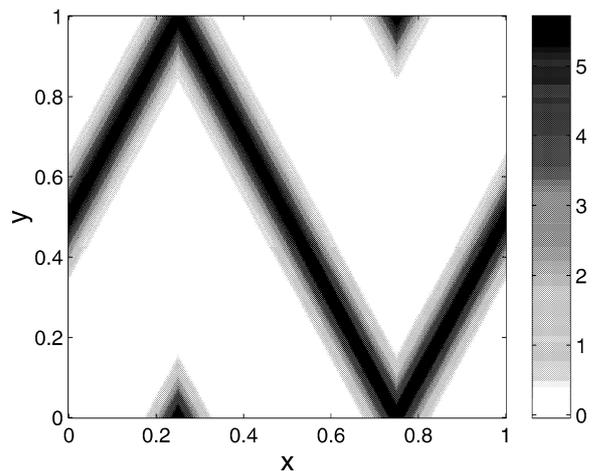

Fig. 1. Transition kernel $\mathcal{P}_N(f(x), y)$ for the map defined by (9) with $\epsilon = 0$ and the noise strength characterized by $N = 20$, which gives $\sigma \approx 0.0694$.

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[1] For linear piecewise expanding 1D maps it was shown in [15] that $r = \lim_{n \to \infty} \sqrt[n]{1/\lambda_f^n}$, where λ_f is the smallest absolute value of the derivative of f.
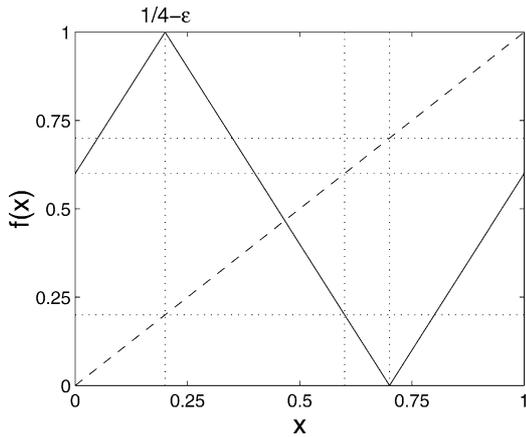
Fig. 2. Map given by Eq. (9) for $\epsilon = 1/20$ together with its Markov partition denoted by dotted lines.
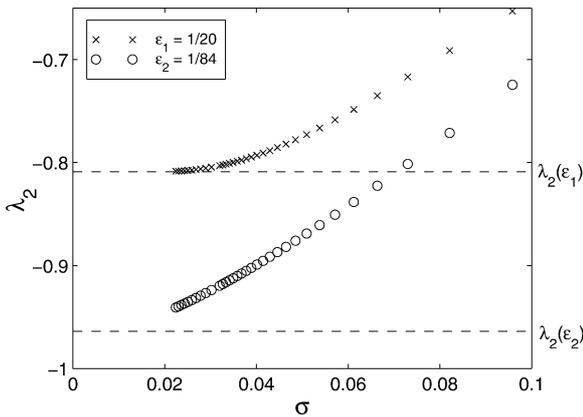

Fig. 3. Subleading eigenvalue λ_2 as a function of noise width σ for the map defined by (9) plotted for $\epsilon = 1/20$ and $\epsilon = 1/84$ together with the deterministic limit ($\sigma \to 0$) represented by horizontal lines.

Markov partition. In this case

$$\mathbf{T} = \begin{pmatrix} 0 & 0 & 1/4 & 3/4 \\ 0 & 1/2 & 1/8 & 3/8 \\ 1 & 0 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 \end{pmatrix} \tag{10}$$

and $\lambda_2 = -(1 + \sqrt{5})/4 \approx -0.809$, while for $\epsilon = 1/84$ we have $\lambda_2 \approx -0.9638$. Fig. 3 presents the second eigenvalue λ_2 of the FP-operator for the system with noise obtained by numerical diagonalization of $(N + 1)$-dimensional matrix $\mathbf{D}$ as a function of the
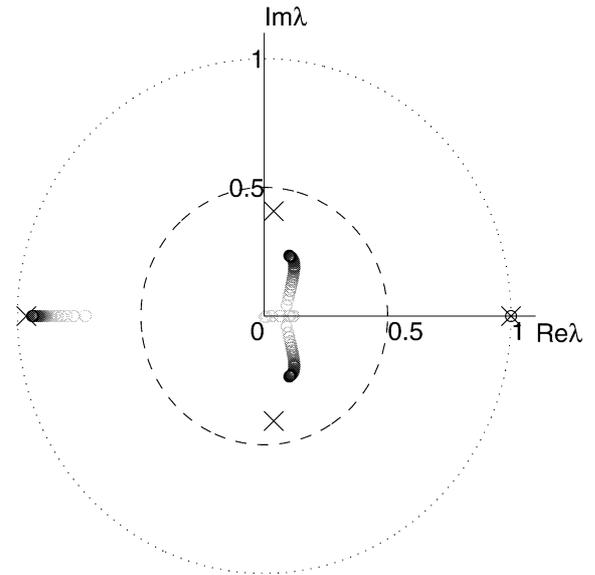

Fig. 4. Plot of four largest eigenvalues in the complex plane as a function of the noise strength for $\epsilon = 1/84$. The darker color, the smaller noise, which changes from $\sigma_1 \approx 0.1$ to $\sigma_2 \approx 0.02$ ($N_1 = 10$, $N_2 = 200$). Crosses represent the eigenvalues of the transition matrix $\mathbf{T}$ with the largest moduli, while dashed circle of radius 1/2 represents the essential spectrum.

noise width σ for two different values of ϵ. With decreasing noise strength σ the second eigenvalue tends to its deterministic counterpart (represented by dashed horizontal lines). For $\epsilon = 1/20$ a fair approximation is obtained already for $N = 200$ ($\sigma \approx 0.02$), whereas for $\epsilon = 1/84$ we need a smaller noise strength to obtain satisfactory results. Fig. 4 shows the motion of four eigenvalues with the largest moduli in the complex plane as the noise strength is decreased. Convergence of the eigenvalue outside essential spectrum to its deterministic counterpart is much faster then these of the two other eigenvalues belonging to the essential spectrum (one may even question, whether they at all converge to the deterministic values). In the same way we expect that eigenvectors corresponding to the eigenvalues outside essential spectrum tend to their deterministic counterparts. Fig. 5 presents the second eigenvector obtained from the matrix representation of the FP-operator in the noisy system ($\epsilon = 1/84$) for $\sigma \approx 0.02$ (dashed line) and from the transition matrix of the Markov partition. These eigenvectors satisfy $\int_0^1 w_i(x)\, dx = 0$, $i = 2, 3, \ldots, N + 1$, and may
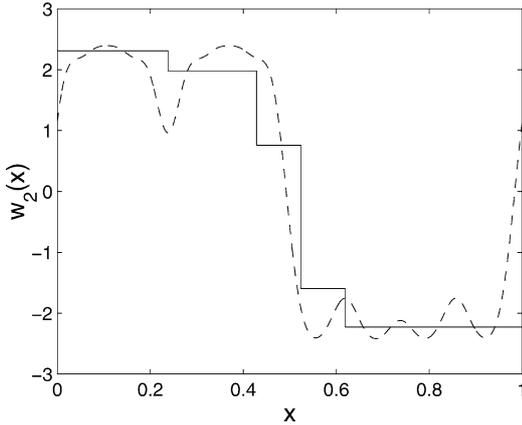
Fig. 5. Eigenvector corresponding to the second eigenvalue obtained from the matrix representation of the FP-operator of noisy system ($\epsilon = 1/84$) for $\sigma \approx 0.02$ (dashed line) and from the transition matrix **T** of the deterministic system (solid line).

be considered as corrections to the invariant measure $w_1(x) = \rho(x) = 1$.

Eventually, we can compare the values of the subleading eigenvalue of the spectrum with the decay rate of the autocorrelation function. The correlation function of any function $h(x)$, defined by a time average

$$C_h(n) = \lim_{t \to \infty} \frac{1}{T} \sum_{t=1}^{T} h\big(x(t+n)\big) h\big(x(t)\big), \qquad (11)$$

is called autocorrelation function in the simplest case $h(x) = x$. For an ergodic system it can be expressed by an average over the phase space with respect to the invariant measure $\rho(x)$,

$$C(n) = \int_{\Omega} dx \, \rho(x) x(n) x,$$

where $\quad x(n) = f^n(x)$. $\qquad (12)$

In the case considered, in which the dynamics is composed of two parts—deterministic and stochastic (see Eq. (2))—one has to average additionally over different realizations of the stochastic perturbation. Let us consider first the one-step correlation function

$$C(1) = \int \langle x\big(f(x) + \xi\big)\rangle \rho(x) \, dx, \qquad (13)$$

where the angle brackets denote averaging over different realizations of the noise (ξ). Employing property

(3c) we can write

$$C(1) = \int \mathcal{P}\big(f(x), x'\big) x x' \rho(x) \, dx \, dx'$$

$$= \sum_k \int u_k\big(f(x)\big) \tilde{v}_k(x') x x' \rho(x) \, dx \, dx'$$

$$= \sum_k \left(\int \tilde{v}_k(x') x' \, dx' \right)$$

$$\times \left(\int u_k\big(f(x)\big) x \rho(x) \, dx \right)$$

$$= \vec{v} \cdot \vec{u}, \qquad (14)$$

where we have defined $\vec{v}$ and $\vec{u}$ as vectors of the integrals $\int dx' \, \tilde{v}_k(x') h(x')$ and $\int dx \, \rho(x) u_k(f(x)) h(x)$ for $k = 0, \ldots, N$. In an analogous way we obtain the correlation function of any observable h for longer delay times,

$$C_h(n) = \int \mathcal{P}\big(f(x_1), x_2\big) \ldots \mathcal{P}\big(f(x_n), x_{n+1}\big)$$

$$\times h(x_1) h(x_{n+1}) \rho(x_1) \, d^{n+1}x$$

$$= \vec{v} \mathbf{D}^{n-1} \vec{u}, \quad n > 0, \qquad (15)$$

where $(N + 1)$-dimensional matrix **D** is already defined in (8). Thus one may expect the correlation function to be composed of a $N + 1$ exponentially decaying modes determined by the moduli of the eigenvalues of the matrix representation of the FP-operator. An example of such a situation is presented in Fig. 6. The autocorrelation function is calculated by choosing 10^6 points according to the uniform distribution in Ω and evolving them by system f given by (9) with $\epsilon = 1/84$ (and the noise characterized by $N = 50$). The value of the decay exponents obtained from the best exponential fit (see inset in Fig. 6) agrees up to 2% with the one we get from the second eigenvalue of the matrix representation of the FP-operator. Results for the decay of correlation obtained in this way for a system subjected to the stochastic perturbation (4) may be extrapolated ($N \to \infty$) for the deterministic system. Thus eigenvalues of the FP-operator (1), stable with respect to noise, determine the decay of correlations in a realistic physical system.

Note that the observed decay of correlation was studied for an ensemble of points distributed uniformly in Ω, that is, according to the invariant measure of the system. This seems to be more natural approach
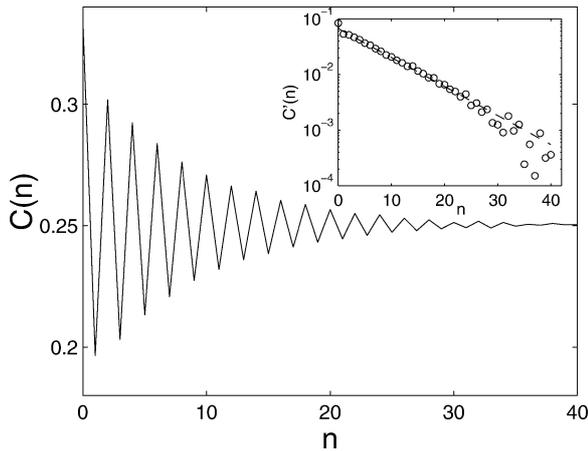
Fig. 6. Correlation function for map (9) with $\epsilon = 1/84$ and width of noise $\sigma \approx 0.045$ ($C(0)$ is just the variance of the invariant measure and in this case $C(0) = \int_0^1 x^2 \, dx = 1/3$). The inset presents $C'(n) = |C(n) - C(\infty)|$ in the semilogarithmic scale. Exponential fit gives $\log C' \approx -0.1198n$, while the subleading eigenvalue of the matrix **D** leads to $\log |\lambda_2| = -0.1201$.

than in the case recently analyzed by Weber et al. [16, 17], in which the initial density was determined by a selected eigenvector, while the observed decay rate was governed by the corresponding eigenvalue.

4. Conclusions

The aim of this Letter is twofold. We point out that by introducing a stochastic noise into a deterministic system and later tending with its strength to zero one may distinguish a "physically" important part of the spectrum of the associated FP-operator. In this way we suggest to define a "physical" spectrum of a classical map, as this robust with respect to stochastic perturbations. Moreover, we provide a method for approximation of the spectrum and eigenvectors of the FP-operator by applying a suitable noise decomposable in the sense of (3c). This technique seems to be more justified from the physical point of view than just truncating the infinite-dimensional matrix representation to some finite dimension, e.g., [16] (thus effectively introducing some perturbation), since one constructs a system with noise of known properties and can decrease the amplitude of the perturbation in a controlled way. Another methods of introducing noise into de-

terministic systems to analyze their spectral properties were recently used in [18,19].

We demonstrated that the eigenvalues of the FP-operator located outside the essential spectrum are robust not only against local perturbations as proved in [13], but also against non-local perturbations of form (4). Moreover, these stable eigenvalues have a direct physical meaning: they determine the rate of the exponential decay of correlation in the system. Thus our approach of analyzing dynamical system with a stochastic perturbation of a variable strength allows one to identify the physically important part of the FP-spectrum without mathematical ambiguities of selecting the space, in which this operator acts.

On the other hand, it would be interesting to analyze spectral properties of dynamical systems in presence of a stochastic perturbation fulfilling properties (3c) but different than (4) studied in this Letter. We expect the eigenvalues of the FP-operator not belonging to its essential spectrum to be weakly dependent on the specific form of the probability distribution $\mathcal{P}(\xi)$, but this conjecture requires further verification.

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