

## Quantum scars on a sphere

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Specific features of a quantum map that can be connected to periodic orbits of the corresponding classical system are investigated. The model we studied is the kicked top for which it is known that many statistical properties of the quantum system, such as distributions of eigenvalues and eigenvectors, are well described by the random matrix theory. It is shown that other statistical measures of eigenvector statistics (e.g., Shannon entropy) reveal deviations from predictions based on ensembles of Gaussian matrices. These differences may be associated with periodic orbits. A formula, valid for an arbitrary quantum map, which can be helpful in associating scarred wave functions with a particular periodic orbit, is given. A comparison is made with the semiclassical results.

### I. INTRODUCTION

Several properties of quantum chaotic systems may be well described by means of random matrix theory<sup>1,2</sup> (RMT). In particular, the statistics of eigenvalues<sup>3-5</sup> and eigenvectors<sup>5-8</sup> conform to the predictions of ensembles of random matrices. On the other hand, it is also possible to find for a given quantum model specific features that could be associated with the dynamics of a corresponding classical system. For example, only short-range correlations of the spectrum of a quantum system usually agree with the universal behavior, while long-range correlations are heavily influenced by the periodic orbits of the underlying classical model.<sup>9</sup>

The fact that classical periodic orbits affect the quantum spectra may be understood via the semiclassical periodic orbits theory,<sup>10</sup> which expresses the leading (in the semiclassical limit) fluctuating part of the density of states in terms of a sum over contributions from the periodic orbits. Problems with the convergence of this sum severely restrict its application in predicting the individual eigenvalues,<sup>11</sup> however a partial summation enables one to reproduce a *smoothed* density of states.<sup>12</sup>

Periodic orbits manifest themselves also in the shape of some of the eigenfunctions of the system. These wave functions show a partial localization (an increased probability density, the so-called quantum scars) in the vicinity of some periodic orbit.<sup>13</sup> The localization is related to properties of the classical motion close to the orbit and may be explained quantum mechanically in the language of Feshbach-type resonance scattering theory.<sup>14,15</sup> The semiclassical theory of quantum scars has also been developed<sup>16</sup> yielding the averaged, over a small energy interval, wave function. This wave function is a semiclassical analog of the periodic orbit resonance wave function in the quantum viewpoint.<sup>15</sup> Recently, it has been suggested<sup>17-20</sup> that the analysis of the Wigner or Husimi distribution of eigenstates in the phase space helps to associate the scars with the periodic orbits responsible for the localization, and that it may lead to a better understanding of the scarring phenomenon.

The study of scars has been primarily restricted to autonomous systems.<sup>13-18</sup> In that case one may estimate the energies at which the most relevant (i.e., the least unstable) periodic orbits may scar the wave functions by applying the semiclassical quantization conditions.<sup>10,21</sup> For some systems it has even been possible to predict the energies of scarred states by an adiabatic analysis approximately valid for localized wave functions.<sup>22-24</sup> Also the size of scars as a function of  $\hbar$  has been investigated.<sup>25</sup>

Much less is known about the effects related to periodic orbits for time-dependent systems, and especially for quantum maps. The semiclassical quantization, analogous to that of Gutzwiller, has been proposed,<sup>26,27</sup> but not really tested. Scarring has been observed for wave functions of the hydrogen atom in a microwave field,<sup>28</sup> for the quantum kicked rotor,<sup>29</sup> and for the quantum Baker map.<sup>20</sup>

The aim of this work is to investigate further the relation of the wave functions of systems described by quantum maps to classical periodic orbits. As a model system, we have chosen the kicked top for which it is known that many statistical properties of the quantum system are well described by the random-matrix theory.<sup>5,8</sup> We show how the periodic orbits affect statistical properties of the eigenvector basis expansion coefficients. Linking the Husimi distribution directly to the properties of the semiclassical propagator, we claim that one can find for a given (relatively short) periodic orbit a corresponding part of the spectrum, where the probability of finding scarred eigenfunctions is enhanced.

### II. HUSIMI DISTRIBUTION OF A SCARRED EIGENFUNCTION

Let us consider a system defined by a quantum map:

$$U: |\psi\rangle \rightarrow |\psi'\rangle = U|\psi\rangle, \quad (1)$$

where  $U$  is a unitary Floquet operator and  $|\psi\rangle$  is the wave function. Let  $|\gamma\rangle$  denote a coherent state in the sense of Perelomov<sup>30</sup> for a corresponding symmetry group of the system. In order to find the corresponding

classical system, it is convenient to define a family of operators  $U_\chi$ , where the parameter  $\chi$  is proportional to  $\hbar$ .<sup>31</sup> In the limit  $\chi \rightarrow 0$ , the coherent state  $|\gamma\rangle$  determines (complex) coordinates  $\gamma$  in the classical manifold, and the quantum map

$$|\gamma'\rangle = U_\chi |\gamma\rangle \quad (2)$$

corresponds to the classical one,

$$\gamma' = R(\gamma). \quad (3)$$

The dimension of the classical map  $R$  depends on the representation of the symmetry group used to construct the quantum map,<sup>32,33</sup> and the dimension  $N$  of the matrix representing the quantum operator  $U$  is proportional to  $1/\chi$ .

Let  $\Gamma$  be a classical periodic orbit ( $K$  cycle):  $\Gamma := \{\gamma_i, i=0, K-1; \gamma_0 = R^K(\gamma_0)\}$ . We are interested in the behavior of the eigenfunctions of the evolution operator  $U$  in the vicinity of the corresponding coherent state  $|\gamma_0\rangle$ . Defining  $M$ -step propagators as

$$F_M = \langle \gamma_0 | U^M | \gamma_0 \rangle, \quad (4)$$

one gets

$$F_M = \sum_{n=1}^N H_n(\gamma_0) e^{iM\varphi_n}, \quad (5)$$

where  $H_n = |\langle \phi_n | \gamma_0 \rangle|^2$  is the Husimi representation<sup>34,35</sup> of the  $n$ th eigenfunction  $|\phi_n\rangle$  corresponding to the quasienergy  $\varphi_n$ . Since

$$F(\omega) = \sum_{M=-L}^L e^{-i\omega M} F_M = \sum_{n=1}^N H_n(\gamma_0) \sum_{M=-L}^L e^{iM(\varphi_n - \omega)}, \quad (6)$$

the Fourier transform of the coefficients  $F_M$  contains an important piece of information: the eigenfunction with quasienergy close to a peak in  $F(\omega)$  is scarred by the orbit  $\Gamma$ . The problem of finding the eigenfunctions that are influenced by the periodic orbit  $\Gamma$  would therefore be solved if one could get  $N$  coefficients  $F_M$  with a sufficient precision using a semiclassical approximation.

### III. NUMERICAL STUDY OF THE KICKED TOP

The periodically kicked top has proved to be a useful tool for studies on quantum chaos.<sup>36-38</sup> The compact classical phase space and the finite size of the matrix representing the quantum evolution operator feature an interesting model to study the connection of classical periodic orbits and local properties of quantum eigenmodes.

The dynamical variables of a top are the three components of the angular momentum operator that obey the commutation relations  $[J_i, J_j] = i\epsilon_{ijk} J_k$ . The squared angular momentum is conserved:  $J^2 = j(j+1)$  with  $j$  integer or half-integer. The quantum number  $j$  fixes the dimension of the Hilbert space as  $N = 2j+1$ . The classical limit is approached with  $j \rightarrow \infty$ . The particular evolution operator

$$U = \exp \left[ -i \frac{k}{2j} J_z^2 \right] \exp \left[ -i \frac{\pi}{2} J_y \right] \quad (7)$$

defines a quantum system that, in general, pertains to the orthogonal universality class of random matrices and displays characteristic linear level repulsion.<sup>5</sup> The appropriate coherent states are now the so-called SU(2) (or angular momentum) coherent states,<sup>30</sup> with the following expansion in the  $|j, m\rangle$  basis:<sup>39</sup>

$$|\gamma\rangle = (1 + \gamma\gamma^*)^{-j} \sum_{m=0}^{2j} \binom{2j}{m}^{1/2} \gamma |j, j-m\rangle. \quad (8)$$

The corresponding classical map then reads

$$R(\gamma) = \frac{1+\gamma}{1-\gamma} \exp \left[ -ik \frac{\gamma + \gamma^*}{1 + \gamma\gamma^*} \right], \quad (9)$$

where  $\gamma$  is a complex number.

In order to analyze the local properties of the quantum system, it has been suggested<sup>37</sup> that the expansion of a coherent state in the eigenbasis of the system

$$|\gamma\rangle = \sum_{n=1}^N c_n |\phi_n\rangle \quad (10)$$

be studied. Statistical properties of this expansion can be described by the number of relevant eigenstates  $M_r$  (minimal number of eigenstates exhausting normalization up to a given number  $r$  equal to, say, 0.99), the sum of moduli  $S \equiv \sum_{n=1}^N |c_n|$ , and the Shannon entropy  $H_s$  of the coherent state  $H_s \equiv -\sum_{n=1}^N |c_n|^2 \ln |c_n|^2$ . The above quantities can be easily evaluated for ensembles of random matrices,<sup>40</sup> for which components of a typical vector are described by the  $\chi^2$  distribution.

Instead of working in the complex  $\gamma$  plane it is convenient to consider the surface of a sphere parametrized by two angles  $\theta, \varphi$ , where  $\gamma = \exp(i\varphi) \tan(\theta/2)$ . We have constructed coherent states along the meridian ( $\varphi = 3\pi/4, 0 \leq \theta \leq \pi$ ), diagonalized numerically the operator  $U$  and computed the three indicators defined above. To present them on the same graph, the following scaled quantities have been used:  $D_1 = M_{0.99}/N$ ,  $D_2 = S/\sqrt{N}$ , and  $D_3 = H_s/\ln(N/2)$ . Figure 1 presents

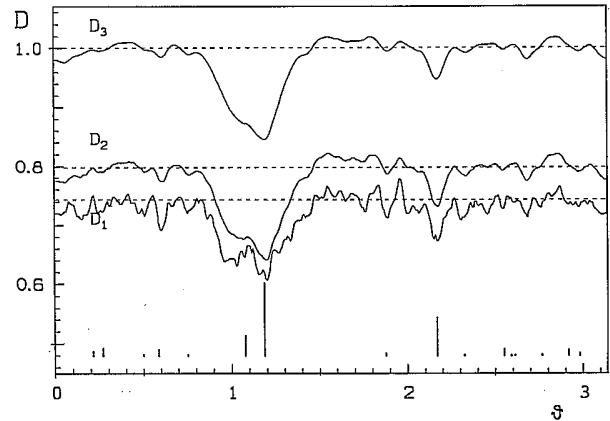


FIG. 1. Number of eigenstates exhausting 0.99 of the normalization ( $D_1$ ), sum of absolute values of eigenbasis expansion coefficients ( $D_2$ ), and Shannon entropy ( $D_3$ ) for the coherent state  $\gamma = \exp(i3\pi/4) \tan(\theta/2)$ . The dashed lines correspond to predictions of the random matrix theory. Parameters of the kicked top:  $k+8.0, j=200$ .

indicators  $D_1$ ,  $D_2$ , and  $D_3$  as a function of  $\theta$  with  $k=8.0$ . For these values of parameter the classical system displays full scale chaos and the statistics of the eigenvalues follows exactly the predictions of the orthogonal ensemble (this has been checked with an accuracy sufficient to distinguish between the Wigner surmise and the exact results<sup>41</sup>). Also, the statistics of the eigenvectors corresponds to predictions of the RMT,<sup>42,43</sup> which are denoted in the picture by dashed horizontal lines. On the other hand, quite large fluctuations around the mean values (in particular for the percentage of relevant eigenstates  $D_1$ ) can be observed. The minima of each curve coincide with the position of periodic orbits in the classical phase space, which are denoted by vertical lines. The lengths of these lines are proportional to  $w = \exp(-\lambda)$ , where  $\lambda$  is the greater of two stability exponents of the corresponding periodic orbit. It is clear that the most stable orbits influence strongly the statistical properties of the eigenvector expansion studied.

Since the number of relevant eigenstates for coherent states in the vicinity of the periodic orbit is smaller than that resulting from the RMT, some eigenfunctions carry unusually large probability. In other words, they are scarred by a corresponding periodic orbit. This information is obtained without plotting any of the eigenstates.

Let us now draw our attention to one particular unstable fixed point  $\gamma_0 = R(\gamma_0)$  of the classical map  $R$  given by

$$\gamma_0 = \frac{a}{1-\alpha} \left[ 1 + i \cot \frac{k\alpha}{2} \right], \quad (11)$$

where  $\alpha$  is a solution of the transcendental equation:

$$\alpha^2 \left[ 1 + \sin^2 \frac{k\alpha}{2} \right] = \sin^2 \frac{k\alpha}{2}.$$

We shall calculate the coefficient  $F_1 = \langle \gamma_0 | U | \gamma_0 \rangle$  applying the explicit expansion of the coherent state (8):

$$\begin{aligned} \langle \gamma_0 | U | \gamma_0 \rangle &= \left[ \frac{1-\gamma_0}{1+\gamma_0\gamma_0^*} \right]^{2j} e^{-ikj/2} \\ &\times \sum_{n=0}^{2j} \binom{2j}{n} \left[ \gamma_0^* \frac{1+\gamma_0}{1-\gamma_0} e^{ik} \right]^n e^{-(ik/2j)n^2}. \end{aligned} \quad (12)$$

Using a Gaussian representation of  $\exp(-ik^2/2j)$  we arrive at

$$\begin{aligned} \langle \gamma_0 | U | \gamma_0 \rangle &= e^{i(\pi/4)} \left[ \frac{j}{2\pi k} \right]^{1/2} \left[ \frac{1-\gamma_0}{1+\gamma_0\gamma_0^*} \right]^{2j} \\ &\times e^{-ikj/2} \int_{-\infty}^{\infty} dt e^{jV(t)}, \end{aligned} \quad (13)$$

where

$$V(t) = -\frac{i}{2k} t^2 + 2 \ln \left[ 1 + \gamma_0^* \frac{1+\gamma_0}{1-\gamma_0} e^{t+ik} \right]. \quad (14)$$

In the semiclassical limit ( $j \gg 1$ ), the integral is calculated in the saddle-point approximation yielding

$$\langle \gamma_0 | U | \gamma_0 \rangle_{sc} = (\mu_1 + \mu_2 - 2)^{-1/2} \left[ \frac{1-\gamma_0}{\sqrt{2}} \right]^{(2j+1)} \exp \left[ ik \left\{ (2j+1) \left[ \left( \frac{\gamma_0\gamma_0^*}{1+\gamma_0\gamma_0^*} \right)^2 - \frac{1}{4} \right] - \frac{1}{4} \left[ \frac{1-\gamma_0\gamma_0^*}{1+\gamma_0\gamma_0^*} \right] \right\} \right], \quad (15)$$

where  $\mu_1 = e^{\lambda_1}$  and  $\mu_2 = e^{\lambda_2}$ ,  $\lambda_1$  and  $\lambda_2$  being the stability exponents of the classical orbit  $\Gamma = \{\gamma_0\}$ . We can also calculate higher-order propagators using the resolution of identity for the coherent states

$$I = \frac{2j+1}{4\pi} \int \frac{d \operatorname{Re}(\eta) d \operatorname{Im}(\eta)}{(1+\eta\eta^*)} |\eta\rangle \langle \eta| \quad (16)$$

and approximating resulting integrals by two-dimensional saddle-point contributions. Finally one gets

$$\langle \gamma_0 | U^M | \gamma_0 \rangle_{sc} = (\mu_1^M + \mu_2^M - 2)^{-1/2} \left[ \frac{1-\gamma_0}{\sqrt{2}} \right]^{M(2j+1)} \exp \left[ ikM \left\{ (2j+1) \left[ \left( \frac{\gamma_0\gamma_0^*}{1+\gamma_0\gamma_0^*} \right)^2 - \frac{1}{4} \right] - \frac{1}{4} \left[ \frac{1-\gamma_0\gamma_0^*}{1+\gamma_0\gamma_0^*} \right] \right\} \right]. \quad (17)$$

Semiclassical approximations of the coefficients  $F_M^{sc} = \langle \gamma_0 | U^M | \gamma_0 \rangle_{sc}$  obtained in this way agree reasonably well with the exact values of  $F_M$  obtained from the numerical analysis of the quantum system (7) only for some first values of  $M$ . It is not very surprising, since  $F_M^{sc}$  decreases in modulus with  $M$  as  $e^{-M\lambda}$ , where  $\lambda$  is the greater of two stability exponents. Moreover, it is known<sup>36</sup> that quantum evolution follows the corresponding classical dynamics of a chaotic system only for times of the order of  $\ln(j)$ . Since to obtain a reasonable resolution of the Fourier transform (6) (of the order of  $2\pi/j$ ) about  $j$  coefficients are needed, one should not expect ex-

traordinary results. Nevertheless a semiclassical method can at least give some information in which part of the spectrum the probability of scarring by a given periodic orbit is large.

Figure 2 shows the Fourier transform (6) obtained with  $j=50$ ,  $L=100$  for  $\gamma_0$  defined by Eq. (11) (which, for  $k=8.0$ , corresponds to  $\theta=2.169$  and  $\varphi=5.498$ ). The smooth line results from the semiclassical formula (17), while the spiked curve corresponds to a fully quantum calculation. Observe that the semiclassical curve provides a good approximation to a *smoothing* of the full quantum calculation. The peaks in the latter curve

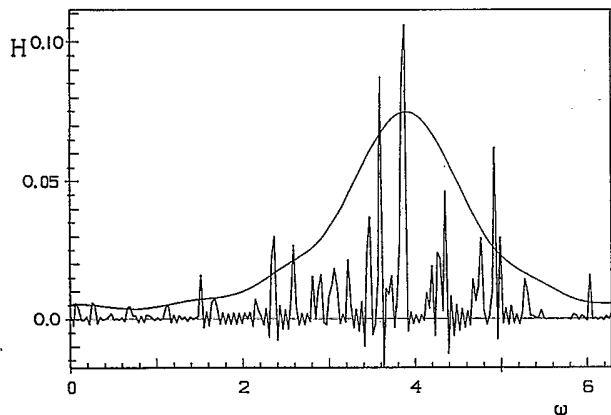


FIG. 2. Quantum-mechanical (spiked line) and semiclassical (smooth line) Fourier transforms of the propagator matrix elements between coherent states corresponding to the classical fixed point for  $k=8.0$  and  $j=50$ .

indeed coincide with those eigenfunctions that are strongly scarred by the investigated orbit  $\gamma_0$ .

The coherent state  $|\gamma_0\rangle$  has been expanded in the system's eigenbasis. The four eigenstates that give the largest contribution to this coherent state are drawn in Husimi representation in Fig. 3. The unstable fixed point  $\gamma_0$  is denoted by an asterisk; the three crosses denote the

symmetric orbits. One indeed notices pronounced scarring in the vicinity of the fixed point considered and its symmetric images. Careful study of the plots reveals that the scarring maxima decay slowly along the stable and unstable manifolds of the orbit. This result agrees with observations made for the quantum rotor.<sup>29</sup> The quasienergy eigenvalues are approximately equal to 3.59, 3.88, 3.87, and 4.92 for each eigenfunction, respectively, and correspond to the peaks in Fig. 2. Note that the states drawn in Figs. 3(a) and 3(c) belong to a different symmetry class than do the states plotted in Figs. 3(b) and 3(d). Analyzing the statistical properties of the spectrum, one must to consider each class separately.

#### IV. CONCLUDING REMARKS

Up to now relatively little has been known about how properties of eigenfunctions of quantum maps can be connected to specific features of the underlying classical dynamical systems. From our analysis of the kicked top, it follows that even when many statistical properties of a quantum system, such as distributions of eigenvalues and eigenvector components, are excellently described by the random matrix theory, fingerprints of classical dynamics are clearly visible in the behavior of other statistical measures.

Explicit evaluation of the matrix elements of the semiclassical propagator in the coherent state representation

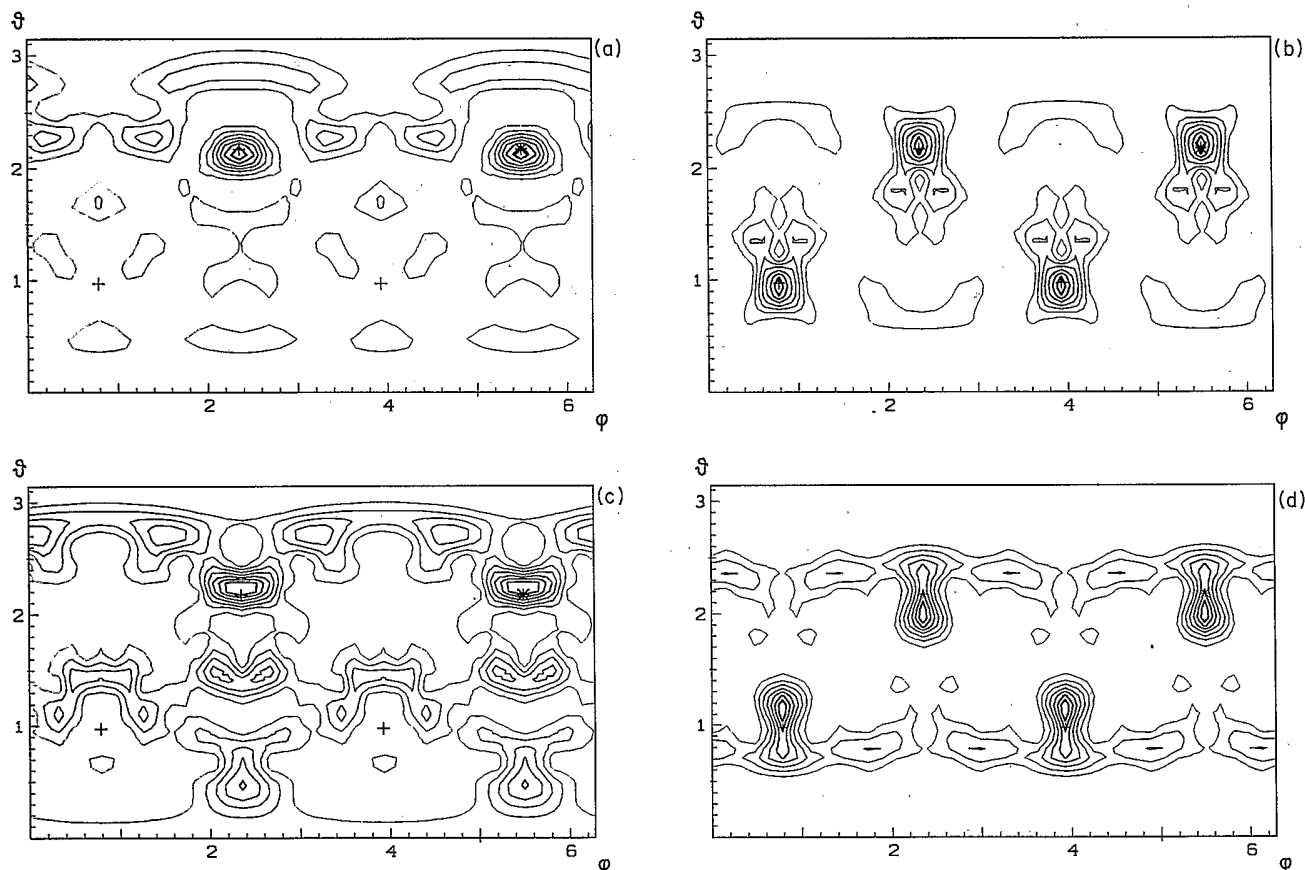


FIG. 3. The Husimi distributions of the eigenfunctions that are most intensively scarred by the fixed point. Level contours for two functions from each parity class are shown. The fixed point is denoted by an asterisk; three crosses correspond to its symmetric images.

has enabled us to estimate the range of quasienergies in which the wave-function scarring is most probable. While the precision obtained is by far not sufficient to pin down the scarred wave functions, one can give at least some information about where the eigenvalues of such functions are located within the spectrum. Some of the numerically obtained wave functions located in the "suspected" energy range indeed show strong localization on the periodic orbit as visualized by their Husimi distributions.

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