

Bounds on entanglement in qudit subsystemsVivien M. Kendon,^{1*} Karol Życzkowski,^{2†} and William J. Munro³¹*Optics Section, Blackett Laboratory, Imperial College, London SW7 2BW, United Kingdom*²*Center for Theoretical Physics, Polish Academy of Sciences, Al. Lotników 32/44, 02-668 Warszawa, Poland*³*Hewlett-Packard Laboratories, Filton Road, Stoke Gifford, Bristol BS34 8QZ, United Kingdom*

(Received 18 March 2002; revised manuscript received 31 May 2002; published 12 December 2002)

The entanglement in a pure state of N qudits (d -dimensional distinguishable quantum particles) can be characterized by specifying how entangled its subsystems are. A generally mixed subsystem of m qudits is obtained by tracing over the other $N - m$ qudits. We examine the entanglement in the space of mixed states of m qudits. We show that for a typical pure state of N qudits, its subsystems smaller than $N/3$ qudits will have a positive partial transpose and hence are separable or bound entangled. Additionally, our numerical results show that the probability of finding entangled subsystems smaller than $N/3$ falls exponentially in the dimension of the Hilbert space. The bulk of pure state Hilbert space thus consists of highly entangled states with multipartite entanglement encompassing at least a third of the qudits in the pure state.

DOI: 10.1103/PhysRevA.66.062310

PACS number(s): 03.67.Lx, 03.65.Ud

I. INTRODUCTION

Quantum information is a rapidly developing field exploiting the peculiar entanglement properties of quantum states. Its applications can be broadly divided into two general types: quantum computation [1,2] and quantum communication [3]. Quantum communication (including quantum cryptography and quantum teleportation) can be framed in terms of repeated use of pairs of entangled qubits—the entanglement properties of two qubits have been well characterized, and a number of analytical measures of entanglement are known [4–9]. However, for multiqubit systems, which are essential for quantum computation, few entanglement measures can be calculated even for pure states. Despite these difficulties, arrays of qubits have been the focus of recent attention [10–15], though often only pairwise entanglement has been considered in such systems. The more general question of entanglement in multiqubit systems (d -dimensional quantum particles) is also important, since most real systems (e.g., atoms) have more than two states.

In this paper, we tackle the question of entanglement in larger qudit systems by deriving a bound on the size of typical entangled mixed states that are subsystems of larger pure states. Previous numerical work by some of us [16] suggested that the qubit subsystems somewhat less than half the size of the pure state, typically have no usable entanglement. In other words, since pure states are typically highly entangled, the entanglement is distributed in multipartite entanglement involving around half or more of the qubits. This work was limited by computational resources to small numbers of qubits (less than 14). Here we provide analytical bounds for any number of qubits, and also extend the analysis to any dimension of qudits.

We emphasize that we are interested in the average properties of all possible pure states. It is easy to construct states with entanglement properties that lie outside our bounds. For

example, the so-called W states (symmetric superposition of one qubit in state $|1\rangle$ with the rest in state $|0\rangle$) have only pairwise entanglement [15] for any value of N . However, such states are of small measure and do not contribute significantly to the average properties of pure states sampled from the whole of the Hilbert space.

To further motivate our investigations, we note a connection to the question of the origin of the speedup in quantum computing. Recently Jozsa and Linden in Ref. [17] argue that for an exponential speedup over classical computation, the state of a quantum computer using pure states will have multipartite entanglement encompassing an arbitrarily large fraction of the total number of qubits in the computation. However, they also point out that the entanglement cannot be inferred to be the cause of the speedup, rather, it is one of a number of properties of the typical states in the Hilbert space of the quantum computer which are necessary for such a speedup. Our work confirms this property of typical states in Hilbert space, and also provides an analytical lower bound on the number of parties in the multipartite entanglement for a given size of Hilbert space.

The paper is organized as follows. First, in Sec. II, we introduce clear definitions of the system we are considering and the entanglement measures we will use to analyze it. Next in Sec. III, we derive the analytic bounds, followed by comparison with numerical results in Sec. IV. We finish with conclusions in Sec. V.

II. DEFINITIONS AND ENTANGLEMENT MEASURES

In a finite-dimensional Hilbert space \mathcal{H} , there exists a natural, unitarily invariant, Fubini-Study measure μ_{FS} , induced by the Haar measure on the unitary group. A system of N qudits has a Hilbert space of size d^N . We will investigate entanglement in such systems by sampling randomly from the set of all pure N -qudit states according to the natural measure μ_{FS} and calculating their average, which is to say, typical properties. This will give us useful information about the majority of states in Hilbert space. Having chosen a random pure state of N qudits, we can divide it into two sub-

*Electronic address: Viv.Kendon@ic.ac.uk

†Electronic address: karol@cft.edu.pl

systems by partial tracing. A subsystem of size m is obtained by partial tracing over the other $N-m$ qudits. This partitioning could represent the system of interest plus the environment, for example, or two different parts of one system. From the mathematical point of view, one obtains in this way a certain probability measure $\mu_{N,m}$ in the space of mixed quantum states acting in a d^m -dimensional Hilbert space induced by the natural measure μ_{FS} on the space of pure states in \mathcal{H}_{d^N} [18]. This is a useful way to sample randomly from mixed states, since unlike μ_{FS} for pure states, there is no unique natural measure for the space of mixed states.

A. Entropy of subsystems

Consider now a pure state quantum system described by a density operator $\rho = |\phi\rangle\langle\phi|$ acting in a composite Hilbert space $\mathcal{H}_{MK} = \mathcal{H}_M \otimes \mathcal{H}_K$ of dimension MK . The reduced density matrices defined by the partial trace $\rho_M := \text{tr}_K \rho$ and $\rho_K := \text{tr}_M \rho$ characterize both subsystems. It is well known that the von Neuman entropies of both subsystems are equal, $S_M = S(\rho_M) = -\text{tr} \rho_M \ln \rho_M = S(\rho_K) = S_K$. The value of S_M shows how entangled the two subsystems are with each other. If $S_M = 0$, there is no entanglement and the composite state may be factorized, $\rho = \rho_M \otimes \rho_K$, but this is not a typical case. The mean entropy of a subsystem averaged over the natural measure μ_{FS} is given by

$$\langle S_M \rangle = \sum_{j=K+1}^{MK} \frac{1}{j} - \frac{M-1}{2K} \approx \ln M - \frac{M}{2K}, \quad (1)$$

where $M \leq K$ and the approximation holds for $0 \ll M < K$. This result was first conjectured by Page [19] and later proved in Refs. [20,21].

In our case the subsystems consist of m and $N-m$ qudits, so substituting $M = d^m$ and $K = d^{N-m}$ into Eq. (1) we obtain the mean entropy $\langle S \rangle_{N,m}$ of the subsystem analyzed, where we are now using the subscripts to remind us of the number of qudits in the pure state (N) as well as the number in the subsystem (m). Equation (1) tells us that on average, the smaller subsystem has nearly maximal entropy, showing that the two subsystems are highly entangled with each other, a typical pure state is highly entangled, see also Ref. [22]. The particular value of $S_{N,m}$ provides a good measure of entanglement between the two subsystems, but says little about entanglement between the qudits within a single subsystem. Asymptotically, for $(N \rightarrow \infty)$, $m \leq N/2$ will tend to be maximally mixed and therefore separable, but to obtain useful results for finite N , we need further information on the size of the region around the maximally mixed state that contains separable states.

B. Purity and mixedness

Next we will characterize how mixed a typical m -qudit state ρ_m might be. A convenient measure is the *purity* [23] defined by

$$r(\rho_m) := \text{tr}(\rho_m^2). \quad (2)$$

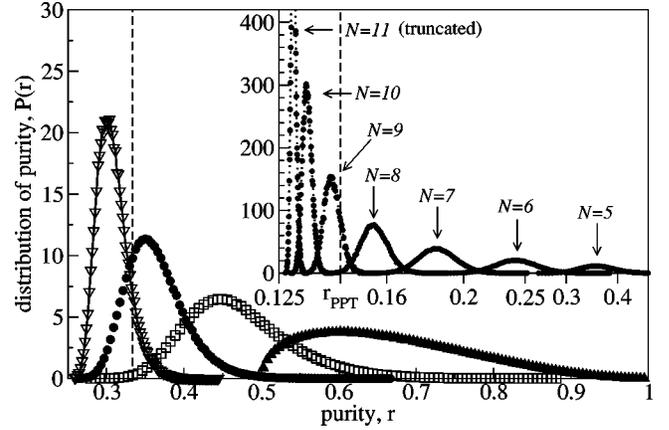


FIG. 1. The numerically obtained distribution of the purity r (main graph) mixed states of $m=2$ qubits obtained by partial tracing of random pure states of (right to left) $N=3, 4, 5$, and 6 qubits, and (inset) mixed states of $m=3$ qubits for $N=5-10$. The vertical dashed lines indicate the values of r_{PPT} from Eq. (5).

Pure states are defined to have $r(\rho_m) = 1$, while mixed states have $1/d^m \leq r(\rho_m) < 1$. We will also refer to $R := 1/\text{tr}(\rho_m^2) \equiv 1/r$ as the *inverse participation ratio* (IPR) of the mixed state ρ_m . The IPR thus ranges from 1 (pure) to d^m (maximally mixed), and larger IPR means the states are more mixed, while larger purity means the states are more nearly pure.

Consider random states drawn according to the natural measure μ_{FS} on the space of pure states in a MK -dimensional Hilbert space. As before, we apply a partial trace to obtain the reduced density matrices ρ_M and ρ_K . In Refs. [18,24,25], it was shown that the average purity of ρ_M and ρ_K is equal to

$$\langle r \rangle = \frac{M+K}{MK+1}. \quad (3)$$

This result does not depend on the particular way in which the MK -dimensional Hilbert space is decomposed, but only on the initial and the final dimensionality of the spaces. Thus it also holds in the problem we are analyzing here of pure states of N qudits reduced to m qudits by partial tracing. Substituting $M = d^m$ and $K = d^{N-m}$ the average purity of the system reads

$$\langle r \rangle_{N,m} = \frac{d^m + d^{N-m}}{d^N + 1}. \quad (4)$$

In other words, the averaging has been performed with respect to the induced measure $\mu_{N,m}$.

Numerical investigation of the induced measures shows [18] that for $m \leq N/2$ the probability distributions $P(r)$ are concentrated close to the mean value $\langle r \rangle$. This is illustrated in Fig. 1, where $P(r)$ is plotted for $d=m=2$ and $N=3-6$ in the main plot, and for $d=2$, $m=3$, $N=5-10$ in the inset. The scale of the purity axis in the inset has been adjusted (the scale is in fact linear in $R=1/r$) to show the data near the minimum purity more clearly.

Fixing the number m of the qudits in the final system and varying the initial number N of qudits involved in the pure state, we get a sequence of probability measures which sweep the space of the mixed states close to the manifold of a constant purity. Hence it is justified to use the mean value $\langle r \rangle$ as a good estimate of the actual value r for a typical subsystem.

C. Peres criterion, positive partial transpose

We will now describe a simple test for entanglement in multiparticle mixed states due to Peres [26]. Consider a multiparticle mixed state ρ acting in a d^m -dimensional Hilbert space, which describes the system consisting of m qudits. An m -particle system can be split into two parts containing j and $m-j$ particles, respectively. The partial transpose is applied to the j particle part of the system producing $\rho^{T_j} := (\mathbf{1}_L \otimes T_j)\rho$. Here T_j denotes the transpose operation in the d^j -dimensional subspace, while the dimension of the subspace that has not been transposed is $L = d^{m-j}$. We say that the m -qudit state has the positive partial transpose (PPT) property, if ρ^{T_j} is positive for all possible values of j (it is sufficient to take $1 \leq j \leq m/2$) and choices of j particles, i.e., with respect to *all* of its possible splitting into two subsystems. Conversely, a state which fails this test is described as being negative partial transpose (NPT). NPT states have some useful entanglement that could be distilled with some probability into one or more maximally entangled states of two qubits. This test does not distinguish bound entangled states [27,28] from separable states, but since bound entanglement is relatively rare [29], and for most purposes it is the free entanglement that is useful, this still provides a useful characterization of entanglement.

For bipartite systems with Hilbert-space dimension d^2 , it has been shown [5] that all the states sufficiently close to the maximally mixed state, with $R \geq d^2 - 1$, have the PPT property. This condition defines the maximal ball inscribed in the convex body of mixed states. Very recently, it was shown that all the mixed states of a bipartite system belonging to the maximal ball possess not only the PPT property but are also separable [30]. The former result may be generalized for a multipartite case: any state ρ_m (within the d^m -dimensional Hilbert space) for which

$$R \geq R_{\text{PPT}} = d^m - 1 \quad \text{so that} \quad r \leq r_{\text{PPT}} = \frac{1}{d^m - 1} \quad (5)$$

has the PPT property with respect to any possible transpositions of the subsystems. We give the proof of the generalization in Appendix A.

III. BOUNDS ON ENTANGLEMENT IN SUBSYSTEMS

We are now ready to provide an analytical relationship between the size of a subsystem and its typical entanglement properties.

A. Bound on PPT region

Using the fact that the distributions of the purity $P(r)$ are narrow for the regions of interest ($m \leq N/2$), and thus $\langle r \rangle_{N,m}$

is a good estimate of the actual value of r in a typical subsystem, we apply this to the critical value r_{PPT} obtained in the preceding section. If the mean value $\langle r \rangle$ averaged over a certain induced measure is smaller than the critical value r_{PPT} (or $\langle R \rangle > R_{\text{PPT}}$), the majority of random states distributed with respect to this measure are localized inside the maximal ball and are PPT. We therefore combine Eqs. (5) and (4) to calculate a relationship between N and m such that we expect almost all subsystems of m qudits will have a positive partial transpose. We find $\langle r \rangle_{N,m} \leq r_{\text{PPT}}$ when

$$d^N \geq d^{3m} (1 - d^{-m} - d^{-2m}). \quad (6)$$

Taking the logarithm (base d) gives us an estimate of how many qudits the initial pure state should contain, such that after the reduction to m qudits the obtained mixed state is PPT on average,

$$N_{\text{PPT}} \geq 3m + \log_d(1 - d^{-m} - d^{-2m}). \quad (7)$$

The \log_d term is always negative, becoming rapidly smaller for either $m > 2$ or $d > 2$. Hence for the systems initially consisting of

$$N \geq 3m \quad (8)$$

qudits, we expect the subsystems of size m to be PPT mixed states with a considerable probability. For the simplest case of $m = d = 2$, the smallest integer number larger than the right-hand side of Eq. (7) gives $N_{\text{PPT}} = \text{int}[6 - \log_2(16/11)] + 1 = 6$. For a fixed number N this establishes a bound on m below which subsystems of this size have on average no useful entanglement in an N -qudit pure state. This is the main result in this paper.

It is also possible to calculate the maximum value of the von Neuman entropy for states on the surface of the maximal ball, see Appendix B. This value S_c , given in Eq. (B1), can then be combined with Eq. (1) for the average entropy of a subsystem to obtain a similar relationship between N and m . The result is (as it should be) the same as Eq. (8), but with the restriction $m \geq 0$ from the approximate form of Eq. (1). Our derivation as presented using Eqs. (5) and (4) is applicable to all $0 < m < N$.

B. Estimation of the transition region

The above result is a bound on the average properties of typical states which identifies the range of parameter values ($N \geq 3m$), where there is a high probability that a subsystem of size m is PPT. We would also like to characterize the range of N, m for which with high probability the subsystems of size m are entangled. This region is not simply the inverse of the PPT region because while the criterion we are using in Eq. (5) is sufficient for ensuring the subsystem is PPT, it is not a necessary condition. There are many states which are PPT that have $r > r_{\text{PPT}}$ right up to separable pure states with $r = 1$.

Since we do not, in general, know the size of the region of PPT states outside the maximal ball defined by Eq. (5), our approach is to identify a region of the body of mixed states

of a considerable size, which contains entangled states only. There is a one parameter family of mixed states (known as the generalized Werner states) containing m qudits defined as a mixture of the maximally entangled pure state $|\Psi\rangle$ and the maximally mixed state I_{d^m} ,

$$\rho_\varepsilon = \frac{(1-\varepsilon)}{d^m} I_{d^m} + \varepsilon |\Psi\rangle\langle\Psi|, \quad (9)$$

where

$$|\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^d |i\rangle_1 \otimes \cdots \otimes |i\rangle_m \quad (10)$$

and ε is a real parameter between 0 and 1 specifying the proportions of the mixture. These states have been shown to be entangled for [31–33]

$$\varepsilon > \varepsilon_{ent} = \frac{1}{d^{m-1} + 1} \quad (11)$$

and strictly separable for $\varepsilon \leq \varepsilon_{ent}$ (there are no bound entangled states in this family). This boundary at ε_{ent} defines a set of entangled states of positive measure [34] as can be seen as follows. The states ρ_ε lie on a line that is on an axis of rotational symmetry in the set of mixed states. This line intersects the set of separable states at $\varepsilon = \varepsilon_{ent}$. The set of separable states is convex, so beyond the hyperplane normal to the ρ_ε line there exists a set of entangled states of a positive measure (see also Ref. [35]).

It is straightforward to show that the purity for ρ_ε at this boundary point ($\varepsilon = \varepsilon_{ent}$) is given by

$$r_{ent} = \frac{d^m + d^2 + 2d}{(d^m + d)^2}. \quad (12)$$

Thus there exists a greater-than-zero probability to encounter entangled states with $r > r_{ent}$. However, we do not know that entangled states actually dominate over PPT states for this value of r . Since Werner states are among the most entangled states [36–38] for a given purity r , we might actually guess that this estimate will turn out to be too tight. Nonetheless, it provides a useful independent check on our previously derived bound, so we present it anyway.

Equating $\langle r \rangle_{N,m}$ from Eq. (4) with r_{ent} from Eq. (12) allows us to establish an estimate for the entangled side of the transition between the entangled and PPT states. It is easily shown that $\langle r \rangle_{N,m} \geq r_{ent}$ when

$$N_{ent} \leq 3m - 2 + \log_d[1 + (2d + 1)d^{-m} + (d + 2)d^{1-2m}]. \quad (13)$$

For all d and m , the \log_d term is positive and tending to zero for increasing d and m . Hence for all finite N with

$$N \leq 3m - 2, \quad (14)$$

we expect the probability of finding the subsystem of size m entangled (not PPT) to be nonzero. Looking at the smallest

case [39], $d = m = 2$, we have $N_{ent} = \text{int}[4 + \log_2(11/4)] = 5$. Let us emphasize again, that this is an estimation only; for this (or lower) values of N the probability of finding entangled subsystems of size m is nonzero, but it does not rule out the existence of substantial numbers of PPT subsystems of size m , so the transition might occur effectively for even smaller values of N .

Now let us fix the initial size of the pure states of N qudits, while we decrease the size of the final system of m qudits. For $m = N$, the probability of finding a separable pure state is equal to zero. As m is reduced, the states obtained by partial tracing over $N - m$ qudits become increasingly mixed, and the probability of finding PPT states increases. Putting both results together, we may characterize quantitatively a transition region

$$3m - 2 \leq N \leq 3m, \quad (15)$$

which does not depend on the qudit size d , in which we estimate that PPT subsystems come to dominate over entangled subsystems. (Note that the inequalities have reversed because we are identifying the region where the ratio of PPT to NPT subsystems of size m is of order unity, rather than the regions where one case dominates over the other.) These are surprisingly tight estimates of the boundary between the entangled and PPT states in terms of the number of qudits in the state. However, it must be remembered that the number of qudits is a logarithmic measure of the size of the Hilbert space. The addition of one qudit enlarges the Hilbert space by a factor of d .

IV. NUMERICAL COMPARISONS

In Ref. [16], for the qubit case $d = 2$, it was shown numerically that entanglement in a mixed m -qubit subsystem of a typical N -qubit pure state falls off sharply towards zero with increasing N for values of $m \leq 5$ and $N \leq 13$. These numerical results suggest the transition from NPT states dominating to PPT states dominating is complete by $N \approx 2m + 3$, which falls outside the range of Eq. (15) for $m > 4$. This is at the edge of the results presented in Ref. [16], so we ran further numerical studies to confirm this divergence between the numerical and analytical results and also, since our analytical results apply for any dimension of the constituent quantum particles d , to test qutrits and confirm they follow the same pattern.

Using straightforward C programs optimized for efficiency to generate the large datasets required for useful statistics, we generated random pure states in the d^N -dimensional Hilbert space, traced over $N - m$ qudits, and analyzed whether the remaining mixed state, with a d^m -dimensional Hilbert space, had a positive or negative partial transpose (PPT or NPT). In Fig. 2, we represent the transition region in the $N - m$ plane for qubits up to $m = 6$, $N = 15$ and qutrits up to $m = 3$, $N = 8$, combining data from Ref. [16] with our data for $N = 14$ to 15 (qubits) covering the transition for $m = 6$, and qutrits for $N = 4$ to 8. Our results confirm the trend. For $m > 4$, the transition from NPT to PPT for a given subsystem size m is completed for $N < 3m - 2$.

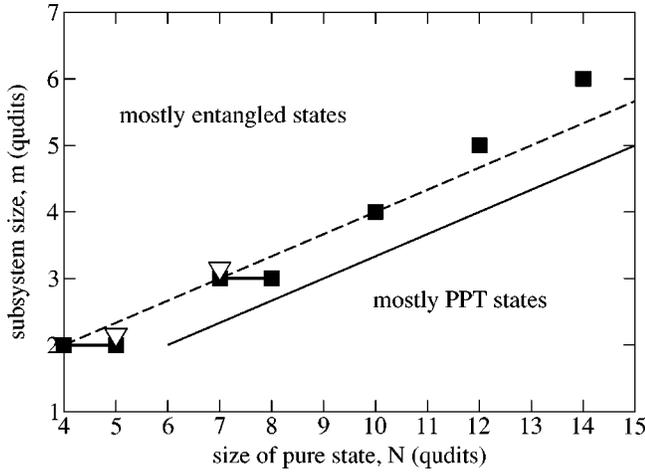


FIG. 2. Numerical results showing the transition from entangled to PPT subsystems. For each m , points are shown for which $P_{\text{PPT}} \in (1\%, 99\%)$ for qubits (■) and qutrits (▽). The solid/dashed lines represent the range in Eq. (15).

This is due to our estimates not taking into account the full size of the set of PPT states, being based only on the maximal ball inscribed in this set.

There is one more feature of this transition that requires comment. As noted in Ref. [16], the transition is very sharp, the proportion of PPT subsystems of a given size m becomes essentially zero to numerical accuracy with the addition of just a few qudits to the pure state. It is not obvious *a priori* that the transition should be sharp. One might reasonably expect it to be smoother, with a typical pure state of N qudits containing finite amounts of m -partite entanglement for all $2 \leq m \leq N$. In order to obtain such a sharp transition, the proportion of NPT subsystems of given size m must fall exponentially in the size of the Hilbert space d^N of the pure state. This is hard to check numerically, but the data we have do support this, see Fig. 3. This sharp transition and expo-

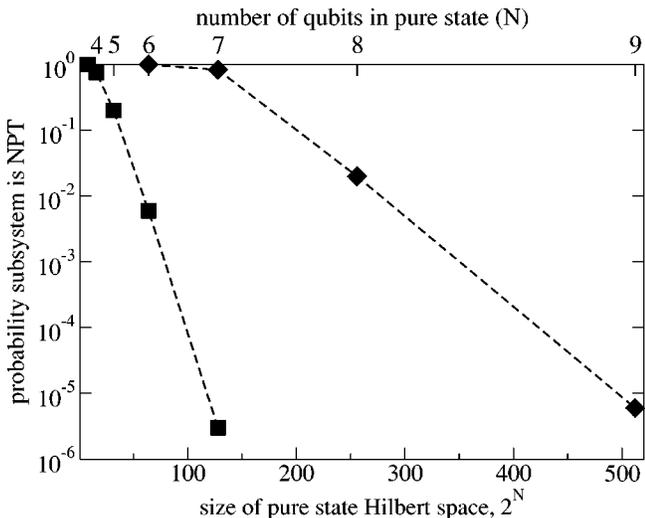


FIG. 3. Probability of finding that a subsystem of m qudits is entangled in random pure states sampled uniformly over the Haar measure for qubits for $m=2$ (■) and $m=3$ (filled ◆).

ponential fall off allows us to assert that this bound on the multipartite nature of entanglement in typical states of a Hilbert space corresponds to the distinction made in Ref. [17] between the sets of states that can be simulated efficiently classically, and sets of states that cannot. It also justifies our statement in the Introduction that exceptions to our bound (8) do not contribute significantly to the average properties of typical pure states.

V. DISCUSSION AND CONCLUSIONS

We have obtained analytically a lower bound on the size of subsystem of a typical pure state that can be expected to contain any entanglement within the subsystem. For a typical pure state of N qudits, its subsystems of size $m < N/3$ can be expected to be PPT, i.e., to have no usable entanglement within them. We then checked numerically and confirmed that the transition from NPT to PPT subsystems actually occurs at significantly larger m nearer to $N/2$, which is the asymptotic limit as $N \rightarrow \infty$. Our results emphasize the importance of multipartite entanglement in quantum-information processing. The types of states necessary for exploring the main bulk of Hilbert space with quantum computers are highly entangled, but the entanglement will not be evident if only a small subset of the qudits are examined. For a quantum computer operating in a pure state, at least one-third of the qubits would need to be processed to detect the entanglement present.

ACKNOWLEDGMENTS

It is a pleasure to thank Christof Zalka for useful discussions. This work was funded in part by the European project EQUIP (IST-1999-11053) and QUIPROCON (IST-1999-29064). V.K. was funded by the UK Engineering and Physical Sciences Research Council, Grant No. GR/N2507701. K.Ż. acknowledges the Polish KBN Grant No. 2P03B-072-19, and is grateful to the Blackett Laboratory for the hospitality during his stay in London.

APPENDIX A: PPT PROPERTY FOR MIXED STATES OF MULTIPARTITE SYSTEMS

In this appendix we prove the following:

Proposition. Any mixed state ρ of m qudits which satisfies the condition

$$R(\rho) = \frac{1}{\text{tr}\rho^2} \geq d^m - 1 \tag{A1}$$

has the PPT property, i.e., all its partially transposed matrices $\rho_m^{T_j}$ are positive.

We start the proof by invoking an algebraic

Lemma. Let A be an $n \times n$ nonzero Hermitian matrix and consider the real number $\alpha := \text{tr}A / (\text{tr}A^2)^{1/2}$. If $\alpha \geq \sqrt{n-1}$ then $A \geq 0$.

Proof. The proof of this is given by Mehta [40] inequality (9.21), p. 217. Let us apply it for the analyzed density matrix partially transposed in an arbitrary way, $A = \rho^{T_j}$, which re-

mains a Hermitian matrix of size $n=d^m$. Any operation of partial transpose T_j does not change the traces, so $\text{tr}\rho^{T_j} = \text{tr}\rho = 1$ and $\text{tr}(\rho^{T_j})^2 = \text{tr}\rho^2 = 1/R$. Therefore, the coefficient $\alpha^2 = 1/\text{tr}\rho^2$ is just equal to the inverse participation ratio R . If a mixed state ρ satisfies the condition (A1), the assumption $\alpha \geq \sqrt{n-1}$ is fulfilled. Hence matrices ρ^{T_j} are positive for all possible operations T_j of partial transpose, so the state ρ is PPT. ■

It is illuminating to discuss a simple geometric interpretation of the condition (A1), which defines a certain subset of the convex body of mixed states \mathcal{M} acting in $n=d^m$ -dimensional Hilbert space. The center of the (n^2-1) -dimensional set \mathcal{M} is given by the maximally mixed state $\rho_* = 1/n$. Introducing the Hilbert-Schmidt metric

$$D_{\text{HS}}(\rho_1, \rho_2) = (\text{tr}(\rho_1 - \rho_2)^2)^{1/2}, \quad (\text{A2})$$

it is not difficult to compute the distance of any state ρ with eigenvalues $\{x_i\}$ to the center of \mathcal{M} ,

$$D_{\text{HS}}^2(\rho, \rho_*) = \sum_{i=1}^n \left(x_i - \frac{1}{n}\right)^2 = \frac{1}{R} - \frac{1}{n}. \quad (\text{A3})$$

Hence the condition $R = \text{const}$ determines a set of points equidistant from ρ_* .

The boundary of \mathcal{M} is defined by the condition $\det \rho = 0$, so any state belonging to it has at least one eigenvalue equal to zero. The state ρ_a with the spectrum $\{1/(n-1), 1/(n-1), \dots, 1/(n-1), 0\}$ is closest to ρ_* among the states belonging to the boundary of \mathcal{M} . Its inverse participation ratio $R(\rho_a) = n-1$ coincides with the critical value from the constraint (A1). Thus all states satisfying it form the maximal ball inscribed into the convex body of mixed states \mathcal{M} and centered at ρ_* .

APPENDIX B: VON NEUMANN ENTROPY ON THE BOUNDARY OF THE MAXIMAL BALL

In this appendix, we outline the proof of the following: *Proposition.* If the von Neuman entropy of a state ρ is

larger than S_c , then ρ belongs to the maximal ball and its inverse participation ratio R is larger than $n-1$, where n is the dimension of the Hilbert space of ρ and S_c is given by

$$S_c = \ln n - \frac{2}{n} \ln 2 + \frac{(n-2)}{n} \ln \left[\frac{n-1}{n-2} \right]. \quad (\text{B1})$$

In the asymptotic limit $n \rightarrow \infty$, $S_c \rightarrow \ln n$, i.e., maximal.

Proof. State ρ_a , with the spectrum

$$\lambda_a = \left\{ \frac{1}{n-1}, \dots, \frac{1}{n-1}, 0 \right\}, \quad (\text{B2})$$

belongs to the boundary of the set of mixed states and to the maximal ball inscribed in it since

$$R = \frac{1}{\text{tr}(\rho^2)} = n-1, \quad (\text{B3})$$

see Appendix A. Its entropy $S(\rho_a) = \ln(n-1)$ is the smallest among all states located at the surface of the hypersphere $R = n-1$. To show this it suffices to perturb the spectrum keeping R constant and compute the entropy variation. Due to the symmetry of the problem, the state ρ_b , which belongs to this hypersphere and has the largest entropy, is located opposite to ρ_a with respect to the maximally mixed state $\rho_* = 1/n$, and we have

$$(\rho_b + \rho_a)/2 = \rho_*. \quad (\text{B4})$$

This gives the spectrum of ρ_b ,

$$\lambda_b = \left\{ \frac{n-2}{n(n-1)}, \dots, \frac{n-2}{n(n-1)}, \frac{2}{n} \right\}, \quad (\text{B5})$$

which evidently belongs to the maximal ball $R = n-1$.

Computing its entropy $S(\rho_b)$, one obtains the critical value S_c given by Eq. (B1). ■

-
- [1] D.P. DiVincenzo, *Science* **270**, 255 (1995).
[2] V. Vedral and M.B. Plenio, *Prog. Quantum Electron.* **22**, 1 (1998).
[3] B. Schumacher, *Phys. Rev. A* **54**, 2614 (1996).
[4] W.K. Wootters, *Phys. Rev. Lett.* **80**, 2245 (1998).
[5] K. Życzkowski, P. Horodecki, A. Sanpera, and M. Lewenstein, *Phys. Rev. A* **58**, 883 (1998).
[6] M. Horodecki, *Quantum Inf. Comput.* **1**, 3 (2001).
[7] G. Vidal and R.F. Werner, *Phys. Rev. A* **65**, 032314 (2002).
[8] C.H. Bennett, D.P. DiVincenzo, J.A. Smolin, and W.K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).
[9] V. Vedral, M.B. Plenio, M.A. Rippin, and P.L. Knight, *Phys. Rev. Lett.* **78**, 2275 (1997).
[10] P. Stelmachovic and V. Bužek (unpublished).
[11] R. Raussendorf and H.J. Briegel, *Phys. Rev. Lett.* **86**, 5188 (2001).
[12] D. Gunlycke, V.M. Kendon, V. Vedral, and S. Bose, *Phys. Rev. A* **64**, 042302 (2001).
[13] W.K. Wootters, e-print quant-ph/0001114.
[14] K.M. O'Connor and W.K. Wootters, *Phys. Rev. A* **63**, 052302 (2001).
[15] M. Koashi, V. Bužek, and N. Imoto, *Phys. Rev. A* **62**, 050302(R) (2000).
[16] V.M. Kendon, K. Nemoto, and W.J. Munro, *J. Mod. Opt.* **49**, 1709 (2002).
[17] R. Jozsa and N. Linden, e-print quant-ph/0201143.
[18] K. Życzkowski and H.-J. Sommers, *J. Phys. A* **34**, 7111 (2001).
[19] D.N. Page, *Phys. Rev. Lett.* **71**, 1291 (1993).
[20] S. Sen, *Phys. Rev. Lett.* **77**, 1 (1996).
[21] S.K. Foong and S. Kanno, *Phys. Rev. Lett.* **72**, 1148 (1994).
[22] M.A. Nielsen and J. Kempe, *Phys. Rev. Lett.* **86**, 5184 (2001).
[23] D. Weaire and A.R. Williams, *J. Phys. C* **10**, 1239 (1977).

- [24] P. Zanardi, C. Zalka, and L. Faoro, Phys. Rev. A **62**, 030301(R) (2000).
- [25] E. Lubkin, J. Math. Phys. **19**, 1028 (1978).
- [26] A. Peres, Phys. Rev. Lett. **77**, 1413 (1996).
- [27] M. Horodecki, P. Horodecki, and R. Horodecki, Phys. Rev. Lett. **80**, 5239 (1998).
- [28] P. Horodecki, Phys. Lett. A **232**, 333 (1997).
- [29] K. Życzkowski, Phys. Rev. A **60**, 3496 (1999).
- [30] L. Gurvits and H. Barnum, e-print quant-ph/0204159.
- [31] P. Rungta, W.J. Munro, K. Nemoto, P. Deuar, G.J. Milburn, and C.M. Caves, in *Directions in Quantum Optics: A Collection of Papers Dedicated to the Memory of Dan Walls*, edited by H.J. Carmichael, R.J. Glauber, and M.O. Scully (Springer-Verlag, Berlin, 2001), pp. 149–164.
- [32] P. Deuar, W.J. Munro, and K. Nemoto, J. Opt. B: Quantum Semiclassical Opt. **2**, 225 (2000).
- [33] A.O. Pittenger and M.H. Rubin, e-print quant-ph/0001110.
- [34] By “positive measure” we mean that we have divided the Hilbert space into two regions with measures of comparable size in $\mu_{N,m}$, i.e., the ratio of the measures of the two parts is finite (neither infinite nor zero).
- [35] R. Lockhart and M. Steiner, e-print quant-ph/000909.
- [36] W.J. Munro, D.F.V. James, A.G. White, and P.G. Kwiat, Phys. Rev. A **64**, 030302 (2001).
- [37] S. Ishizaka and T. Hiroshima, Phys. Rev. A **62**, 022310 (2000).
- [38] F. Verstraete, K. Audenaert, T.D. Bie, and B.D. Moor, Phys. Rev. A **64**, 012316 (2001).
- [39] For two qubits, ε_{ent} is at the boundary of the maximal ball (PPT implies separable in this case, see also Ref. [30]), which is why the exact results for the bounds for two qubits are the same [right-hand side of Eqs. (8) and (13) are equal]. But this isn’t so for $d > 2$ or $m > 2$, hence we get a different value for the bound from this method, in general. The states in between are maximally entangled mixed states, see, Refs. [35–37].
- [40] M.L. Mehta, *Matrix Theory* (Hindustan Publishing, Delhi, 1989).