Negativity of the Wigner function as an indicator of non-classicality

Anatole Kenfack[1,2,5] **and Karol Życzkowski**[3,4]

[1] Max Planck Institute for the Physics of Complex Systems, Nöthnitzerstrasse 38, 01187 Dresden, Germany
[2] Physics Department, University of Dschang, PO Box 67, Dschang, Cameroon
[3] Instytut Fizyki imienia Mariana Smoluchowskiego, Uniwersytet Jagielloński, ulica Reymonta 4, 30-059 Kraków, Poland
[4] Centrum Fizyki Teoretycznej, Polska Akademia Nauk, Aleja Lotników 32/44, 02-668 Warszawa, Poland

E-mail: kenfack@mpipks-dresden.mpg.de

Abstract
A measure of non-classicality of quantum states based on the volume of the negative part of the Wigner function is proposed. We analyse this quantity for Fock states, squeezed displaced Fock states and cat-like states defined as coherent superposition of two Gaussian wavepackets.

Keywords: quasiprobability distribution functions, non-classicality, negativity of Wigner functions, Fock states, squeezed states, generalized Fock states, Schrödinger cat state

(Some figures in this article are in colour only in the electronic version)

1. Introduction

Analysing pure quantum states in an infinite dimensional Hilbert space it is useful to distinguish a family of *coherent states*, localized in the classical phase space and minimizing the uncertainty principle. These quantum analogues of points in the classical phase space are often considered as 'classical' states. For an arbitrary quantum state one may pose a natural question, to what extent is it 'non-classical' in a sense that its properties differ from that of coherent states? In other words, is there any parameter that may legitimately reflect the degree of non-classicality of a given quantum state? This question was motivated by the first observation of non-classical features of electromagnetic fields such as sub-Poissonian statistics, antibunching and squeezing. Additionally, it is well known that the interaction of (non)linear devices with quantum states may cause them to flip from one state to another; for instance, nonlinear devices may produce non-classical states from their interaction with the vacuum or a classical field. A systematic survey of non-classical properties of quantum states would be worthwhile because of the current ever increasing number of

experiments in nonlinear optics. An earlier attempt to shed some light on the non-classicality of a quantum state was pioneered by Mandel [1], who investigated radiation fields and introduced a parameter **q** to measure the deviation of the photon number statistics from the Poissonian distribution, characteristic of coherent states.

In general, to define a measure of non-classicality of quantum states one can follow several different approaches [2]. Distinguishing a certain set $\mathcal{C}$ of states (e.g. the set of coherent states $|\alpha\rangle$), one looks for the distance of an analysed pure state $|\psi\rangle$ to this set, by minimizing a distance $d(|\psi\rangle, |\alpha\rangle)$ over the entire set $\mathcal{C}$. Such a scheme based on the trace distance was first used by Hillery [3, 4], while other distances (the Hilbert–Schmidt distance [5, 6] or the Bures distance [7, 8]) were later used for this purpose. The same approach is also applicable to characterize mixed quantum states: minimizing the distance of the density ρ to the set of coherent states is related [6, 9] to the search for the maximal fidelity (the Hilbert–Schmidt fidelity $\text{Tr}(\rho\sigma)$ or the Bures–Uhlmann fidelity (Tr $\sqrt{\rho^{1/2}\sigma\rho^{1/2}})^2$) with respect to any coherent state, $\sigma = |\alpha\rangle\langle\alpha|$. On the same footing, the Monge distance introduced in [10, 11] may be applied to describe to what extent a given mixed state is close to the manifold of coherent states.

[5] Author to whom any correspondence should be addressed.

Yet another way of proceeding is based on the generalized (Cahill) phase space representation R_τ of a pure state, which interpolates between the Husimi (Q), the Wigner (W) and the Glauber–Sudarshan (P) representations. The Cahill parameter τ is proportional to the variance of a Gaussian function one needs to convolute with P representation to obtain R_τ [12]. In particular for $\tau = 1, 1/2, 0$ one obtains the Q-, W- and P-representations, respectively. By construction the Q representation is non-negative for all states, while the Wigner function may also admit negative values, and the P representation may be singular or may not exist.

The smoothing effect of R_τ is enhanced as τ increases. If τ is large enough so that R_τ becomes a positive definite regular function, thus acceptable as a classical distribution function, then the smoothing is said to be complete. The greatest lower bound τ_m for the critical value was adopted by Lee [13, 14], as the *non-classical depth* of a quantum state, and this approach was further developed in [15–17]. The limiting value, $\tau_m = 1$, corresponds to the Q function which is always acceptable as a classical distribution function. The lowest value, $\tau_m = 0$, is ascribed to an arbitrary coherent state because its P function is a Dirac delta function, so its ϵ-smoothing becomes regular. The range of τ_m is thus $\tau_m \in [0, 1]$.

If the Husimi function of a pure state admits at least one zero $Q(\alpha_0) = 0$, then a Cahill R_τ distribution with a narrower smearing, $\tau < 1$, becomes negative in the vicinity of α_0. Therefore the classical depth for such quantum states is maximal, $\tau_m = 1$ [15]. The only class of states for which Q representation has no zeros are the squeezed coherent states for which τ_m is a function of the squeezing parameter s. In the limiting case $s = 0$ one obtains the standard coherent state for which the $R_0 = P$ distribution is a Dirac delta function, that is $\tau_m = 0$.

A possible way to distinguish a classical state is to require that its P-representation exists and is everywhere non-negative. Such an approach was advocated in [18] and further explored in [19], while a recent work [20] establishes a link between the task of classifying all states with positive P-representation and the 17th Hilbert problem concerning positive polynomials.

A closely related approach to characterizing quantum states is based on properties of their Wigner functions in phase space $\{p, q\}$. One can prove that the Wigner function is bounded from below and from above [12]. In the normalization $\int \int W(q, p) \, dq \, dp = 1$ used later in this work, such a bound reads $|W(q, p)| \leqslant 1/\pi\hbar$. Further bounds on integrals of the Wigner function were derived in [21], while an entropy approach to the Wigner function was developed in [22, 23].

In order to interpret the Wigner function as a classical probability distribution one needs to require that W is non-negative. As found by Hudson in 1974 [24], this is the case for coherent or squeezed vacuum states only. A possible measure of non-classicality may thus be based on the negativity of the Wigner function which may be interpreted as a signature of quantumness.

The negativity of the Wigner function has been linked to non-locality, according to the Bell inequality [25], while investigating the original Einstein–Podolsky–Rosen (EPR) state [26]. In fact Bell argued that the EPR state will not exhibit non-local effects because its Wigner function is everywhere positive, and as such will allow for a hidden variable description of correlations. However, it is now demonstrated [27, 28] that the Wigner function of the EPR state, though positive definite, provides direct evidence of non-locality. This violation of Bell's inequality holds true for the regularized EPR state [29] and also for a correlated two-mode quantum state of light [30].

It is also worth recalling that the Wigner function can be measured experimentally [31], including the measurements of its negative values [32]. The interest put on such experiments has triggered a search for operational definitions of the Wigner functions, based on experimental setup [33, 34].

The aim of this paper is to study a simple indicator of the non-classicality, which depends on the *volume* of the negative part of the Wigner function. To demonstrate a potential use of such an approach we investigate certain families of quantum states. The non-classicality indicator is defined in section 2. The Schrödinger cat state, being constructed as coherent superposition of two Gaussian wavepackets, is analysed in section 3 while section 4 is devoted to Fock states and to the squeezed displaced Fock states. Finally in section 5, a brief discussion of results and perspectives is given.

2. The non-classicality indicator

The Wigner function of a state $|\psi\rangle$ defined by [35, 36]

$$W_\psi(q, p) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} dx \left\langle q - \frac{x}{2} | \psi \right\rangle \left\langle \psi | q + \frac{x}{2} \right\rangle \exp(ipx) \quad (1)$$

satisfies the normalization condition $\int \int W_\psi(q, p) \, dq \, dp = 1$. Hence the doubled volume of the integrated negative part of the Wigner function may be written as

$$\delta(\psi) = \int \int [|W_\psi(q, p)| - W_\psi(q, p)] \, dq \, dp$$
$$= \int \int |W_\psi(q, p)| \, dq \, dp - 1. \quad (2)$$

By definition, the quantity δ is equal to zero for coherent and squeezed vacuum states, for which W is non-negative. Hence in this work we shall treat δ as a parameter characterizing the properties of the state under consideration. Similar quantities related to the volume of the negative part of the Wigner function were used in [37–39] to describe the interference effects which determine the departure from classical behaviour.

Furthermore, a closely related approach was recently advocated by Benedict and collaborators [40, 41]. Their measure of the non-classicality of a state $|\psi\rangle$ reads

$$\nu(\psi) = 1 - \frac{I_+(\psi) - I_-(\psi)}{I_+(\psi) + I_-(\psi)} \quad (3)$$

where $I_+(\psi)$ and $I_-(\psi)$ are the moduli of the integrals over those domains of the phase space where the Wigner function is positive and negative, respectively. The normalization condition implies $I_+ - I_- = 1$, so that $\nu = 2I_-/(2I_- + 1)$ leads to $0 \leqslant \nu < 1$. Using this notation we may rewrite (2) as $\delta = I_+ + I_- - 1 = 2I_-$ and obtain a simple relation between both quantities

$$\nu = \frac{2I_-}{1 + 2I_-} = \frac{\delta}{1 + \delta} \quad (4)$$

with $\delta = \nu/(1 - \nu)$. It turns out that both quantities are equivalent in the sense that they induce the same order in the space of pure states: the relation $\delta(\psi_1) > \delta(\psi_2)$ implies $\nu(\psi_1) > \nu(\psi_2)$. However, from a pragmatic point of view there exists an important difference between both quantities.

To compute explicitly the quantity (3) one faces a difficult task to identify appropriately the domains in which the integration has to be carried out. On the other hand, knowing the Wigner function $W(q, p)$ of a quantum state, it is straightforward to get its absolute value and to evaluate numerically the integration (2).

Let us emphasize again that the Hilbert space containing all pure states is huge, so one should not expect to characterize the non-classical features of a quantum state just by a single scalar quantity. Our approach focuses on a particular issue: whether the Wigner function is positive and may be interpreted as a classical probability distribution. Therefore, the proposed indicator δ should be considered as a tool complementary to those worked out earlier and reviewed above.

3. The Schrödinger cat state

The Schrödinger cat state owes its name to the Schrödinger's famous Gedankenexperiment [42]. In this experiment, the cat paradoxically turns out to be simultaneously in two macroscopically distinguishable states, namely *dead* and *alive*. The Schrödinger cat state is then defined as a superposition of two such states [37]. In our work we construct similar 'cat states' by choosing two coherent states $\phi_\pm$ localized in two distant points of the configuration space, $\pm q_0$. The wavefunction of such a state reads in the position representation

$$\Psi(q) = \frac{N}{\sqrt{2}}[\phi_+(q) + \phi_-(q)] \quad (5)$$

where

$$\phi_\pm(q) = \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}(q \pm q_0)^2 + i\frac{p_0}{\hbar}(q \pm q_0)\right). \quad (6)$$

From now on atomic units are used ($m = \hbar = \omega = 1$). In other words we measure the size of the product pq in units of $\hbar$. The classical limit $\hbar \to 0$ means the action pq characteristic of the system is many orders of magnitude larger than $\hbar$. A glance at equation (6) reveals that the phase, governed by p_0, is of great importance in that it induces oscillations on the wavefunction, as can be seen in figure 1. Note that the normalization constant N depends on the location of the centres (q_0, p_0) of both coherent states that make up the cat state. Therefore one sees that the Wigner function may depend not only on the distance $2q_0$ between both states, but also on their momentum, p_0. So far, the studies on the cat states [34] have usually been restricted to the case of standing cats, $p_0 = 0$. In this paper we demonstrate that the parameter p_0 influences the shape of the Wigner function, in particular, if $q_0 \sim 1$ and both packets are not spatially separated.

Inserting (5) into the Wigner function (1) one obtains

$$W_\Psi(q, p) = W_+(q, p) + W_-(q, p) + W_{int}(q, p). \quad (7)$$

Here

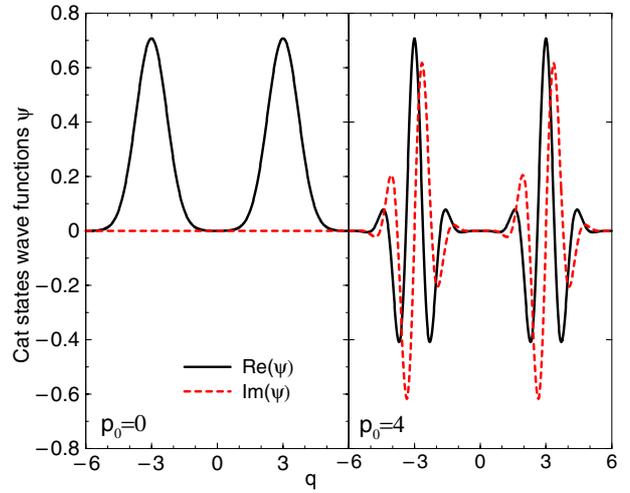$$W_\pm(q, p) = \frac{N^2}{2\pi} \exp(-(q \pm q_0)^2 - (p - p_0)^2) \quad (8)$$

Figure 1. Schrödinger cat states wavefunctions plotted with $p_0 = 0$ (left) and with $p_0 = 4$ (right). Dashed and solid curves represent the imaginary and the real parts of the wavefunction, respectively. Notice that the envelopes of both wavefunctions do coincide.

represents two peaks of the distribution centred at the classical phase space points $(\pm q_0, p_0)$, while

$$W_{int}(q, p) = \frac{N^2}{\pi} \cos(2pq_0) \exp(-q^2 - (p - p_0)^2) \quad (9)$$

stands for the interference structure which appears between both peaks. Normalizing (5) yields

$$N = (1 + \cos(2p_0q_0) \exp(-q_0^2))^{-1/2}. \quad (10)$$

Making use of the formula (7) for the Wigner function of the cat state $|\Psi\rangle$, its non-classicality parameter

$$\delta(\Psi) = \int \int |W_+(q, p) + W_{int}(q, p) + W_-(q, p)| \, dq \, dp - 1 \quad (11)$$

may be approximated by

$$\delta(\Psi) \approx N^2 \left[1 + \int \frac{dp}{\sqrt{\pi}} |\cos(2pq_0)| \exp(-(p - p_0)^2)\right] - 1. \quad (12)$$

Strictly speaking the right-hand side of equation (12) forms an upper bound for $\delta(\Psi)$, which may be practically used as its fair approximation. Because of the oscillations of the absolute value of cosine, it is difficult to perform the integration analytically. In the special case $q_0 = 0$, the superposition of coherent states (5) reduces to a single coherent state and correspondingly (12) leads to $\delta(\Psi) = 0$.

Figure 2 shows plots of the Wigner function of the cat states for several values of the separation q_0 and the momentum p_0. One clearly sees the formation of the quantum interference structure halfway between the two humps as the separation distance q_0 increases. The frequency of the interference structure increases with the separation [34].

For intermediate separations ($0 < q_0 \leqslant 4$), the Wigner function changes its structure with p_0, see figures 2(b) and (d). However, for a larger separation distance, $q_0 > 4$, the Wigner function for $p_0 = p_1 \neq 0$ may be approximated by the Wigner function for the state with $p_0 = 0$ translated by a constant vector $\Delta p = p_1$.
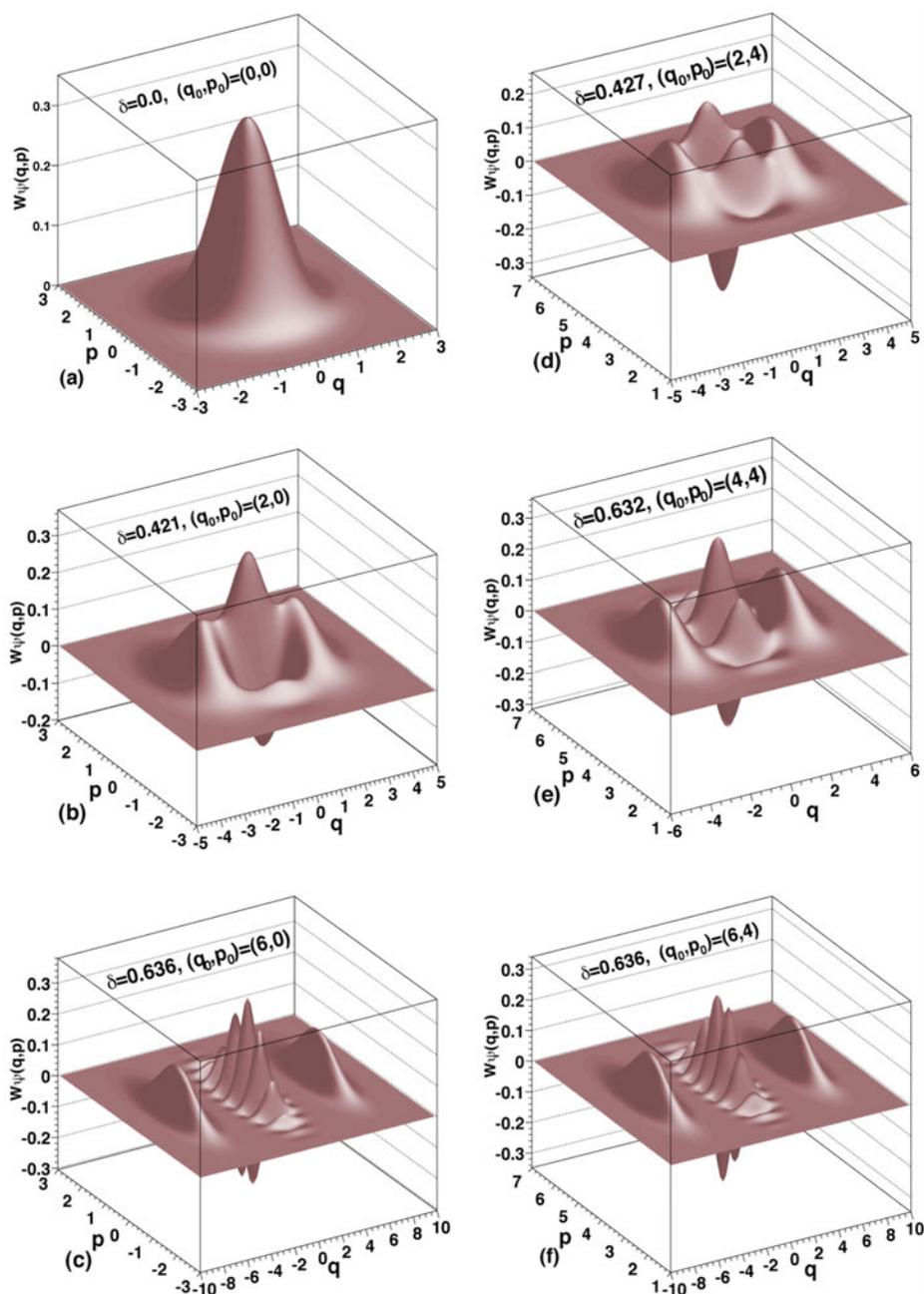
Figure 2. Plots of the Wigner functions of the Schrödinger cat states (7). Each panel is labelled by the separation distance q_0, the momentum p_0 and the resulting indicator δ. Observe that for intermediate separations, $q_0 \sim 1$, the indicator δ changes with p_0. The left column shows the 'standing cats' ($p_0 = 0$) while the cats in motion ($p_0 = 4$) are represented in the right column.

In the case of 'standing cats', ($p_0 = 0$), the indicator δ increases monotonically with the separation q_0, and reflects the presence of the interference patterns at $q = 0$—see figure 3(k). The growth of the non-classicality saturates at $q_0 \approx 4$, as the interference patterns become practically separated from both peaks, and the parameter δ tends to the limiting value, $\delta_{max} \approx 0.636$. In the limit $q_0 \to \infty$ the oscillations of the cosine term in equation (12) become rapid and a crude approximation $|\cos(q_0 p_0)| \approx 1$ gives an explicit upper bound $\delta \leqslant 2N^2 - 1 \approx 1$.

This picture gets more complicated for the states with $p_0 \neq 0$, in particular for a small separation distance, ($0 <$ $q_0 \leqslant 4$). In this case, δ exhibits oscillations as shown in figure 3. Notice that δ does not become zero; this is in contrast to what the *eye test* tends to show in figure 3. To shed some light on this behaviour we have chosen to plot in figure 4 the Wigner function for which $\delta(q_0)$ achieves extremal values. For instance, δ at $q_0 = 0.725$ (figure 4(h)) is smaller than at $q_0 = 0.4$ (figure 4(g)) or 1.175 (figure 4(i)). This is due to the interference structure, which is not symmetric with respect to the reflection $p \to -p$, in contrast to the case of cats with $p_0 = 0$.

As shown in figure 3, the frequency of oscillations increases with p_0, but the limiting value $\delta(q_0 \to \infty)$ does
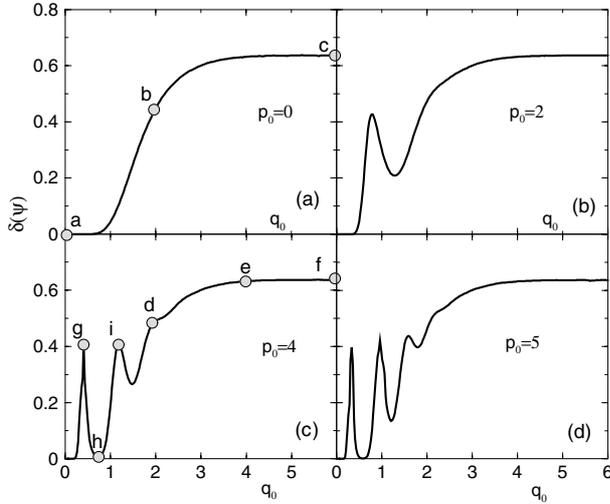
Figure 3. Indicator δ of the Schrödinger cat state $|\psi\rangle$ as a function of the separation distance q_0 and several values of p_0 as labelled on each panel. Grey dots ((a)–(f)) refer to labels of individual panels of figure 2 while grey dots ((g)–(i)) refer to that of figure 4.

not depend on the initial momentum p_0. This can also be demonstrated investigating the dependence of the quantity δ as a function of p_0. As follows from equation (12), the indicator δ displays regular oscillations with the period $p_{osc} =$

π/q_0—see figure 5. In other words a non-zero separation parameter q_0 breaks the translational invariance in momentum and introduces a characteristic momentum scale $p_{osc} \sim 1/q_0$. Note that the amplitudes of the oscillations decrease fast with q_0, so that for well separated cats with $q_0 > 4$ the quantity δ is practically independent of p_0.

4. Generalized Fock states

Let us consider the squeezed displaced Fock state defined by

$$|\beta, \eta, n\rangle = D(\beta)S(\eta)|n\rangle, \qquad (13)$$

where $|n\rangle$ is the original Fock state and $n = 0, 1, 2, \ldots$. The displacement $D(\beta)$ and the squeezed $S(\eta)$ operators are defined by [34, 43]

$$D(\beta) := \exp(\beta a^\dagger - \beta^* a) \qquad \text{and}$$
$$S(\eta) := \exp(\tfrac{1}{2}(\eta^* a^2 - \eta a^{\dagger 2})), \qquad (14)$$

where a and $a^\dagger$ are usual photon annihilation and creation operators, respectively. The complex variable β represents the magnitude and angle of the displacement. Similarly, writing the complex number in its polar form, $\eta = s \exp(i\phi)$, it is easy to see that the radius s plays the role of the squeezing strength while the angle ϕ indicates the direction of squeezing. It was shown in [12] that the displacement operators $D(\beta)$ form a
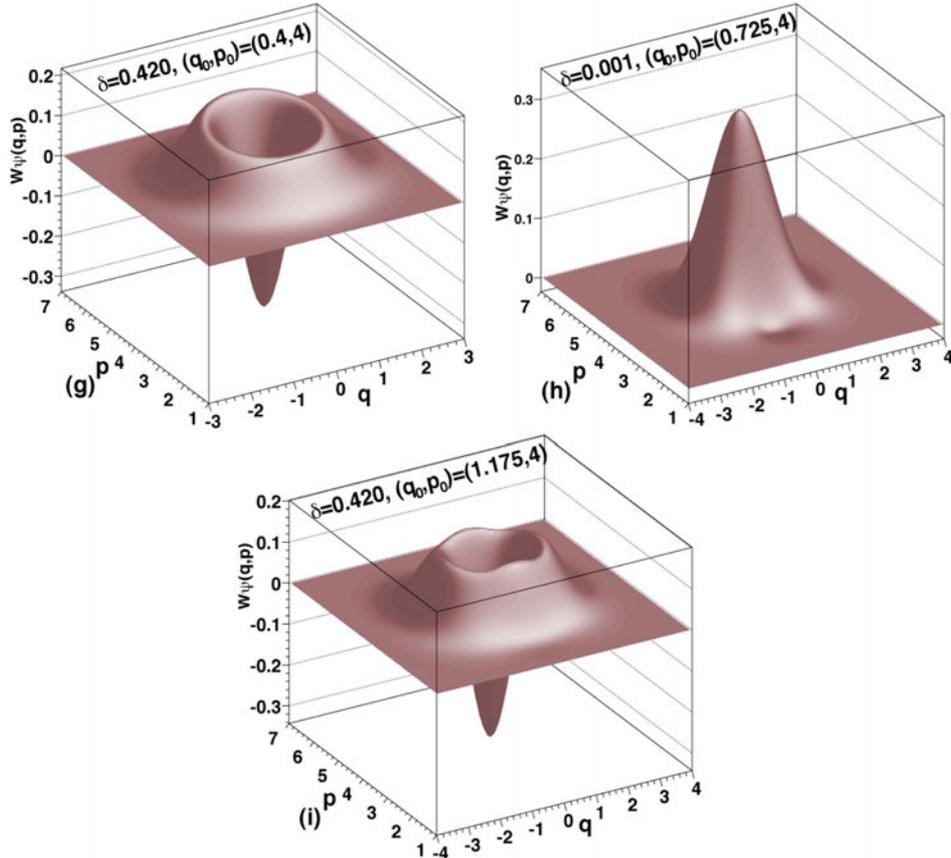

Figure 4. As in figure 2. The values of q_0 labelling each panel correspond to the successive extrema ((g)–(i)) of the indicator δ plotted in figure 3(c) as a function of q_0. The Wigner function for the 'cats in motion' as in the right column of figure 2, for the selected values of p_0, for which the dependence $\delta(q_0)$ achieves its extrema.
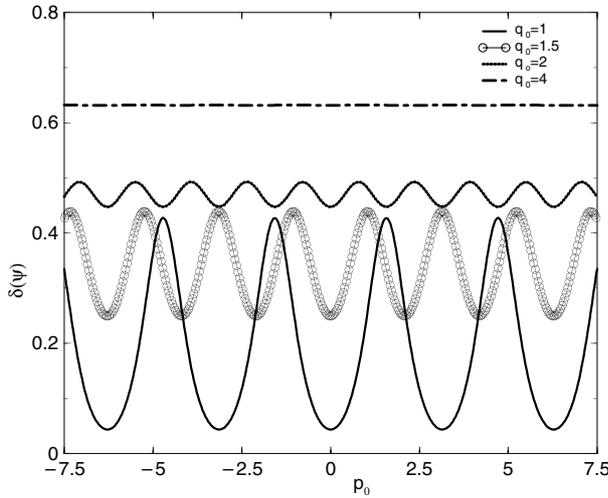
Figure 5. Indicator δ of the Schrödinger cat state $|\Psi\rangle$ as a function of the momentum p_0 for certain values of the separation q_0.

complete set of operators. Thus any bounded operator F, (for which the Hilbert–Schmidt norm $\|F\| = \sqrt{\mathrm{Tr}(F^\dagger F)}$ is finite), can be expressed in the form $F = \int f(\xi)D^{-1}(\xi)\,\mathrm{d}^2\xi/\pi$ in which the weight function $f(\xi) = \mathrm{Tr}(FD(\xi))$ is unique and square-integrable. Given that every density operator is bounded ($\mathrm{Tr}(\rho^\dagger \rho) = \mathrm{Tr}(\rho^2) \leqslant 1$), one may write an arbitrary density operator $\rho = \int \chi(\xi)D^{-1}(\xi)\,\mathrm{d}^2\xi/\pi$. Here the weight function $\chi(\xi) = \mathrm{Tr}(\rho D(\xi))$ is just the expectation value of the displacement operator commonly known as the characteristic function. The complex Fourier transform of $\chi(\xi)$ defines the Wigner function

$$W(\alpha) = \int \frac{\mathrm{d}^2\xi}{\pi} \chi(\xi) \exp(\xi^*\alpha - \xi\alpha^*). \qquad (15)$$

One may therefore express $\chi(\xi)$ in terms of the Wigner function by performing the inverse Fourier transform as

$$\chi(\xi) = \int \frac{\mathrm{d}^2\alpha}{\pi} W(\alpha) \exp(\xi\alpha^* - \xi^*\alpha), \qquad (16)$$

so that upon substitution into the density operator expression above, one gets

$$\rho = \int \frac{\mathrm{d}^2\alpha}{\pi} W(\alpha)T(\alpha). \qquad (17)$$

The operators $T(\alpha)$ denote

$$T(\alpha) = \int \frac{\mathrm{d}^2\xi}{\pi} \exp(\xi\alpha^* - \xi^*\alpha)D^{-1}(\xi)$$
$$= 2D(\alpha)(-1)^{a^\dagger a}D^{-1}(\alpha), \qquad (18)$$

so that the Wigner function may be interpreted as a weight function for the expansion of the density operator in terms of the operators $T(\alpha)$ [12]. These operators are Hermitian, $T = T^\dagger$, and possess the same completeness properties as the displacement operators $D(\alpha)$. It turns out that the Wigner distribution function can now be written as follows

$$W(\alpha) = 2\,\mathrm{Tr}(\rho D(\alpha)(-1)^{a^\dagger a}D^{-1}(\alpha)). \qquad (19)$$

Making use of the parity operator $(-1)^{a^\dagger a} = \exp(\mathrm{i}\pi a^\dagger a)$, one finally shows that

$$W(\alpha) = 2\sum_{n=0}^{\infty}(-1)^n\langle n|D^{-1}(\alpha)\rho D(\alpha)|n\rangle, \qquad (20)$$

since $\exp(\mathrm{i}\pi a^\dagger a)|n\rangle = \exp(\mathrm{i}\pi n)|n\rangle$, with n being the photon number.

In the case of the squeezed displaced Fock states, $\rho = |\beta, \eta, n\rangle\langle\beta, \eta, n|$, explicit calculations of matrices elements [44] generated from (20) provides us with the following expression of the Wigner function

$$W_n(\alpha) = \frac{2}{\pi}(-1)^n \exp(-2|b|^2)L_n(4|b|^2), \qquad (21)$$

with $b = \cosh(s)(\alpha^* - \beta^*) + \exp(-\mathrm{i}\phi)\sinh(s)(\alpha - \beta)$. Here L_n denotes the Laguerre polynomial of the nth order.

The Wigner function (21) allows us to compute the non-classicality parameter $\delta(|\beta, \eta, n\rangle)$ for a given displaced squeezed Fock state $|\beta, \eta, n\rangle$. In what follows certain special cases will be investigated such as squeezed displaced vacuum states, pure Fock states and squeezed displaced Fock states. It will be therefore convenient to represent the complex variable α by the position and momentum coordinates, $\alpha = \frac{1}{\sqrt{2}}(q + \mathrm{i}p)$, and treat likewise the displacement operator, $\beta = \frac{1}{\sqrt{2}}(q_0 + \mathrm{i}p_0)$.

Substituting $\beta = \eta = 0$ in equation (21) yields the Wigner function for the Fock state $|n\rangle$,

$$W_n(q, p) = \frac{(-1)^n}{\pi} \exp[-(q^2 + p^2)]L_n[2(q^2 + p^2)]. \qquad (22)$$

This allows us to evaluate analytically the indicator $\delta(|n\rangle)$, for $n = 1, 2, 3, 4$

$$\delta(|0\rangle) = 0 \text{ (vacuum)}$$

$$\delta(|1\rangle) = \frac{4}{\mathrm{e}^{1/2}} - 2 \approx 0.426\,1226$$

$$\delta(|2\rangle) = 4((2 + \sqrt{2})\mathrm{e}^{-1-\frac{1}{\sqrt{2}}} + (-2 + \sqrt{2})\mathrm{e}^{-1+\frac{1}{\sqrt{2}}}) \approx 0.728\,99$$

$$\delta(|3\rangle) \approx 0.976\,67$$

$$\delta(|4\rangle) \approx 1.191\,38, \tag{23}$$

since the zeros of the Laguerre polynomials are available up to the fourth order. For larger n we computed the quantity $\delta(|n\rangle)$ numerically and plotted these in figure 6. The indicator δ grows monotonically with n as the number of zeros of the Laguerre polynomial L_n increases with n. For $n \in [1, 250]$ this dependence may be approximated by $\frac{1}{2}\sqrt{n}$. Hence, the larger the quantum number n, the less the Wigner function $W_{|n\rangle}$ can be interpreted as a classical distribution function.

Setting $n = 0$ in (21) one obtains a squeezed coherent state or squeezed vacuum state. Choosing the squeezing angle $\phi = 0$, one sees that the Wigner function is a Gaussian centred at the displacement vector (q_0, p_0) with the shape determined by the squeezing parameter s,

$$W_0(q, p) = \frac{1}{\pi} \exp\left(-\mathrm{e}^{2s}(q - q_0)^2 - \frac{1}{\mathrm{e}^{2s}}(p - p_0)^2\right). \qquad (24)$$

In such a case the Wigner function remains everywhere non-negative for any choice of the squeezing and displacement
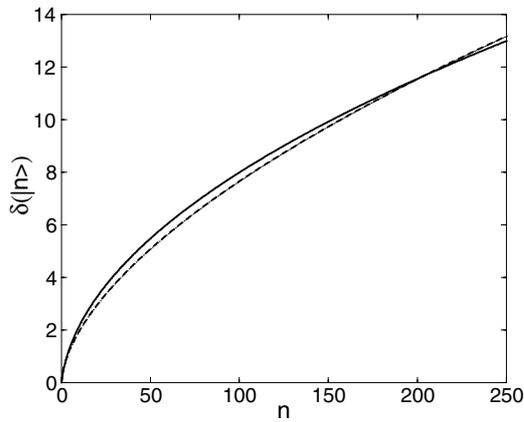
Figure 6. The non-classicality indicator $\delta(|n\rangle)$ of the Fock states versus the quantum number $n \leqslant 250$ (solid curve). The dashed curve represents $\frac{1}{2}\sqrt{n}$ plotted for comparison.

parameters [24], so that the non-classicality indicator vanishes, $\delta(|\beta, s, 0\rangle) = 0$. Note that the displacement of any state in phase space does not change the shape of the Wigner function, so the quantity δ is independent of the displacement operator $D(\beta)$.

Furthermore, the squeezing operator $S(\eta)$ influences the shape of the Wigner function, but does not lead to a change in the volume of its negative part. Therefore, the parameter δ does not also depend on the squeezing. As an illustration

we have chosen the squeezed ($|\alpha| = s, \phi = \pi/6$) displaced ($\beta = 0$) third photon ($n = 3$) state, $|0, s \exp(i\pi/6), 3\rangle$. The contour plots of the Wigner function of such a state are shown in figure 7 for some values of the squeezing parameter s. The indicator δ is equal to 0.9762, irrespective to the squeezing strength. If squeezing is strong enough, the ring-like Wigner function collapses to a quasi one dimensional object with a cigar form.

The squeezed vacuum is often described as a non-classical state [34]. Since the quantity δ does not depend on squeezing, it should not be interpreted as the only parameter which characterizes the non-classicality. To describe the non-classical features of the squeezed states one may use, for instance, the non-classical depth [13, 15, 17].

5. Concluding remarks

In this work we have proposed a simple indicator of non-classicality which measures the volume of the negative part of the Wigner function. Although the proposed coefficient δ is a function of the related quantity ν, recently introduced by Benedict, Czirják *et al* [40, 41], it is much easier to compute numerically.

The quantity (2) was used to analyse exemplary quantum states, including the Schrödinger cat states. The non-classicality δ increases with the separation between the classical points defining the cat state. This growth saturates, if the separation distance is so large that the quantum interference
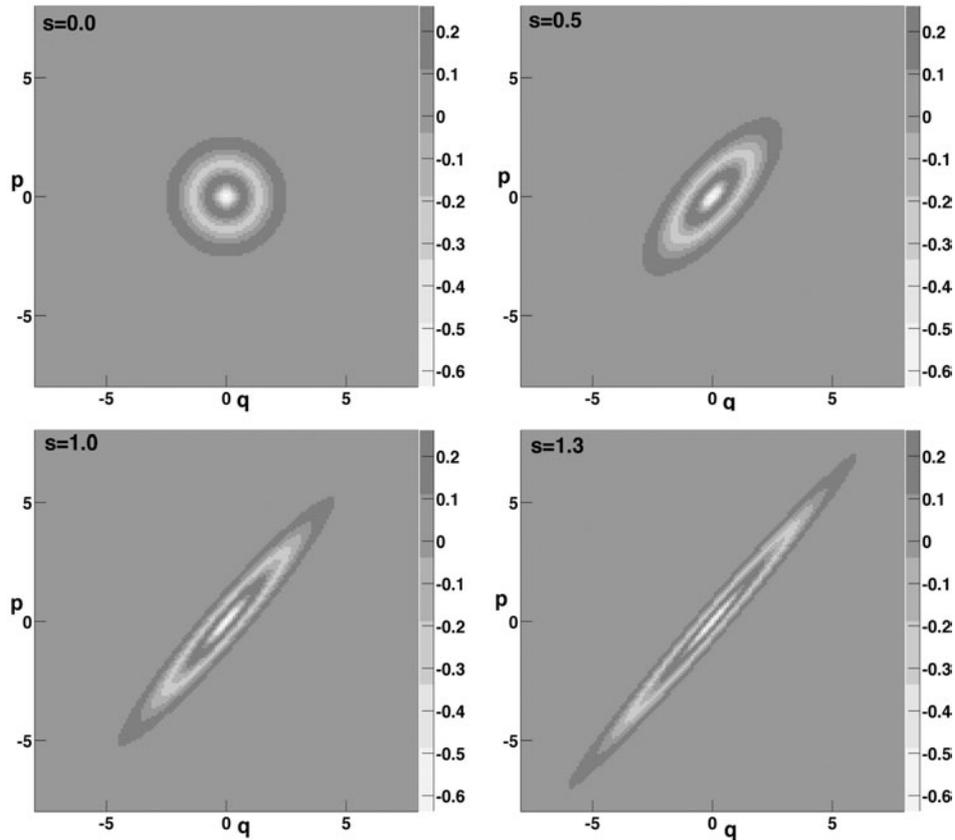

Figure 7. Contour plots of the Wigner functions of the squeezed Fock states $|0, s \exp(i\pi/6), 3\rangle$ labelled by the squeezing strengths s. Irrespective of s the indicator $\delta \approx 0.9762$.

patterns are well isolated from both main peaks of the distributions. Moreover, for a non-zero momentum $p_0 \neq 0$, the quantity δ undergoes oscillations until the separation distance becomes so large that both packets are separated from the interference patterns. Asymptotically, if the separation is large enough, the indicator δ does not depend on p_0 and tends to a constant value, $\delta_{max} \approx 0.636$.

In the case of Fock states $|n\rangle$, the quantity δ equals zero for the coherent vacuum state $|0\rangle$ and grows monotonically with the quantum number n. If a quantum state is displaced by the Glauber operator $D(\beta)$, the shape of the Wigner function and the non-classicality parameter do not change. Although the squeezing operator $S(\eta)$ changes the shape of the Wigner function, our results obtained for the squeezed Fock states show that the non-classicality δ does not depend on squeezing. Since the non-classical depth of a state is a function of the squeezing strength, it is clear that there is no direct relation between τ_m and δ, so both quantities characterizing quantum states may be regarded as complementary.

The results presented in this work were obtained for pure states of infinite dimensional Hilbert space with use of the standard harmonic oscillator coherent states. It is worth emphasizing that our approach is also suited to analysing mixed quantum states. Furthermore, one may study the similar problem for quantum states of a finite dimensional Hilbert space, which was originally tackled in [40]. In such a case one defines the Husimi function with the help of the $SU(2)$ spin coherent states, while the Wigner functions may be obtained by expanding the density matrix in the complete basis of the rotation operators [45–47]. The Wigner function for finite dimensional systems may also be defined in alternative ways— see [48–54] and references therein. Studying the volume of the negative part of the Wigner function, defined according to any of these approaches, one may get interesting information concerning the non-classical properties of the state analysed. For instance some recent attempts [55, 39, 56] try to link the negativity of the Wigner function with the entanglement of analysed quantum states defined on a composed Hilbert space, or with the violation of the Bell inequalities.

Acknowledgments

References

[1] Mandel L 1979 *Opt. Lett.* **4** 205
[2] Dodonov V V 2002 *J. Opt. B: Quantum Semiclass. Opt.* **4** R1
[3] Hillery M 1987 *Phys. Rev.* A **35** 725
[4] Hillery M 1989 *Phys. Rev.* A **39** 2994
[5] Dodonov V V, Man'ko O V, Man'ko V I and Wünsche A 2000 *J. Mod. Opt.* **47** 633
[6] Dodonov V V and Renó M B 2002 *Phys. Lett.* A **308** 249
[7] Marian P, Marian T A and Scutaru H 2002 *Phys. Rev. Lett.* **88** 153601
[8] Marian P, Marian T A and Scutaru H 2003 *Phys. Rev.* A **68** 062309
[9] Avelar A T, Baseia B and Malbouisson J M C 2003 *Preprint* quant-ph/0308161
[10] Życzkowski K and Słomczyński W 1998 *J. Phys. A: Math. Gen.* **31** 9055
[11] Życzkowski K and Słomczyński W 2001 *J. Phys. A: Math. Gen.* **34** 6689
[12] Cahill K E and Glauber R J 1969 *Phys. Rev.* **177** 1882
[13] Lee C T 1991 *Phys. Rev.* A **44** R2775
[14] Lee C T 1992 *Phys. Rev.* A **45** 6586
[15] Lütkenhaus N and Barnett S M 1995 *Phys. Rev.* A **51** 3340
[16] Marchiolli M A, Bagnato V S, Guimaraes Y and Baseia B 2001 *Phys. Lett.* A **279** 294
[17] Malbouisson J M C and Baseia B 2003 *Phys. Scr.* **67** 93
[18] Richter Th and Vogel W 2002 *Phys. Rev. Lett.* **89** 283601
[19] Wünsche A 2004 *J. Opt. B: Quantum Semiclass. Opt.* **6** 159
[20] Korbicz J, Cirac J I, Wehr J and Lewenstein M 2004 *Preprint* quant-ph/0408029
[21] Bracken A J, Doebner H-D and Wood J G 1999 *Phys. Rev. Lett.* **83** 3758
[22] Manfredi G and Feix M R 2000 *Phys. Rev.* E **62** 4665
[23] Włodarz J J 2003 *Int. J. Theor. Phys.* **42** 1075
[24] Hudson R L 1974 *Rep. Math. Phys.* **6** 249
[25] Bell J S 1987 *Speakable and Unspeakable in Quantum Mechanics* (Cambridge: Cambridge University Press) pp 196–200
[26] Einstein A, Podolsky B and Rosen N 1935 *Phys. Rev.* **47** 777
[27] Banaszek K and Wódkiewicz K 1998 *Phys. Rev.* A **58** 4345
[28] Cohen O 1997 *Phys. Rev.* A **56** 3484
[29] Banaszek K and Wódkiewicz K 1999 *Acta. Phys. Slovaca* **49** 491
[30] Banaszek K and Wódkiewicz K 1999 *Phys. Rev. Lett.* **82** 2009
[31] Smithey D T, Beck M, Raymer M G and Faridani A 1993 *Phys. Rev. Lett.* **70** 1244
    Dunn T J, Walmsley I A and Mukamel S 1995 *Phys. Rev. Lett.* **74** 884
    Breitenbach G, Schiller S and Mlynek J 1997 *Nature* **387** 471
    Banaszek K, Radzewicz C and Wódkiewicz K 1999 *Phys. Rev.* A **60** 674
[32] Kurtsiefer Ch, Pfau T and Mlynek J 1997 *Nature* **386** 150
    Nogues G, Rauschenbeutel A, Osnaghi S, Bertet P, Brune M, Raimond J M, Haroche S, Lutterbach L G and Davidovich L 2000 *Phys. Rev.* A **62** 054101
    Lvovsky A I, Hansen H, Aichele T, Benson O, Mlynek J and Schiller S 2001 *Phys. Rev. Lett.* **87** 0504021
[33] Lougovski P, Solano E, Zhang Z M, Walther H, Mack H and Schleich P W 2003 *Phys. Rev. Lett.* **91** 010401
    Lamb W E 1969 *Phys. Today* **22** 23
    Wódkiewicz K 1984 *Phys. Rev. Lett.* **52** 1064
    Royer A 1985 *Phys. Rev. Lett.* **55** 2745
    Banaszek K and Wódkiewicz K 1996 *Phys. Rev. Lett.* **76** 4344
[34] Leonhardt U 1997 *Measuring the Quantum State of Light* (Cambridge: Cambridge University Press)
[35] Wigner E P 1932 *Phys. Rev.* **40** 749
[36] Hillery M, O'Connell R F, Scully M O and Wigner E P 1984 *Phys. Rep.* **106** 123
[37] Schleich W P 2001 *Quantum Optics in Phase Space* (Weinheim: Wiley–VCH)
[38] Białynicki-Birula I, Cirone M A, Dahl J P, Fedorov M and Schleich W P 2002 *Phys. Rev. Lett.* **89** 0604041
[39] Dahl J P, Mack H, Wolf A and Schleich W P 2004 unpublished
[40] Benedict M G and Czirják A 1999 *Phys. Rev.* A. **60** 4034

[41] Földi P, Czirják A, Molnár B and Benedict M G 2002 *Opt. Express* **10** 376
[42] Schrödinger E 1980 *Proc. Am. Phil. Soc.* **124** 323
[43] Mandel L and Wolf E 1995 *Optical Coherence and Quantum Optics* (Cambridge: Cambridge University Press)
[44] Abd Al-Kader G M 2003 *J. Opt. B: Quantum Semiclass. Opt.* **5** S228
     Venkata Satyanarayana M 1985 *Phys. Rev.* D **32** 400
[45] Agarwal G S 1981 *Phys. Rev.* A **24** 2889
[46] Dowling J P, Agarwal G S and Schleich W P 1994 *Phys. Rev.* A **49** 4101
[47] Heiss S and Weigert S 2000 *Phys. Rev.* A **63** 012105
[48] Wootters W K 1987 *Ann. Phys.* **176** 1

     Gibbons K S, Hoffman M J and Wootters W K 2004 *Preprint* quant-ph/0401155
[49] Galetti D and De Toledo Piza A F R 1988 *Physica* A **149** 267
[50] Vaccaro J A and Pegg D T 1990 *Phys. Rev.* A **41** 5156
[51] Leonhardt U 1995 *Phys. Rev. Lett.* **74** 4101
[52] Miranowicz A, Leoński W and Imoto N 2001 *Adv. Chem. Phys.* **119** 155
[53] Mukunda N, Chaturvedi S and Simon R 2004 *Phys. Lett.* A **321** 160
[54] Wootters W K 2004 *IBM J. Res. Dev.* **48** 99
[55] Galvão E 2004 *Preprint* quant-ph/0405070
[56] Bengtsson I 2004 *Preprint* quant-ph/0406174