k-uniform mixed states

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We investigate the maximum purity that can be achieved by k-uniform mixed states of N parties. Such N-party states have the property that all their k-party reduced states are maximally mixed. A scheme to construct explicitly k-uniform states using a set of specific N-qubit Pauli matrices is proposed. We provide several different examples of such states and demonstrate that in some cases the state corresponds to a particular orthogonal array. The obtained states, despite being mixed, reveal strong nonclassical properties such as genuine multipartite entanglement or violation of Bell inequalities.

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I. INTRODUCTION

Since quantum correlations are both a basic resource in quantum information processing and a fundamental phenomenon related to foundations of quantum mechanics, their characterization becomes of great importance for practical as well as strictly theoretical reasons [1]. For the simplest system of two qubits, the Bell states [2] play a special role. They are also known as maximally entangled states, because they exhibit strong two-qubit quantum correlations and at the same time their single-qubit reduced states are maximally mixed. A great deal of attention has recently been paid to the identification of entangled states that generalize that concept, the pure states of N-partite systems, such that, tracing out arbitrary $N-k$ subsystems, the remaining k subsystems are maximally mixed (see, e.g., Refs. [3,4] for pioneering works). Such pure states are called k uniform. By construction, the integer number k cannot exceed $N/2$ and the states for $k=\lfloor N/2\rfloor$ are referred to as absolutely maximally entangled (AME) [5]. They are a natural generalization of maximally entangled Bell states ($k=1$).

While AME states for five and six qubits have been constructed explicitly [6–8], such states do not exist for systems consisting of four [4] and seven qubits [9]. Moreover, it has been shown that there exist no AME states for systems with a larger number of qubits [10,11]. Interestingly, if the local dimension is chosen to be large enough, AME states always exist [12]. For example, it has been proven that there exist AME states for three and four qudits, for every prime $d>2$ [13]. A necessary condition [8,14] for the existence of an N-partite AME state of arbitrary dimension is given by

$$N \leqslant \begin{cases} 2(d^2-1) & \text{for } n \text{ even} \\ 2d(d+1)-1 & \text{for } n \text{ odd.} \end{cases} \tag{1}$$

Since for many cases one cannot construct pure k-uniform states, one can ask the following question: What is the highest possible purity of a k-uniform state for a given number of parties N?

In this paper we address the problem of finding k-uniform states with the highest possible purity for which the corresponding pure AME states do not exist. We begin with the reformulation of the k uniformity of states with the use of a correlation tensor. We proceed with describing the method of explicit construction of k-uniform states using N-qubit Pauli operators. Then we describe the relation between the construction presented and the notion of orthogonal arrays. In the following we give specific examples of k-uniform N-qubit states, which also are numerically proven to be of the highest purity with respect to given values k and N. After discussing the properties of the k-uniform states with regard to entanglement and quantum Fisher information, we present an example of a specific quantum circuit which enables generation of the respective k-uniform state. We then briefly mention the results for k-uniform qudit states with higher dimensionality of subsystems, after which we summarize.

II. CORRELATIONS OF k-UNIFORM STATES

An arbitrary state of N qubits can be represented as

$$\rho = \frac{1}{2^N}\sum_{\mu_1,\dots,\mu_N=0}^{3} T_{\mu_1,\dots,\mu_N}\sigma_{\mu_1}\otimes\cdots\otimes\sigma_{\mu_N}, \tag{2}$$

where σ_μ are Pauli matrices and $T_{\mu_1,\dots,\mu_N}=\mathrm{Tr}(\rho\sigma_{\mu_1}\otimes\cdots\otimes\sigma_{\mu_N})$ are real coefficients called correlation tensor elements, which we will refer to simply as correlations. Let us now define a length of correlations among r subsystems

$$M_r(\rho) = \sum_{\pi} \sum_{i_1,i_2,\dots,i_r=1}^{3} T^2_{\pi(i_1 i_2,\dots,i_r)}, \quad (3)$$

where $\pi(i_1,\dots,i_r)$ stands for all permutations of r nonzero indices on N positions. For a k-uniform state of N particles ρ_N^k we have

$$M_r(\rho_N^k) = 0 \quad (4)$$

for all $1 \leqslant r \leqslant k$. In other words, k-uniform states do not have any k-partite correlations or correlations between a smaller number of parties, i.e., $T_{\pi(i_1 i_2,\dots,i_r)} = 0$ for $r \leqslant k$. With this notation, the purity of a given N-qubit state is given by

$$\mathrm{Tr}\rho^2 = \frac{1}{2^N} \sum_{i_1,i_2,\dots,i_N=0}^{3} T^2_{i_1 i_2,\dots,i_N} = \frac{1}{2^N}\left(1 + \sum_{r=1}^{N} M_r(\rho)\right). \quad (5)$$

Furthermore, because of Eq. (4), the sum can be reduced only to the last $N-k-1$ elements

$$\mathrm{Tr}(\rho_N^k)^2 = \frac{1}{2^N}\left(1 + \sum_{r=k+1}^{N} M_r(\rho)\right). \quad (6)$$

For a given purity, the total length of correlations $\sum_{r=1}^{N} M_r(\rho) = 2^N \mathrm{Tr}\rho^2 - 1$ is fixed and state independent. The absence of correlation for $r \leqslant k$ results in the fact that all available correlations occur between a large number of qubits ($r > k$). This, combined with a relatively high purity, can manifest strong nonclassical properties, for instance, the genuine multipartite entanglement.

III. STATES FROM GENERATORS

Below we present a scheme for constructing k-uniform states from particular sets of N-qubit Pauli matrices. These building blocks resemble the generators used within the framework of the stabilizer formalism [15,16]. For further convenience, if not stated otherwise, we will use the simplified notation for multiqubit Pauli operators

$$\sigma_0 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_3 \otimes \cdots \equiv \mathbb{1}XYZ\dots. \quad (7)$$

Let us now suppose that there exists a set of N-qubit Pauli operators

$$\mathcal{G} = \{G_1,\dots,G_m\} \quad (8)$$

such that these operators have the following properties: (i) mutual commutation, $[G_i, G_j] = 0$ for all i, j; (ii) independence, $G_1^{i_1}\cdots G_m^{i_m} \sim \mathbb{1}$ only for $i_1 = \cdots = i_m = 0$ with $i_j = \{0,1\}$; and (iii) k uniformity, $G_1^{i_1}\cdots G_m^{i_m}$ $(i_j = \{0,1\})$ results in N-qubit Pauli operator (7) containing the identity operators on at most $N-k-1$ positions. The last property distinguishes our approach from the standard stabilizer formalism (see, e.g., [17]). In the literature, m is called the rank of the stabilizer group. Stabilizer groups with $m = N$ are called full rank, whereas stabilizer groups with $m < N$ are rank deficient. The elements of such a set $\mathcal{G}$ will be called generators. We can use them to generate a k-uniform state by summing all possible products of the elements from $\mathcal{G}$,

$$\rho = \frac{1}{2^N} \sum_{j_1,\dots,j_m=0}^{1} G_1^{j_1}\cdots G_m^{j_m}. \quad (9)$$

The above construction leads to a valid physical state by virtue of the following argument.

Consider a set of m mutually commuting N-qubit Pauli operators $\mathcal{G} = \{G_1,\dots,G_m\}$. Let us rewrite the state (9) in the form

$$\rho = \frac{1}{2^N}(\mathbb{1} + G_1)(\mathbb{1} + G_2)\cdots(\mathbb{1} + G_m). \quad (10)$$

Therefore, we see that the eigenvalues of ρ can be written in the form

$$\lambda^i = \frac{1}{2^N}(1 + \lambda_1^i)(1 + \lambda_2^i)\cdots(1 + \lambda_m^i), \quad (11)$$

where $\lambda_j^i = \pm 1$ is the ith eigenvalue of the jth generator in the common eigenbasis of mutually commuting operators from the set $\mathcal{G}$. Note that from (9) we have $\mathrm{Tr}\rho = 1$, while λ^i are either 0 or 2^{m-N}, and hence ρ constitutes a physical state with exactly 2^{N-m} nonzero eigenvalues. Naturally, the case $m = N$ corresponds to a pure state with exactly one eigenvalue equal to 1.

Now the state (9) has 2^m nonvanishing correlations equal to ± 1 and its purity can be calculated simply as

$$\mathrm{Tr}\rho^2 = \frac{1}{2^N} 2^m = 2^{m-N}. \quad (12)$$

Note that the larger the set $\mathcal{G}$, the higher the purity of the outgoing state. We observe that the problem of constructing k-uniform states is therefore directly related to the problem of finding the largest possible set of generators $\mathcal{G}$. Consequently, in the case of $k = \lfloor \frac{N}{2} \rfloor$ and $m = N$ the construction leads to an AME state with purity equal to 1.

Due to the construction method, we expect to obtain k-uniform states of high purity. In all cases considered (up to $N = 6$) we have numerical evidence that there are no k-uniform states of higher purity (see Appendix A for details).

IV. ORTHOGONAL ARRAYS

In general, in order to determine $\mathcal{G}$, we have to search the full set of 4^N N-qubit operators. However, we observe that it is possible to construct a set of generators with the help of orthogonal arrays. Orthogonal arrays [18,19] are combinatorial arrangements, tables with entries satisfying given orthogonal properties. An orthogonal array OA(r, N, l, s) is a table composed of r rows and N columns with entries taken from $0,\dots,l-1$ in such a way that each subset of s columns contains all possible combination of symbols with the same number of repetitions. The number of such repetitions is called the index of the OA; if $l^s = r$ the orthogonal array is of index unity.

Suppose we wish to find $\mathcal{G}$ for a k-uniform state of N qubits. For this purpose we can use an orthogonal array OA$(r, \mathbf{N}, \mathbf{4}, s)$ with four levels (corresponding to four different Pauli matrices). In doing so, we treat each row of the OA as a string of indices $a_1,\dots,a_N$ $(a_i \in \{0,1,2,3\})$, which defines the specific N-qubit Pauli operator $A_1,\dots,A_N$

($A_i \in \{\mathbb{1}, X, Y, Z\}$) using the convention $0 \to \mathbb{1}$, $1 \to X$, $2 \to Y$, and $3 \to Z$. After performing this operation we end up with a set of r operators from which we have to choose the largest set $\mathcal{G}$ such that its elements meet conditions (i) and (ii) from (8). Those conditions guarantee that the desired state is physical and determine its purity. The parameter k for which property (iii) from (8) holds does not depend explicitly on the construction presented but rather on a particular example of the OA. The maximal number of $\mathbb{1}$'s in each row of the OA equals $s-1$, which may suggest the $N-s$ uniformity of the obtained state. In condition (iii), however, we require that the number of $\mathbb{1}$'s is limited not only for generators but also for all elements of the form $G_1^{i_1}, \ldots, G_m^{i_m}$. In some of the examples presented (see Secs. V D, V F, and V H) the number of $\mathbb{1}$'s is also limited by $s-1$ for all such elements. Hence the desired states are indeed $N-s$ uniform. In other examples, however, the uniformity of the desired state is slightly smaller than the prediction from the generators. Although the states obtained from the OA of index unity coincide with $(N-s)$-uniform states, the precise connection has to be established. In general, the relation between uniformity k and quantities s and N seems to be irregular.

It is well known that in the simplest case of four qubits there is no 2-uniform pure state [4]. However, relaxing the assumption that the desired state is pure, the orthogonal array OA(16,4,4,2) can be utilized to construct the mixed 4-qubit 2-uniform state, which leads to the set of operators

$$\begin{aligned}
&0000 \to \mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1}, \quad 0111 \to \mathbb{1}XXX,\\
&0222 \to \mathbb{1}YYY, \quad 0333 \to \mathbb{1}ZZZ,\\
&1012 \to X\mathbb{1}XY, \quad 1103 \to XX\mathbb{1}Z,\\
&1230 \to XYZ\mathbb{1}, \quad 1321 \to XZYX,\\
&2023 \to Y\mathbb{1}YZ, \quad 2132 \to YXZY,\\
&2201 \to YY\mathbb{1}X, \quad 2310 \to YZX\mathbb{1},\\
&3031 \to Z\mathbb{1}ZX, \quad 3120 \to ZXY\mathbb{1},\\
&3213 \to ZYXZ, \quad 3302 \to ZZ\mathbb{1}Y.
\end{aligned} \tag{13}$$

Within this set one can find $m=3$ operators conforming to the properties from Sec. III, which constitute the set $\mathcal{G}$, e.g.,

$$G_1 = \mathbb{1}YYY, \quad G_2 = XZYX, \quad G_3 = YXZY, \tag{14}$$

and by virtue of Eq. (9) lead to the state ρ_4^2 of purity $\frac{1}{2}$.

V. EXAMPLES

Below we present examples of k-uniform states with the highest possible purity for several cases of k and N. In each case we provide generators from the set $\mathcal{G}$, which uniquely define the corresponding k-uniform state. All examples are summarized in Fig. 1.

$k \backslash N$	4	5	6	7	8	9
1	1	1	1	1	1	1
2	$\frac{1}{2}$	1	1	1		
3	$\frac{1}{2^2}$	$\frac{1}{2}$	1	$\frac{1}{2}$		
4		$\frac{1}{2^4}$	$\frac{1}{2^3}$			
5			$\frac{1}{2^4}$	$\frac{1}{2^4}$		$\frac{1}{2^5}$
6				$\frac{1}{2^6}$	$\frac{1}{2^6}$	
7					$\frac{1}{2^6}$	$\frac{1}{2^7}$
8						$\frac{1}{2^8}$

FIG. 1. Purity of the states presented in Sec. V. States that can be obtained by one of two general procedures described in Sec. V A are indicated by horizontal and vertical arrows.

A. General schemes

When the verification of the properties (i)–(iii) for generators in Sec. III becomes computationally demanding, in some particular cases we can employ simple schemes for the construction of k-uniform N-qubit states. (i) The first method, presented in detail in [20,21], can be implemented if another particular $(k-1)$-uniform state is known ($k-1$ is even). The method eliminates all correlations between an odd number of subsystems and does not change the remaining ones. Since k is odd for even $k-1$, the k-partite correlations vanish and the state becomes k uniform. To this end, we evenly mix the original state ρ_N^{k-1} with its antistate

$$\rho_N^k = \tfrac{1}{2}\big(\rho_N^{k-1} + \bar{\rho}_N^{k-1}\big), \tag{15}$$

where the antistate $\bar{\rho}_N^{k-1} = \sigma_y^{\otimes N}\mathrm{conj}(\rho_N^{k-1})\sigma_y^{\otimes N}$, with $\mathrm{conj}(\cdot)$ denoting complex conjugation. (ii) We can also obtain k-uniform states by tracing out any of the subsystems of the N-qubit k-uniform state, which leads to the k-uniform $(N-1)$-qubit state. In both methods the purity of the resulting state is reduced by $\frac{1}{2}$. These methods, however, do not guarantee that the states obtained are of the highest possible purity.

B. Case of N arbitrary and $k=1$

The 1-uniform pure state is the N-qubit Greenberger-Horne-Zeilinger (GHZ) state $|\mathrm{GHZ}\rangle = 1/\sqrt{2}(|0\ldots0\rangle + |1\ldots1\rangle)$, for which the $m=N$ generators are

$$G_1 = ZX\cdots XX, \quad G_2 = XZ\cdots XX, \tag{16}$$

$$\vdots$$

$$G_{N-1} = XX\cdots ZX, \quad G_N = XX\cdots XZ. \tag{17}$$

C. Case of N arbitrary and $k = N - 1$

For $k = N - 1$ only N-partite correlations are possible, and hence the generators cannot have the identity operator $\mathbb{1}$ on any position. For N odd, the set $\mathcal{G}$ consists of only one generator ($m = 1$)

$$G_1 = Z \cdots Z, \tag{18}$$

while for N even, the set $\mathcal{G}$ consists of two generators $m = 2$,

$$G_1 = X \cdots X, \quad G_2 = Z \cdots Z. \tag{19}$$

For even N the states can be written in the form

$$\rho_N = \frac{1}{2^N}\left(\mathbb{1}^{\otimes N} + (-1)^{N/2}\sum_{j=1}^{3}\sigma_j^{\otimes N}\right) \tag{20}$$

and are known as the generalized bound entangled Smolin states [22,23]. They are a useful quantum resource for multiparty communication schemes and have been experimentally demonstrated [24].

D. Case of $N = 4$ and $k = 2$

Since in this case a pure AME state does not exist [4], we cannot have four generators and so the set $\mathcal{G}$ consists of $m = 3$ elements

$$G_1 = XXXX, \quad G_2 = YYYY,$$
$$G_3 = \mathbb{1}XYZ. \tag{21}$$

The above construction yields the symmetric mixture of two pure states

$$|\varphi_1\rangle = \frac{1}{\sqrt{2}}(|\phi_1\rangle + |\phi_2\rangle),$$
$$|\varphi_2\rangle = \frac{\sigma_x^{\otimes 4}}{\sqrt{2}}(|\phi_1\rangle - |\phi_2\rangle), \tag{22}$$

where

$$|\phi_1\rangle = \tfrac{1}{2}(|0010\rangle + |1110\rangle + i|1000\rangle - i|1001\rangle),$$
$$|\phi_2\rangle = \tfrac{1}{2}(|1111\rangle - |0011\rangle + i|0100\rangle + i|0101\rangle),$$

and $\sigma_x^{\otimes 4}$ is a flip operation on all particles. Notice that each of the states given in Eq. (22) is almost 2 uniform. More accurately, four out of $\binom{4}{2}$ of its reductions to two qubits are maximally mixed. The remaining reductions are given in the standard basis by

$$\frac{1}{4}\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \quad \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$$

for $|\varphi_1\rangle$ and $|\varphi_2\rangle$, respectively. Observe that the sum of those matrices is proportional to $\mathbb{1}$, which is relevant to the fact that the mixture of $|\varphi_1\rangle$ and $|\varphi_2\rangle$ is 2 uniform.

E. Case of $N = 5$ and $k = 2$

The five-qubit pure AME state is described by $m = 5$ generators

$$G_1 = \mathbb{1}XYXY, \quad G_2 = \mathbb{1}ZXX\mathbb{1},$$
$$G_3 = XYY\mathbb{1}Z, \quad G_4 = XZYZY, \quad G_5 = ZXZ\mathbb{1}X. \tag{23}$$

The explicit formula of the state is

$$\frac{1}{\sqrt{8}}(|01111\rangle + |10011\rangle + |10101\rangle + |11100\rangle$$
$$- (|00000\rangle + |00110\rangle + |01001\rangle + |11010\rangle)),$$

which is equivalent to the AME(5,2) state constructed via the link with quantum error correction codes [25].

F. Case of $N = 5$ and $k = 3$

The 5-qubit 3-uniform mixed state can be obtained from OA(16, 5, 4, 2), which leads to the $m = 4$ generators

$$G_1 = \mathbb{1}XXXX, \quad G_2 = \mathbb{1}YYYY,$$
$$G_3 = X\mathbb{1}XYZ, \quad G_4 = Y\mathbb{1}YZX. \tag{24}$$

The corresponding state is of the form (22), with

$$|\phi_1\rangle = \tfrac{1}{2}(|00101\rangle + |01010\rangle + i|00110\rangle + i|01001\rangle),$$
$$|\phi_2\rangle = \tfrac{1}{2}(|10000\rangle + |11111\rangle - i|10011\rangle - i|11100\rangle).$$

and has purity $\frac{1}{2}$.

An interesting property of the state is the fact that it contains only four-qubit correlations. Nevertheless, the state is genuinely five-qubit entangled.

G. Case of $N = 6$ and $k = 2$

The 6-qubit 2-uniform pure state can be described by $m = 6$ generators

$$G_1 = XXYYZZ, \quad G_2 = XXZZYY,$$
$$G_3 = XZZXXZ, \quad G_4 = XYYX\mathbb{1}Z,$$
$$G_5 = YX\mathbb{1}ZXY, \quad G_6 = YYYY\mathbb{1}\mathbb{1} \tag{25}$$

and is equivalent to the state presented in [26].

H. Case of $N = 6$ and $k = 3$

The six-qubit AME state can be obtained from OA(64, 6, 4, 3), which leads to the $m = 6$ generators

$$G_1 = \mathbb{1}\mathbb{1}ZZZZ, \quad G_2 = \mathbb{1}XYZ\mathbb{1}X,$$
$$G_3 = \mathbb{1}ZXY\mathbb{1}Z, \quad G_4 = XYZ\mathbb{1}Z\mathbb{1},$$
$$G_5 = Z\mathbb{1}Z\mathbb{1}XY, \quad G_6 = ZYYZZY. \tag{26}$$

Formula (9) gives a pure AME(6,2) state

$$|\phi\rangle = \tfrac{1}{4}(|000110\rangle + |011100\rangle + |100000\rangle + |111010\rangle$$
$$- |001001\rangle - |010011\rangle - |101111\rangle - |110101\rangle$$
$$+ i|000101\rangle + i|010000\rangle + i|101100\rangle + i|111001\rangle$$
$$- i|001010\rangle - i|011111\rangle - i|100011\rangle - i|110110\rangle)$$

equivalent to the one found in [13].

I. Case of $N = 6$ and $k = 4$

The 4-uniform 6-qubit mixed state can be described by $m = 3$ generators

$$G_1 = \mathbb{1}XXXXX, \quad G_2 = Y\mathbb{1}YYYY,$$
$$G_3 = ZZ\mathbb{1}XYZ$$

and has purity $\frac{1}{8}$.

J. Case of $N = 7$ and $k = 2$

The 2-uniform 7-qubit pure state can be described by $m = 7$ generators

$$\begin{aligned} &G_1 = \mathbb{1}XXYYZZ, \quad G_2 = \mathbb{1}Z\mathbb{1}\mathbb{1}Z\mathbb{1}Z,\\ &G_3 = XXZXZZX, \quad G_4 = YYZYZZY,\\ &G_5 = \mathbb{1}\mathbb{1}ZXYYX, \quad G_6 = \mathbb{1}YY\mathbb{1}\mathbb{1}YY,\\ &G_7 = Z\mathbb{1}Z\mathbb{1}Z\mathbb{1}Z, \end{aligned} \tag{27}$$

which results in the state of the form [26]

$$\begin{aligned} |\phi\rangle = \frac{1}{\sqrt{8}}(&|0000000\rangle + |0110011\rangle + |1011010\rangle + |1101001\rangle\\ &- |1111111\rangle - |0111100\rangle - |1010101\rangle - |1100110\rangle). \end{aligned}$$

K. Case of $N = 7$ and $k = 3$

Since in this case a pure AME state does not exist, we cannot specify seven generators. Here, however, we can employ the scheme for eliminating all the correlations of the rank given by an even number. Therefore, using the above 7-qubit 2-uniform state ρ_7^2, we can construct the 7-qubit 3-uniform mixed state ρ_7^3, which is described by $m = 6$ generators

$$\begin{aligned} &G_1 = Y\mathbb{1}YXZXZ, \quad G_2 = \mathbb{1}XXYYZZ,\\ &G_3 = ZXYYXZ\mathbb{1}, \quad G_4 = ZZ\mathbb{1}YXXY,\\ &G_5 = YY\mathbb{1}Y\mathbb{1}\mathbb{1}Y, \quad G_6 = ZXYZ\mathbb{1}YX. \end{aligned} \tag{28}$$

L. Case of $N = 7$ and $k = 5$

The 7-qubit 5-uniform mixed state can be obtained from the $m = 3$ generators

$$\begin{aligned} &G_1 = \mathbb{1}XXXXXX, \quad G_2 = X\mathbb{1}XYYYY,\\ &G_3 = YYY\mathbb{1}XYZ. \end{aligned}$$

The purity of the state is $\frac{1}{16}$.

M. Case of $N > 7$ and $k = N - 2$

For $k = N - 2$ only $(N - 1)$-partite correlations are possible, and hence the generators have the identity operator $\mathbb{1}$ on at most one position. In either case, at least two generators can be found, i.e., for N odd,

$$G_1 = \mathbb{1}X\cdots X, \quad G_2 = \mathbb{1}Y\cdots Y, \tag{29}$$

while for N even,

$$G_1 = \mathbb{1}X\cdots X, \quad G_2 = X\mathbb{1}Y\cdots Y, \tag{30}$$

leading to purity $1/2^{N-2}$.

N. Case of $N = 9$ and $k = 5$

The 9-qubit 5-uniform mixed state can be obtained from OA(32, 9, 4, 2), which leads to the $m = 4$ generators

$$\begin{aligned} &G_1 = XXXXXXXX\mathbb{1}, \quad G_2 = YYYYYYYY\mathbb{1},\\ &G_3 = \mathbb{1}XYZ\mathbb{1}XYZX, \quad G_4 = \mathbb{1}\mathbb{1}ZZYYXX\mathbb{1}. \end{aligned} \tag{31}$$

The purity of the state is $\frac{1}{32}$.

O. Case of $N = 12$ and $k = 5$

From OA(4096,12,4,5) we can isolate the set of $m = 6$ generators

$$\begin{aligned} &G_1 = XYY\mathbb{1}ZZX\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1},\\ &G_2 = YZZ\mathbb{1}XXY\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1},\\ &G_3 = \mathbb{1}XYY\mathbb{1}ZZX\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1},\\ &G_4 = \mathbb{1}YZZ\mathbb{1}XXY\mathbb{1}\mathbb{1}\mathbb{1}\mathbb{1},\\ &G_5 = XXXXXXXXXXXX,\\ &G_6 = YYYYYYYYYYYY, \end{aligned} \tag{32}$$

which leads to the state of purity $\frac{1}{64}$.

VI. GENUINE MULTIPARTITE ENTANGLEMENT

We also focus on the genuine N-partite entanglement for the considered N-qubit states. We evaluate the entanglement monotone W as proposed in Ref. [27] for the states with $N \leqslant 6$. a nonzero value of W indicates genuine multipartite entanglement for the state considered. We find that most of the states studied exhibit genuine multipartite entanglement. The values of W are presented in Table I. For 7-qubit k-uniform states we derive witnesses using the method designed for the stabilizer states shown in [1,16]. They have the form $\mathcal{W}_7^k = \alpha_7^k\mathbb{1} - \rho_7^k$, with $\alpha_7^1 = \alpha_7^2 = \frac{1}{2}$ and $\alpha_7^3 = \frac{1}{4}$, and prove genuine multipartite entanglement for the states considered: ρ_7^1, ρ_7^2, and ρ_7^3.

VII. FISHER INFORMATION

Let us consider an N-qubit Hamiltonian that allows observers to perform a different evolution on each particle. The local evolutions are generated by the operators $\sigma_{\vec{n}}^{(j)} = \vec{n}^{(j)} \cdot \vec{\sigma}^{(j)}$ $(j = 1, \ldots, N)$. Such a Hamiltonian takes the form

$$\mathcal{H} = \frac{1}{2}\sum_j \sigma_{\vec{n}_j}^{(j)} \tag{33}$$

and is a generalization of a standard collective Hamiltonian for which $\sigma_{\vec{n}_j}^{(j)} = \sigma_{\vec{n}}^{(j)}$ for all j.

For pure states, the quantum Fisher information [28] can be easily calculated as the variance of the Hamiltonian, $F(\rho, \mathcal{H}) = 4\,\mathrm{Tr}[(\Delta\mathcal{H})^2\rho]$. The square of the Hamiltonian is given by

$$\mathcal{H}^2 = \frac{N}{4} + \frac{1}{2}\sum_{i<j}\sigma_{\vec{n}}^{(i)}\sigma_{\vec{m}}^{(j)}. \tag{34}$$

Therefore, the quantum Fisher information can be expressed in terms of correlation tensor elements as (see [29] for comparison with a collective case)

$$\begin{aligned} F(\rho, \mathcal{H}) &= 4\{\mathrm{Tr}(\mathcal{H}^2\rho) - [\mathrm{Tr}(\mathcal{H}\rho)]^2\}\\ &= N + 2\big(T_{n_1n_20\cdots0} + T_{n_10n_30\cdots0} + \cdots + T_{0\cdots0n_{N-1}n_N}\big)\\ &\quad - \big(T_{n_10\cdots0} + T_{0n_20\cdots0} + \cdots + T_{0\cdots0n_N}\big)^2. \end{aligned} \tag{35}$$

Since for k-uniform states (with $k \geqslant 2$) all two- and single-qubit correlation tensor elements vanish, the quantum Fisher

TABLE I. Properties of k-uniform N-qubit states given in Sec. V: purity W, genuine multipartite entanglement monotone M, length of correlations F, quantum Fisher information $F_i = F(\rho, J_i)$ (where J_i is the collective angular momentum operator), white-noise robustness f_{crit}, and probability of violation of local realism p_v.

N	k	purity	W	M	F	f_{crit}	p_v (%)
2	1	1	0.5	$M_2 = 3$	$F_x = 4, F_y = 0, F_z = 4$	0.293	28.32
3	1	1	0.5	$M_3 = 4, M_2 = 3$	$F_x = 3, F_y = 3, F_z = 9$	0.5	74.69
	2	$\frac{1}{4}$	0	$M_3 = 1$	$F_x = 0, F_y = 3, F_z = 3$	0	0
4	1	1	0.5	$M_4 = 9, M_2 = 6$	$F_x = 4, F_y = 4, F_z = 16$	0.647	94.24
	2	$\frac{1}{2}$	0.5	$M_4 = 3, M_3 = 4$	$F_x = F_y = F_z = 4$	0.422	35.11
	3	$\frac{1}{4}$	0	$M_4 = 3$	$F_x = F_y = F_z = 4$	0.292	0.024
5	1	1	0.5	$M_5 = 16, M_4 = 5, M_3 = 10$	$F_x = F_y = F_z = 25$	0.75	99.60
	2	1	0.5	$M_5 = 6, M_4 = 15, M_3 = 10$	$F_x = F_y = F_z = 5$	0.568	99.96
	3	$\frac{1}{2}$	0.5	$M_4 = 15$	$F_x = 5, F_y = 4, F_z = 5$	0.460	63.65
	4	$\frac{1}{16}$	0	$M_5 = 1$	$F_x = 0, F_y = 5, F_z = 5$	0	0
6	1	1	0.5	$M_6 = 33, M_4 = 15, M_2 = 15$	$F_x = 6, F_y = 6, F_z = 36$	0.823	99.97
	2	1	0.5	$M_6 = 10, M_5 = 24, M_4 = 21, M_3 = 8$	$F_x = F_y = F_z = 6$	0.666	> 99.99
	3	1	0.5	$M_6 = 18, M_4 = 45$	$F_x = F_y = F_z = 6$	0.591	100
	4	$\frac{1}{16}$	0	$M_6 = 1, M_5 = 2$	$F_x = 1, F_y = 6, F_z = 5$	0.293	$<10^{-3}$
	5	$\frac{1}{16}$	0	$M_6 = 3$	$F_x = F_y = F_z = 6$	0.293	$<10^{-6}$
7	1	1	> 0	$M_7 = 64, M_6 = 7, M_4 = 35, M_2 = 21$	$F_x = 7, F_y = 7, F_z = 49$	0.875	100
	2	1	> 0	$M_7 = 15, M_6 = 42, M_5 = 42, M_4 = 21, M_3 = 7$	$F_x = F_y = F_z = 7$	0.785	100
	3	$\frac{1}{2}$	> 0	$M_6 = 42, M_4 = 21$	$F_x = F_y = F_z = 7$	0.644	99.16

information

$$F\left(\rho_N^{k\geqslant 2}, \mathcal{H}\right) = N \tag{36}$$

depends only on the number of qubits.

Note that the quantum Fisher information (36) does not depend on a particular choice of the vectors $\vec{n}_j$. This implies that the quantum Fisher information F_{avg} averaged over all directions $\vec{n}$ is also equal to N. This fact can be used to verify the presence of entanglement, because for all product states $F_{\mathrm{avg}} < 2N/3$ [30,31].

The situation for mixed states is more complicated. In this case Eq. (36) provides only an upper bound on the quantum Fisher information. In general, it can be a function of higher-order correlations. In spite of this, in several cases of mixed states we observe similar behavior as for pure states (see Table I).

VIII. BELL VIOLATION

We investigate families of k-uniform states (up to seven qubits) with a numerical method based on linear programming [32]. The method allows us to reveal nonclassicality even without direct knowledge of Bell inequalities for the given problem. For each state we determine the minimal admixture of white noise that is necessary to destroy quantum correlations f_{crit} and the probability of violation of local realism p_v for randomly sampled settings [33]. The results are presented in Table I. In all cases (except the trivial ones), $f_{\mathrm{crit}} > 0$ and we observe a conflict with local realism.

IX. QUANTUM CIRCUITS FOR k-UNIFORM STATES

Recently, in Ref. [34] quantum circuits that generate absolutely maximally entangled states have been designed. We can employ a similar scheme in order to generate mixed k-uniform states. As an example, in the following we present a quantum circuit which results in generating a 2-uniform 4-qubit state. The circuit is presented in Fig. 2 and consists of the Hadamard operations H, the phase gate S, and nonlocal CNOT and SWAP operations defined in a standard way as in Ref. [2]:

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix},$$

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \mathrm{SWAP} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Since the output of quantum circuits is pure states, in order to obtain a mixed state, the last gate is applied at random: With probability $\frac{1}{2}$ we perform the $\bar{X} = X \otimes X$ transformation and with probability $\frac{1}{2}$ we do nothing, so the resulting state is an equal mixture of two original pure states given in Eq. (22).

X. HIGHER-DIMENSIONAL k-UNIFORM STATES

The scheme of generating k-uniform states with the use of a specific set of generators can be extended to

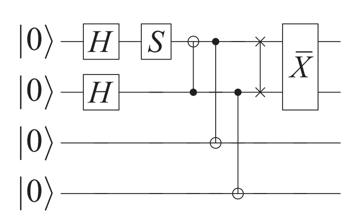

FIG. 2. Scheme for constructing a 2-uniform 4-qubit state. The last transformation applied at random (i.e., with probability $\frac{1}{2}$) outputs an equal mixture of two pure states.

higher-dimensional systems in which, instead of N-qubit Pauli operators G_i, one uses N-qudit operators $G_i^{(d)}$ composed of d-dimensional Weyl-Heisenberg matrices $S_{kl}^{(d)} = (X^{(d)})^k (Z^{(d)})^l$, where $X^{(d)} = \sum_{i=0}^{d} |i\rangle\langle i+1|$ and $Z^{(d)} = \sum_{i=0}^{d} \omega^i |i\rangle\langle i|$, with $\omega = e^{2i\pi/d}$ and $k, l = 0, \dots, d-1$. Then the set of generators $\mathcal{G}^{(d)}$ must also conform to the same set of properties defined in Sec. III. The resulting k-uniform N-qudit state is given by

$$\rho = \frac{1}{d^N} \sum_{j_1,\dots,j_m=0}^{d-1} \left(G_1^{(d)}\right)^{j_1} \cdots \left(G_m^{(d)}\right)^{j_m} \tag{37}$$

and its purity is d^{m-N}. It is worth noting that if a pure k-uniform state does not exist, the highest purity that can be achieved is $1/d$. Already for $d = 3$, this value is relatively small.

Using the above scheme one can construct so-called graph states including the (1-uniform) N-qudit GHZ-type state which is obtained from the $m = N$ generators [35]

$$\begin{aligned}
G_1^{(d)} &= Z^{(d)} X^{(d)} \cdots X^{(d)} X^{(d)},\\
G_2^{(d)} &= X^{(d)} Z^{(d)} \cdots X^{(d)} X^{(d)},\\
&\vdots\\
G_{N-1}^{(d)} &= X^{(d)} X^{(d)} \cdots Z^{(d)} X^{(d)},\\
G_N^{(d)} &= X^{(d)} X^{(d)} \cdots X^{(d)} Z^{(d)}.
\end{aligned} \tag{38}$$

Let us now illustrate the above method with two more examples of constructing k-uniform qutrit states (pure and mixed). For four qutrits, as opposed to the qubit case, there exists the pure AME(4,3) state that can be determined by $m = 4$ generators

$$\begin{aligned}
G_1^{(3)} &= \mathbb{1}^{(3)} Z^{(3)} Z^{(3)} (Z^{(3)})^2,\\
G_2^{(3)} &= \mathbb{1}^{(3)} X^{(3)} X^{(3)} (X^{(3)})^2,\\
G_3^{(3)} &= Z^{(3)} \mathbb{1}^{(3)} Z^{(3)} Z^{(3)},\\
G_4^{(3)} &= X^{(3)} \mathbb{1}^{(3)} X^{(3)} X^{(3)}.
\end{aligned} \tag{39}$$

Another example is a 2-uniform 3-qutrit mixed state defined by the $m = 2$ generators

$$G_1^{(3)} = X^{(3)} X^{(3)} X^{(3)}, \quad G_2^{(3)} = Z^{(3)} Z^{(3)} Z^{(3)} \tag{40}$$

that can be expressed as a symmetric mixture of three pure states

$$\begin{aligned}
|\alpha_1\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle),\\
|\alpha_2\rangle &= \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle),\\
|\alpha_3\rangle &= \frac{1}{\sqrt{3}}(|012\rangle + |201\rangle + |120\rangle).
\end{aligned} \tag{41}$$

Although this state has a relatively low purity $\frac{1}{3}$, it exhibits genuine multipartite entanglement ($W = 1$). Note that the purity of the 2-uniform 3-qutrit state is higher than that for the corresponding qubit state (see Sec. V C), which is equal to $\frac{1}{4}$.

Finally, we used an iterative method based on semidefinite programming [36] to determine the maximal purity of k-uniform qudit states. The method is described in detail in Appendix B. With this algorithm, we first managed to reproduce all purity values up to $N = 6$ parties in Table I. Then we ran the algorithm for higher d values. For three parties ($N = 3$ and $k = 2$), for $d = 3$ we obtained the maximal purity equal to $\frac{1}{3}$ [see Eq. (41)], while for $d = 4$ it reads $\frac{1}{4}$. In addition, we investigated the case of $N = 4$, $k = 3$, and $d = 3$, for which we get purity $\frac{1}{9}$. In this particular case the purity seems to decay with the increase of dimension.

XI. CONCLUSION

We investigated instances of k-uniform states of N qubits, for which it is known that the corresponding absolutely maximally entangled pure states do not exist. The k-uniform states are distinguished by revealing the highest multipartite correlations among all quantum states of the same purity. A general scheme for finding particular sets of N-qubit Pauli operators allows us to construct k-uniform mixed states for this system. We illustrated this method with examples of all k-uniform states up to six qubits. These states were numerically verified to be of the highest purity with respect to any given values of k and N.

We showed that particular mixed k-uniform states can be constructed with the help of orthogonal arrays, but in a different way from the known scheme of utilizing the notion of an OA for constructing pure AME states: In the case of mixed states the key role is played by the correlation tensor elements instead of ket vectors of the pure AME state itself. We also discussed some instances of k-uniform states of three- and four-qudit systems. Here, however, the dimensionality of the total system rises much faster with the number of qudits, making the numerical analysis ineffective for high dimensions.

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APPENDIX A: NUMERICAL METHOD BASED ON NONLINEAR OPTIMIZATION

The k-uniform states were found numerically by searching over the complete set of multipartite quantum states. This procedure requires a nonlinear optimization, which was provided by the NLOPT package. We implemented the PRAXIS optimization routine, which is an algorithm for gradient-free local optimization based on Brent's principal axis method [37], specially designed for unconstrained optimization.

To determine the k-uniform states, we introduce a cost function defined as

$$p_{\max}(N,k) = \max_{\rho}\left[p_N^k - \beta\sum_{i=1}^{k} M_i(\rho)\right], \qquad \text{(A1)}$$

where $p_{\max}$ is the sum over the lengths of all nonzero correlations, maximized over the entire state space of N parties. According to the definition of a k-uniform state, the correlations between the subsystems up to a total of k subsystems should vanish, whereas the rest are incorporated in the term p_N^k, the total length of the nonvanishing part of the correlations. To ensure that the constraint of vanishing correlations has been satisfied, we associate a regression coefficient β with the lengths of correlations among the k subsystems. To efficiently determine the global maximum for the cost function, one takes the constant β large enough, for which the cost part vanishes, and hence $\beta > 2^{N-1}$.

APPENDIX B: NUMERICAL METHOD BASED ON SEMIDEFINITE PROGRAMMING

To find N-party k-uniform states ρ_N^k of high purity $\mathrm{Tr}[(\rho_N^k)^2]$, we use the following iterative procedure based on semidefinite programming. Inputs to the algorithm are the number of parties N, the number of subsystems k with vanishing correlations, and the dimension d of local Hilbert spaces, which is assumed to be constant for all parties. In addition, we fix the parameter $\epsilon \in [0, 1]$, which sets the speed of convergence. The typical value of ϵ used in the algorithm is 0.3.

Our task is to compute the optimal value $P_{\mathrm{opt}} = \max_\rho \mathrm{Tr}(\rho^2)$ over k-uniform states $\rho \in \mathbb{C}^{(d^N)}$. In this problem the objective function is quadratic in the variable ρ and the constraints are either semidefinite ($\rho \geqslant 0$) or linear [k uniformity and normalization $\mathrm{Tr}(\rho) = 1$]. This is computationally a hard problem. However, we note that the optimal value P_{opt} is identical to $\max_{\rho,\sigma} \mathrm{Tr}(\rho\cdot\sigma)$, where optimization is carried out over k-uniform states $\rho, \sigma \in \mathbb{C}^{(d^N)}$. Indeed, it can be shown that for any pair of k-uniform states ρ and σ the state $\rho_{\mathrm{av}} = (\rho+\sigma)/2$ fulfills the relation $\mathrm{Tr}(\rho_{av}^2) \geqslant \mathrm{Tr}(\rho\cdot\sigma)$, which in turn entails the above alternative form for the optimal value P_{opt}. We use this latter form to provide a seesaw-type heuristic method for computing P_{opt}.

To this end, we choose randomly a k-uniform state ρ and maximize $\mathrm{Tr}(\rho\cdot\sigma)$ over k-uniform σ states. Then we fix σ and optimize the same objective function over k-uniform ρ states. Each two steps can be formulated as a semidefinite program, which we repeat again and again until convergence of $\mathrm{Tr}(\rho\cdot\sigma)$ is achieved. Explicitly, the iterative algorithm described above can be written as follows.

1. Generate randomly a k-uniform state $\rho \in \mathbb{C}^{(d^N)}$.
2. Solve the semidefinite program

$$\begin{aligned} P &= \max_{\sigma} \mathrm{Tr}(\rho\cdot\sigma) \\ \text{s.t. } &\rho_\epsilon \geqslant 0,\ \mathrm{Tr}(\rho_\epsilon) = 1, \\ &\sigma = (1-\epsilon)\rho + \epsilon\rho_\epsilon, \\ &\sigma \text{ is } k \text{ uniform}, \end{aligned} \qquad \text{(B1)}$$

where the optimization is carried out over the set of k-uniform density matrices σ and the constraints within the optimization are either linear or semidefinite. For small enough N and d this problem can be solved efficiently.

3. Set $\rho = \sigma$.
4. Repeat steps 2–4 until convergence of the value P is reached. Here P defines a lower bound to the value of P_{opt}.

Note that it may not be easy to generate randomly N-party k-uniform states within step 1. We can sidestep this issue by generating instead a random N-party state and setting $\epsilon = 1$ within the very first iteration. Then step 2 will ensure that σ is k uniform, and hence ρ in step 3 will also be k uniform. Also notice that the value of P is nondecreasing with the sequence of iterations. However, the above optimization may still get stuck in local maxima of the function $\mathrm{Tr}(\rho\cdot\sigma)$. Therefore, we may have to run the above procedure several times before obtaining a global optimal solution for P_{opt}.

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