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Rapid Communications

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Random-matrix theory and eigenmodes of dynamical systems

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We investigate the predictions of random-matrix theory for the eigenvector statistics and compare them with eigenmodes of kicked tops under conditions of classical chaos. The well-known χ_ν^2 distribution finds an interesting application with $\nu=1$, 2, and 4 for the orthogonal, unitary, and symplectic universality class, respectively. The change of the eigenvector statistics accompanying the classical transition from chaotic to regular motion is also considered.

Quantum systems with a chaotic classical limit are known to display spectral fluctuations well described by random-matrix theory[1,2] (RMT). More recently there has been an interest in the eigenvector statistics for Hamiltonians or Floquet operators.[3-8] Again, RMT turns out reliable under conditions of fully developed classical chaos. We shall provide further corraboration to the latter statement in the present Rapid Communication.

Let c_j^n denote the jth component of the nth eigenvector in a generic N-dimensional basis. The squared moduli $\eta=|c_j^n|^2$ obey the statistics given for each universality class by RMT (Refs. 9-11)

$$\tilde{P}_{\mathrm{COE}}(\eta)=\frac{\Gamma\left(\frac{N}{2}\right)}{\Gamma\left(\frac{N-1}{2}\right)}\frac{(1-\eta)^{(N-3)/2}}{\sqrt{\pi\eta}}, \tag{1}$$

$$\tilde{P}_{\mathrm{CUE}}(\eta)=(N-1)(1-\eta)^{N-2}, \tag{2}$$

and eventually

$$\tilde{P}_{\mathrm{CSE}}(\eta)=(N-1)(N-2)\eta(1-\eta)^{N-1}, \tag{3}$$

where COE, CUE, and CSE represent the circular orthogonal, unitary, and symplectic ensembles, respectively.

For $N\gg 1$ the mean $\langle\eta\rangle$ is proportional to $1/N$. To analyze the limit $N\rightarrow\infty$ it is therefore suitable to introduce a rescaled variable y with unit mean, $\langle y\rangle=1$. The above distributions then take the form

$$P_{\mathrm{COE}}(y)=\frac{1}{\sqrt{2\pi y}}e^{-y/2}, \tag{1'}$$

$$P_{\mathrm{CUE}}(y)=e^{-y}, \tag{2'}$$

$$P_{\mathrm{CSE}}(y)=4ye^{-2y}. \tag{3'}$$

The first of these is known as the Porter-Thomas distribution. All three densities are special cases of the so-called χ_ν^2 distribution

$$P_\nu(y)=\left(\frac{\nu}{2}\right)^{(\nu/2)}y^{\nu/2-1}\Gamma^{-1}\left(\frac{\nu}{2}\right)e^{-\nu y/2}. \tag{4}$$

The latter is well known as the distribution of the sum $y=\sum_{i=1}^{\nu}x_i^2$ for ν independent Gaussian random variables x_i with zero mean and variance $1/\sqrt{\nu}$.

It is, indeed, most intuitive to find $\nu=1$, 2, and 4 for the orthogonal, unitary, and symplectic case, respectively, in view of the physical meaning of the quantities η in (1)-(3). In the orthogonal case the eigenvector components are generically real ($\nu=1$), while in the unitary case one confronts complex components ($\nu=2$). On the other hand, in the symplectic case η refers to the occupation probability of a two-dimensional space pertaining to two eigenvectors sharing the same eigenvalue by Kramers' degeneracy ($\nu=4$).

Perhaps the most natural application of Eqs. (1)-(3) or (1')-(3') refers to the eigenvectors of a Hamiltonian H or Floquet operator F in some fixed basis. Equally legitimate is an interpretation for some arbitrarily chosen state $|\phi\rangle$ in the eigenbasis of H or F. We shall, in the following, focus on a variant of the latter case, in which the "arbitrarily chosen" state $|\phi\rangle$ is, in fact, generated from an eigenvector of H or F by a suitable observable A as

$$|\phi\rangle=\frac{A|n\rangle}{|\langle n|AA^\dagger|n\rangle|^{1/2}}. \tag{5}$$

A motivation for studying this case is provided by the fact that the matrix elements $\langle m|\phi\rangle\sim\langle m|A|n\rangle$ sometimes define transition strengths between eigenstates as $|\langle m|\phi\rangle|^2$. We have checked the foregoing predictions of RMT for periodically kicked tops. Such systems are particularly suitable for our purpose, since realizations of all universality classes can be given.[12-14] Moreover, working with periodically driven, rather than autonomous systems, is advantageous since the eigenphases of the Floquet operator tend to have a constant mean density throughout the spectrum.

The dynamical variables of the tops of the three components of the angular momentum operator which obey the commutation relations $[J_i,J_j]=i\epsilon_{ijk}J_k$. The squared angular momentum is conserved, $J^2=j(j+1)$ with j integer or half integer. The quantum number j fixes the dimension of the Hilbert space as $N=2j+1$. The classical limit is approached with $j\to\infty$. A Floquet operator without any antiunitary symmetry (generalized time reversal) is

$$F_1=\exp(-ipJ_y)\exp\left(-i\frac{K_1J_z^2}{2j}\right)\exp\left(-i\frac{K_2J_x^2}{2j}\right), \quad (6)$$

provided all three coupling constants are nonzero. Quadratic level repulsion has been asertained under the condition of classical chaos ($p\approx K_1\approx K_2\approx 5$). For $K_2=0$, on the other hand, there exists an antiunitary symmetry enforcing the linear level repulsion characteristic of the orthogonal universality class, provided the classical analog displays global chaos.[12]

The presumably simplest top pertaining to the symplectic universality class has the Floquet operator

$$F_2=\exp\left(-ip\frac{J_z^2}{j}\right)\exp\left[-\frac{i}{j}(K_1J_z^2+K_2[J_x,J_z]_+ +K_3[J_x,J_y]_+)\right], \quad (7)$$

where $[\,,\,]_+$ denotes an anticommutator and j must be taken half integer.[14] In fact, for integer j value the Floquet operator F_2 belongs to the orthogonal universality class.

We have calculated the matrix elements of J_z in the Floquet eigenbasis normalizing in the sense of (5), i.e., $\langle m|J_z|n\rangle|\langle n|J_z^2|n\rangle|^{-1/2}$. The quantum number j was chosen sufficiently large for the histograms for

$$y=|\langle m|J_z|n\rangle|^2/\langle n|J_z^2|n\rangle, \quad (8)$$

to become reasonably smooth. It may be noteworthy that smaller values of j are more sufficient for this purpose than for obtaining good statistics for properties of eigenvalues. Figure 1 presents the histograms obtained. Figure 1(a) refers to the symplectic case (operator F_2, $j=149.5$, $p=K_1=5.0$, $K_2=2.5$, $K_3=20.0$), while Figs. 1(b) and 1(c) present, respectively, the unitary (operator F_1, $j=250$, $p=1.7$, $K_1=9.0$, $K_2=1.0$) and the orthogonal (operator F_1, $j=250$, $p=1.7$, $K_1=9.0$, $K_2=0.0$) cases. Logarithmic scales were used for the abscissas in order to accentuate the differences between the various distributions. The dashed lines describe the χ_ν^2 distribution with $\nu=4$, 2, and 1. The solid curves were constructed by elevating the parameter ν from discrete to continuous; an optimal value of ν was then sought by minimalizing the integrated mean-square deviation between χ_ν^2 and the histograms obtained. The agreement between the numerical data for the various universality classes and the respective predictions of RMT is impressive. Fixing our attention to the orthogonal case (Floquet operator F_1, $j=250$, $K_2=0.0$, $p=1.7$) we now proceed to a study of the quantum signature of the classical transition from global chaos to predominantly regular motion as the K_1 parameter decreases towards zero. Figure 2 presents the histograms for a succession of three values of K_1. We could not resist the temptation of playing the slightly frivolous game of fitting the χ_ν^2 distribution with $0<\nu<1$ to these histograms. Needless to say, such fits are neither suggested by random-matrix theory nor by any other rational arguments. The optimal fits displayed in Fig. 2 can hardly be said to be successful beyond giving the quantitative tend to flatter distribution for decreasing coupling constant K_1. Especially, they do not grasp the migration of the peak to-

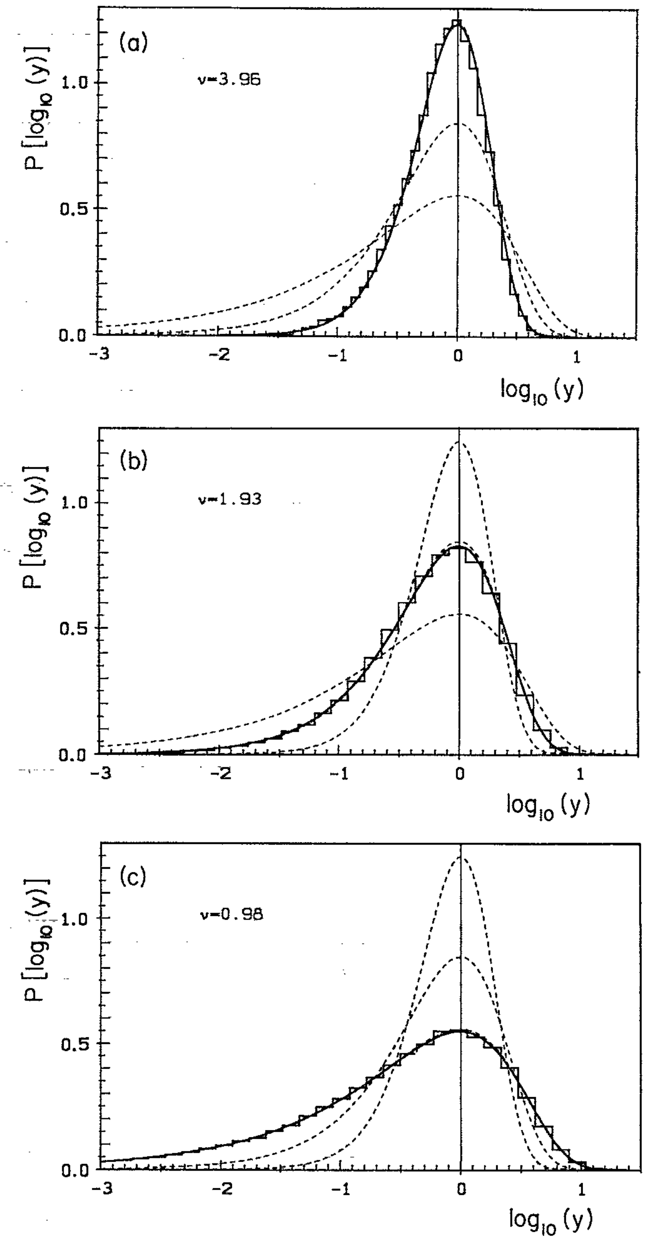

FIG. 1. Histograms of distribution of transition strengths computed for (a) symplectic, (b) unitary, and (c) orthogonal dynamics (see text). Dashed lines denote χ_ν^2 distribution with $\nu=4$, 2, and 1; the solid line shows the best-fit curve.

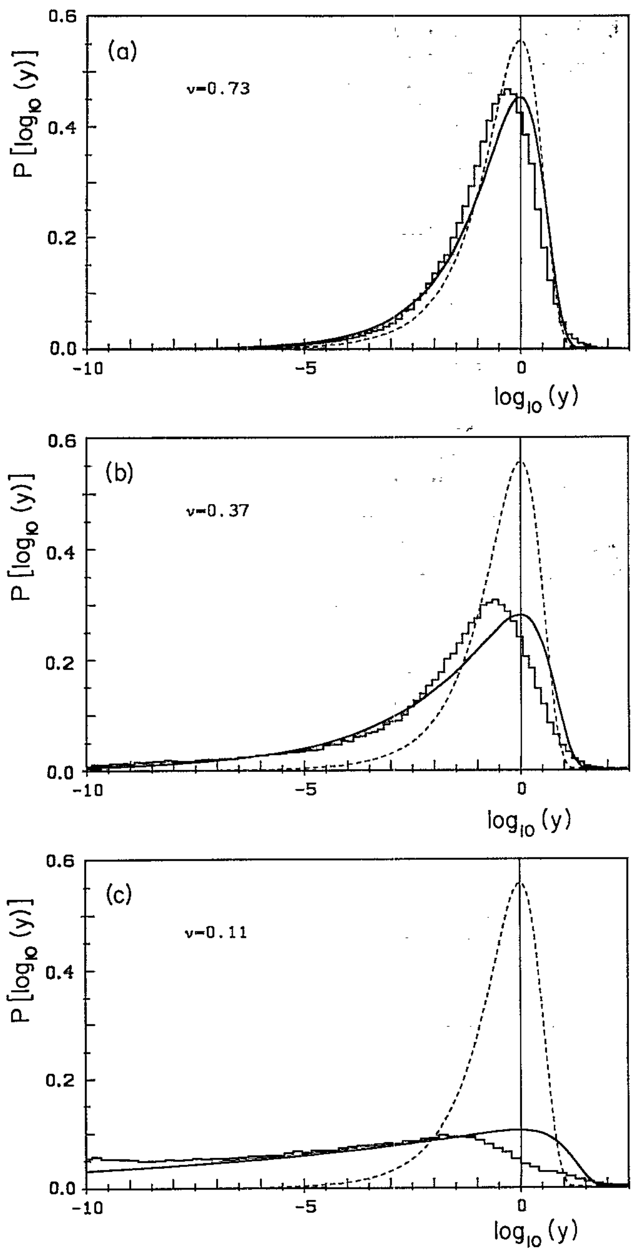

FIG. 2. Transition chaotic-regular motion. Histograms obtained for the F_1 system with (a) $K_2=0.0$, $K_1=3.5$; (b) $K_1=3.0$; and (c) $K_1=2.5$. Note the change of scale. Dashed line shows χ_1^2, solid line shows the best-fit curve.

wards smaller values of y as K_1 is reduced. A theory for the transition is in obvious need.

To investigate the case of strictly regular motion the classically integrable Floquet operator

$$F_3=e^{-i\delta J_z} \tag{9}$$

was considered. In an effort to construct a "generic basis" we have rotated the eigenvectors $|j,m\rangle$ of J_z, $\mathbf{J}^2$, and F_3 as $R_{\alpha,\beta,\gamma}|j,m\rangle$, where R is a unitary operator characterized by three Euler angles α,β,γ.[15] As "transition strengths" we have taken

$$y=|\langle j,m|R^{+}_{\alpha\beta\gamma}e^{-i\delta J_z}R_{\alpha\beta\gamma}|j,m\rangle|^2 . \tag{10}$$

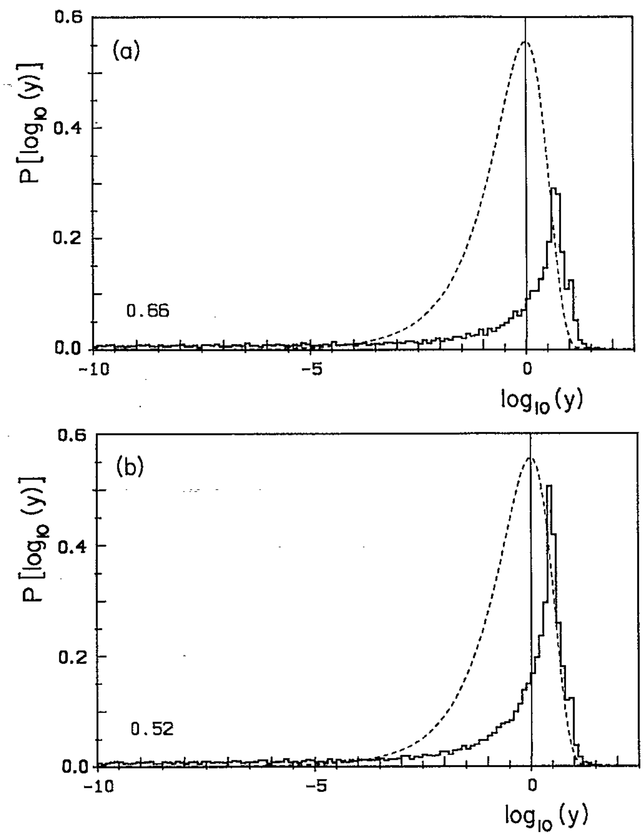

FIG. 3. Histogram of $e^{-i\delta J_z}$ matrix elements, $\delta=0.5$ in basis rotated by Eulers angles (a) $\alpha=0.4$, $\beta=-0.6$, $\gamma=1.0$; and (b) $\alpha=0.9$, $\beta=1.3$, $\gamma=0.3$.

Figure 3 depicts two histograms based on different sets of Euler angles [$\alpha=0.4$, $\beta=-0.6$, $\gamma=1.0$ for Fig. 3(a) and $\alpha=0.9$, $\beta=1.2$, $\gamma=0.3$ for Fig. 3(b)] obtained for the same coupling constant $\delta=0.5$. The moduli of several transition strengths were equal to 10^{-16}, a computational zero at our machine. The percentage of elements smaller than 10^{-10} is denoted in the bottom left-hand corner of each picture; the dashed lines represent the Porter-Thomas distribution.

Clearly, the distribution in Fig. 3 varies significantly with the angle parameters. The peak in histogram at $\log_{10}(y)\sim0.5$ has nothing to do with chaos; rather it should be understood as a remnant of the eigenrepresentation of F_3. The three-parameter rotation $R_{\alpha\beta\gamma}$ seems incapable of producing a generic basis. It is worth recalling that no such peak was observed in Fig. 2(c). Even though that case corresponds to predominantly regular classical motion the eigenbasis of F_1 seems "generic enough" (with respect to the "transition operator" J_z) to avoid any such peak in histogram.

Previous authors[4,5,16,17] have disparate views on the matter just touched upon, including a wide range of the best fits for ν in χ^2_ν distribution describing the transition from chaos to regular dynamics. We would therefore like to emphasize again that the height, width, and localization of the peaks in Fig. 3 are not universal but representation dependent. Only the long flat tail of the distribution $P[\log_{10}(y)]$ is a generic property of regular systems.

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