PHYSICAL REVIEW A

STATISTICAL PHYSICS, PLASMAS, FLUIDS, AND RELATED INTERDISCIPLINARY TOPICS

THIRD SERIES, VOLUME 44, NUMBER 10 15 NOVEMBER 1991

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Manifestation of wave chaos in pseudointegrable microwave resonators

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(Received 17 July 1991)

We analyze the distribution of eigenfrequencies of microwave resonators which correspond to pseudointegrable billiards. The statistical properties of the measured spectra are explained by means of a model of additive random matrices.

PACS number(s): 05.45.+b

The statistical properties of the eigenvalues of quantum systems presently are studied intensively in connection with the belief that they reflect the degree of order in the corresponding classical system. Whereas for integrable systems a Poisson distribution of the level-spacing statistics is observed, the spectra of classically chaotic systems display the Wigner distribution (level repulsion) characteristic of the Gaussian orthogonal ensemble (GOE) of random matrices [1]. This observation has been made in a large number of numerical studies [2]. Similar results have also been obtained experimentally in the distribution of resonances of microwave cavities. For resonators having the form of a classically chaotic billiard (Bunimovich standium or Sinai billiard) the Wigner distribution has been observed for the eigenfrequency spacings [3].

Another type of system that displays chaotic wave behavior is the so-called pseudointegrable system. In this case, the corresponding classical dynamics is not chaotic. Rather, the system possesses integrals of motion and its phase-space trajectory is confined to an invariant manifold. While for the integrable case the manifold is topologically equivalent to a torus (genus g equal to unity), the invariant manifold for the pseudointegrable system is represented by more complicated topological structures [4] that are characterized by $g \geq 2$. Such systems cannot be quantized semiclassically [5]. Recent numerical experiments performed on various quantum pseudointegrable models [5–8] showed that, although for both chaotic and pseudointegrable models linear level repulsion is observed for small level spacings, noticeable differences in the Wigner distribution become apparent to the regime of larger level spacings.

The aim of our paper is to report on the first experimental evidence of wave chaos that is observed in pseudointegrable microwave resonators. The system we are going to discuss consists of a rectangular resonator cavity coupled to the power source by a microwave cable (see Fig. 1). The microwaves were excited by a thin wire antenna perpendicular to the top and bottom faces of the cavity, these faces being parallel to each other with a distance of $d = 8$ mm. The absorption of the microwaves has been measured as a function of frequency. Each absorption maximum that is obtained corresponds to an eigenfrequency of the resonator. For frequencies $\nu < c/2d$, only the first transverse magnetic mode is excited and the resonator can be considered as being two dimensional with the electric-field vector pointing from the top to the bottom of the resonator. The (vector) equation governing the field coincides, therefore, with the standard two-dimensional Schrödinger equation with hard-wall boundary conditions on the cavity boundary [3] (the component of the electric field tangential to the cavity walls has to vanish). The electric field in the cavity therefore can be mapped in a one-to-one manner to the wave function inside the corresponding quantum billard and its eigenfrequencies correspond to the quantum-mechanical eigenvalues. Strictly speaking, the tangential component of the electric field is not exactly equal to zero on the boundary since the electric field penetrates into the cavity walls due to their finite conductivity. The consequences are a broadening of the resonance lines and a systematic loss of closely neighboring eigenfrequencies.

The cavity can be described as a rectangular quantum billiard with Dirichlet boundary conditions. The wire antenna inside the cavity gives rise, however, to wave scattering. Having in mind that the wire is very thin, the antenna can be described as a two-dimensional dipole (line source) with a coupling that depends on the microwave frequency [9]. The stationary wave equation reads

$$-\Delta E_z + \mu(\nu)\delta(\mathbf{x}-\mathbf{x}_0)E_z = \left(\frac{2\pi\nu}{c}\right)^2 E_z \,, \qquad (1)$$

where E_z denotes the tangential component of the electric

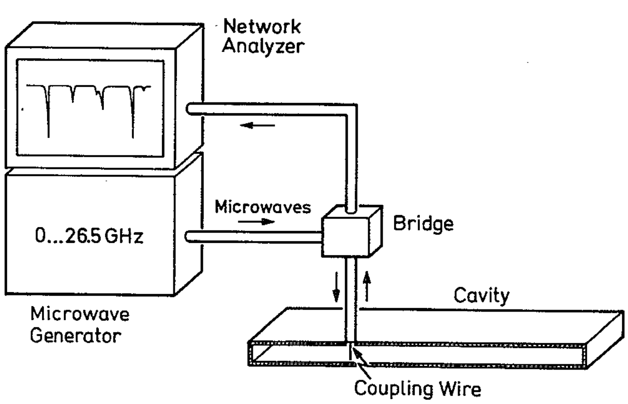

FIG. 1. Scheme of the apparatus. The microwaves are radiated via a gold wire (diameter $D = 0.5$ mm) into a rectangular cavity (for dimensions, see text). The reflected microwave power is registered by means of a scalar network analyzer.

field and the Laplace operator Δ is defined with zero boundary conditions on the cavity boundary. The δ function describes the two-dimensional dipole, $\mathbf{x}_0$ denotes the position of the antenna, and $\mu(\nu)$ stands for the frequency-dependent coupling parameter. Maxwell's equations imply

$$\mu(\nu) = k\nu \,, \qquad (2)$$

with the constant k depending on the conductivity of the wire.

Recent theoretical studies of Eq. (1) have shown that its solutions display wave chaos [6,10] and that the level-spacing statistics differs significantly from the Poisson distribution [8,11]. The corresponding classical billiard is pseudointegrable, since trajectories that are influenced by the point interaction (δ potential representing the wire antenna) are of measure zero relative to the other trajectories which coincide with the trajectories in the integrable rectangular billiard.

It is known that the spectral fluctuations of quantized classically chaotic systems can be well described by ensembles of random matrices. In the pseudointegrable case, the phase-space trajectories are confined to an invariant surface of higher genus. From the topological point of view, we can decompose this surface into several mutually connected tori and then divide the classical trajectories into two groups. Belonging to the first group are trajectories that are confined to one particular torus. These trajectories coincide with trajectories in an integrable system and can be quantized semiclassically. The second group consists of trajectories covering two or more tori. This group cannot be quantized semiclassically and leads to the appearance of wave chaos.

Having this classical picture in mind, it is reasonable to describe the spectral fluctuation in a quantized pseudointegrable model using an additive model of random matrices, where one matrix represents the influence of trajectories belonging to the first group while the other one describes the influence of the trajectories from the second group. The simplest ensemble with such properties is given by [12–14]

$$H = (H^0 + \lambda V)/\sqrt{1+\lambda^2} \,, \qquad (3)$$

where H^0 is a diagonal $N \times N$ random Gaussian matrix each element of which has zero mean value and variance $\langle (H^0_{ij})^2 \rangle = \delta_{ij}$. This matrix has Poissonian level statistics and describes the influence of the integrable trajectories. The matrix V, on the other hand, belongs to the GOE and its elements have the ensemble means $\langle V_{ii}^2 \rangle = 2\langle V_{ij}^2 \rangle = 1/2N$, $i \neq j$. The parameter λ describes the relative influence of the "chaotic" contribution to the ensemble: for $\lambda = 0$ the ensemble (3) has Poissonian spectral statistics, while for $\lambda \to \infty$ the GOE limit is achieved. The normalization factor $(1+\lambda^2)^{-1/2}$ in the definition (3) assures a constant support of the density of eigenvalues in the latter limit [13].

In the most interesting case of a large matrix dimension N it is hardly possible to obtain a direct expression for the level-spacing distribution $P_\lambda(S)$ for the ensemble (3). In the simplest case $N = 2$, however, the spacing distribution can be expressed in terms of the modified Bessel function

$I_0(x)$ and the Tricomi function $U(a,c,x)$ as [13–15]

$$P_\lambda(S) = \left[\frac{Su(\lambda)^2}{\lambda}\right] \exp\left[\frac{-u(\lambda)^2 S^2}{4\lambda^2}\right] \times \int_0^\infty e^{(-\xi^2 - 2\xi\lambda)} I_0\left[\frac{S\xi u(\lambda)}{\lambda}\right] d\xi, \quad (4)$$

where $u(\lambda) = \sqrt{\pi} U(-\frac{1}{2}, 0, \lambda^2)$. For $\lambda = 0$ the above distribution reduces to the exponential e^{-S} while the Wigner surmise is approached in the limit $\lambda \to \infty$. Unlike for the Berry-Robnik or the Brody distributions, we encounter a linear repulsion of eigenvalues for all intermediate values of λ, i.e., $P(S,\lambda) \sim S/\lambda$, for $S \ll \lambda$. Interestingly, the distribution (4) describes not only the level statistics of the 2×2 ensembles, but also gives a satisfactory approximation for $P(S)$ for large matrix dimensions N [13,14]. Moreover, the distribution (4) may be applied to describe the level statistics of the rectangular billiard with a point interaction as was analyzed in Ref. [8]. In order to compare these experimental results with the theoretical predictions, we have solved numerically Eq. (1) with $\mu = 1$ and evaluated the first 23000 levels. The resulting distribution which is in excellent agreement with the predictions of the ensemble (3) is plotted in Fig. 2. The value of the fitted parameter λ varies from zero (Poisson spectrum for zero coupling strength μ) to about 0.61 for the limiting case of infinite coupling constant μ. It is worth noting that $\lambda(\mu = \infty) \neq \infty$ reflecting, thereby, the pseudointegrability of the system ($\lambda = \infty$ corresponds to a classically chaotic system).

A close similarity of the singular-billiard model with the resonator problem discussed above and described by Eq. (1) suggests an analysis of the experimental data with the help of the distribution (4). In order to obtain smooth histograms we have measured the spectra of 69 rectangular billiards of the same width (20.0 cm) but with lengths ranging from 16.5 to 51.0 cm. Since the effective coupling constant is frequency dependent (2), one can expect the statistical properties of the spectrum to change with the frequency. A convincing confirmation of this fact is obtained already by studying the mean level density. For small frequencies $0 < \nu < 10$ GHz, the measured density is significantly smaller than the prediction of the Weyl formula [4], while for higher frequencies $16 < \nu < 18$ GHz the density approaches the expected value. A simple explanation of this fact can be given. For small frequencies the spectrum has, in general, Poisson statistics, so that the probability of finding a level cluster and, in consequence, of losing an eigenfrequency is considerable. With increasing frequency, the effective coupling μ in Eq. (1) becomes stronger and the spacing distribution becomes closer to the GOE prediction. Also, level repulsion occurs and the relative number of missed eigenfrequencies decreases.

To make use of the data obtained for different resonators, a spectral unfolding [14,16] was done separately for each case and the normalized spacings were gathered together for various frequency ranges. Figure 3 represents

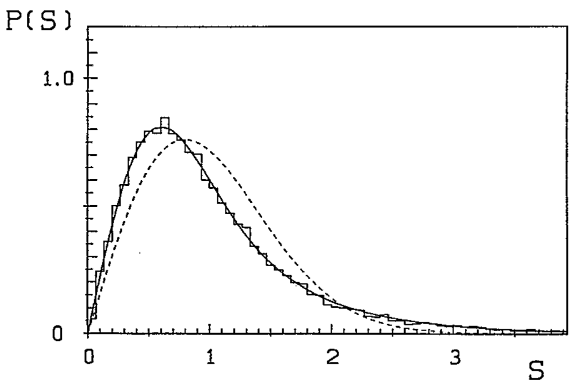

FIG. 2. The level-spacing distribution evaluated numerically for $\mu = 1$. The fitted λ is equal to 0.48. The dashed curve represents the Wigner surmise.

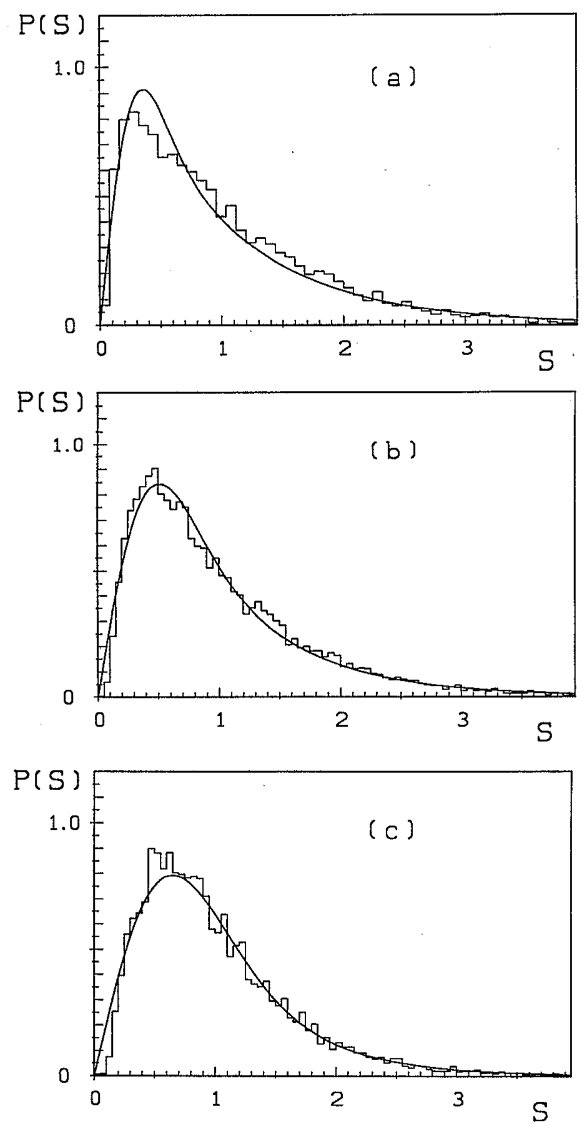

FIG. 3. Level-spacing distribution collected from 69 rectangular billiards of different sizes for the energy ranges (in GHz) (a) $5 < \nu < 10$, (b) $10 < \nu < 15$, and (c) $15 < \nu < 18$. Fitted values of the parameter λ in the distribution (5) are equal to 0.19, 0.34, and 0.58, respectively.

the experimental data obtained for the following frequency ranges (in GHz): 3(a) $5<\nu<10$; 3(b) $10<\nu<15$, and 3(c) $15<\nu<18$. We observe a transition from the Poisson towards the Wigner distribution. The solid line represents the best fit of the distribution (3) with λ equal to 0.19, 0.34, and 0.58, respectively. As expected from Eq. (1), the effective coupling constant λ grows with the frequency ν.

To conclude, we have demonstrated that the spacing distribution of rectangular microwave resonators, corresponding to classically integrable billiards, differs from the Poisson distribution due to the presence of the wire antenna. In other words, the measurement device itself is responsible for the appearance of the wave chaos in the system. The effective coupling between the antenna and the resonator depends on the microwave frequencies. The complete experimental setup, the resonator and the wire antenna, corresponds to a classically pseudointegrable system. Statistical properties of its spectrum may be well described by the distribution (3) resulting from the additive random-matrix model.

Financial support by the Alexander von Humboldt Fellowship and by the project "Quantum Chaos" of Polski Komitet Badán Naukowych is gratefully acknowledged. The work was sponsored by the Deutsche Forschungsgemeinschaft via the Sonderforschungsbereiche "Nichtlineare Dynamik" and "Unordnung und Grosse Fluktuationen. M. Kollmann is thanked for assistance in the analysis of the experimental data.

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