

Genuinely multipartite entangled states and orthogonal arrays

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A pure quantum state of N subsystems with d levels each is called k -multipartite maximally entangled state, which we call a k -uniform state, if all its reductions to k qudits are maximally mixed. These states form a natural generalization of N -qudit Greenberger-Horne-Zeilinger states which belong to the class 1-uniform states. We establish a link between the combinatorial notion of orthogonal arrays and k -uniform states and prove the existence of several classes of such states for N -qudit systems. In particular, known Hadamard matrices allow us to explicitly construct 2-uniform states for an arbitrary number of $N > 5$ qubits. We show that finding a different class of 2-uniform states would imply the Hadamard conjecture, so the full classification of 2-uniform states seems to be currently out of reach. Furthermore, we establish links between the existence of k -uniform states and classical and quantum error correction codes and provide a graph representation for such states.

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I. INTRODUCTION

A multipartite pure state of N qudits is called entangled if it cannot be written as the tensor product of N single-qudit pure states. There exist also states for which at least some subsystems can be factorized. Hence, one distinguishes the biseparable states, for which one can find a splitting with respect to which a given state is separable. A state is called *genuinely entangled* if all subsystems are correlated and the state is not separable with respect to any possible splitting of N subsystems [1,2].

In the case of multipartite systems there exists several different classes of entangled states. In the simplest case of three qubits there are two noncomparable classes of entanglement, called Greenberger-Horne-Zeilinger (GHZ) and W , which are not equivalent with respect to local operations and classical communication [3]. In the case of $N = 4$ qubits the number of inequivalent classes of pure states grows to nine [4] or more [5] if another classification is used. In general, the number of parameters describing entanglement grows exponentially with the number N of subsystems [6].

Investigation of highly entangled states of several qubits, initiated by Gisin and Bechmann-Pasquinucci [7], was then continued in the specific case of four qubits by Higuchi and Sudbery [8]. Later on the issue of identifying multipartite pure states which are distinguished by maximizing certain measures of entanglement was analyzed in [9–14] and was further developed in [15–17].

In the case of bipartite systems, one distinguishes the Bell states and their generalizations for two d -level systems, for which reduced states are maximally mixed. A class of multiqubit pure states with a stronger property, in which every possible one-qubit reduction is maximally mixed, was analyzed in [7,11,18,19] and called perfect maximally

multipartite entangled state. A further important contribution on these issues was provided recently by Arnaud and Cerf [20], who used the name of *k -multipartite maximally entangled* pure states. In the present work we call these states *k -uniform*, following an earlier paper of Scott [21]. These states are distinguished by the fact that their reduction with respect to an arbitrary splitting leaves a k -partite reduced state maximally mixed, so the mean purity of the reduction, averaged over different choices of the ancillary system, is minimal [22]. In particular, 1-uniform states are also *balanced*, which means that the total number of each of the d levels appearing in the representation of the state is equal [1,23].

Construction of genuinely multipartite entangled states is an important open problem in the theory of quantum information, as these states have numerous applications to quantum teleportation, quantum key distribution, dense coding, and error-correcting codes [20,21]. The main goal of this work is to establish a link between orthogonal arrays and k -uniform states and to make use of it to construct families of such states for an arbitrary number of subsystems.

This paper is organized as follows. In Sec. II we introduce k -uniform states and resume the *state of the art*. In Sec. III we present basic concepts of orthogonal arrays useful in this work. In Sec. IV we connect k -uniform states with orthogonal arrays in a natural way. This allows us to prove the existence of families of k -uniform states for N subsystems of d levels each. In Sec. V we present examples of k -uniform states obtained from orthogonal arrays and we study the minimal number of terms required to construct a k -uniform state of N qubits. In Sec. VI we show that the problem of classifying 2-uniform states for qubits contains the Hadamard conjecture. Consequently, the complete classification of 2-uniform states seems to be currently out of reach. Fortunately, a different class of 2-uniform states is possible, so making use of the well-known Hadamard matrices of order 2^m and 3×2^m we construct such states for every N . The paper is concluded in Sec. VII, in which a list of open questions is presented.

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Explicitly constructed families of 2-uniform states of 6 to 15 qubits are listed in Appendix A. In Appendix B we present further examples of k -uniform states of qudits having $d > 2$ levels related to orthogonal arrays. In Appendix C we present an extended construction of further families of k -uniform states.

II. k -MULTIPARTITE MAXIMALLY ENTANGLED PURE STATES

In this section we discuss the concept of k -uniform states [18,20] and present a short review of the state of the art. A pure quantum state of N subsystems with d levels each is called k maximally entangled, written k -uniform, if every reduction to k qudits is maximally mixed. For example, GHZ states belong to the class 1-uniform and W states do not belong to any class of k -uniform states. The general problem to construct k -uniform states is difficult and explicit solutions are known for systems consisting of a few subsystems only. For the convenience of the reader, we present here some of the most important results known in the literature.

We start recalling the notion of local equivalence. Two pure states $|\psi\rangle$ and $|\phi\rangle$ of a bipartite system are called *locally equivalent*, written $|\psi\rangle \sim_{\text{loc}} |\phi\rangle$, if there exists a product unitary matrix $U_A \otimes U_B$ such that $|\psi\rangle = U_A \otimes U_B |\phi\rangle$. In the case of N -partite systems the product unitary matrix factorizes into a tensor product of N unitaries. By definition, if a given state $|\Psi\rangle$ belongs to the class of k -uniform states, so does any other locally equivalent state.

In the case of two qubits, one distinguishes four mutually orthogonal, maximally entangled *Bell states*,

$$|\Phi_2^\pm\rangle = |00\rangle \pm |11\rangle \quad \text{and} \quad |\Psi_2^\pm\rangle = |01\rangle \pm |10\rangle. \quad (1)$$

These locally equivalent states are 1-uniform. Moreover, it is easy to show that any state of this class has to be locally equivalent to the Bell state $|\Phi_2^+\rangle$.

For three qubits the only 1-uniform state is the GHZ state [24],

$$|\text{GHZ}\rangle = |000\rangle + |111\rangle, \quad (2)$$

up a local unitary transformation. For brevity we omit in the paper the normalization factors, provided they are the same for each term.

A class of 1-uniform states of N qudits systems of d levels is easy to find,

$$|\text{GHZ}_N^d\rangle = \sum_{i=0}^{d-1} |i\rangle^{\otimes N}. \quad (3)$$

For three qubits there exists another class of maximally entangled states called W ,

$$|W\rangle = |100\rangle + |010\rangle + |001\rangle. \quad (4)$$

However, the reductions to 1-qubit systems are not maximally mixed, so $|W\rangle$ is neither a 1-uniform state nor a balanced state [23].

In the case of four qubits it is straightforward to write some 1-uniform states. Any state locally equivalent to the four-qubit state $|\text{GHZ}_4^2\rangle$ —a special case of (3)—does the job. Another

example reads

$$|\Phi_4\rangle = |0000\rangle + |0011\rangle + |0101\rangle + |0110\rangle \\ + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle. \quad (5)$$

This state is locally equivalent to the 1-uniform state of four qubits $|\text{HS}\rangle$ [8]. It is worth emphasizing that there exist no 2-uniform states of four qubits [8,12,13,18,25]. This fact was interpreted as a symptom of *frustration* [22], as the requirement that the entanglement is maximal for all possible bipartitions of the system becomes conflicting. The phenomenon of frustration is known in various matrix models including spin glasses [26], for which a phase transition takes place. In the case of $N = 4$ qubit systems, or other systems for which uniform states do not exist, it is interesting to identify special states, for which the average entanglement is maximal. More formally, one looks for states for which the mean purity of the k -partite reduced state, averaged over different choices of the ancillary system, is minimal [22,27]. However, it has been proven that such minimization procedures do not help to uniquely identify maximally entangled states when a k -uniform state does not exist [28]. For five qubits there are 2-uniform states:

$$|\Phi_5\rangle = -|00000\rangle + |01111\rangle - |10011\rangle + |11100\rangle \\ + |00110\rangle + |01001\rangle + |10101\rangle + |11010\rangle. \quad (6)$$

This state, also called $|0_L\rangle$, has been used to distribute quantum information over five qubits [29]. It is worth noting that a k -uniform state is also a k' -uniform state with $0 < k' < k$. Thus, the state $|\Phi_5\rangle$ is also 1-uniform. It is easy to see that there is no 3-uniform state of five qubits, as a k -uniform state of N qubits can exist if [21]

$$k \leq \lfloor N/2 \rfloor, \quad (7)$$

where $\lfloor \cdot \rfloor$ denotes the integer part. If this bound is not satisfied, then the dimension of the reduced state is larger than the size of the ancillary space, so the reduction of a pure state cannot be maximally mixed. Observe that for $N = 2, 3$, and 5 the above inequality is saturated; however, for $N = 4$ it is not the case.

Additionally, there exist 3-uniform states of six qubits [12]:

$$|\Phi_6\rangle = -|000000\rangle + |001111\rangle - |010011\rangle + |011100\rangle \\ + |000110\rangle + |001001\rangle + |010101\rangle \\ + |011010\rangle - |111111\rangle + |110000\rangle \\ + |101100\rangle - |100011\rangle + |111001\rangle \\ + |110110\rangle - |101010\rangle - |100101\rangle. \quad (8)$$

The above examples have been constructed by hand or by computing algorithms. As far as we know, the first expression of a 3-uniform state of six qubits was found by Borrás *et al.* [12]. In general, for $N > 6$ the existence of k -uniform states for $k > 1$ remains open.

In this work we use the combinatorial notion of *orthogonal arrays* and demonstrate that they form a suitable tool for studying genuine multipartite entangled pure states.

III. ORTHOGONAL ARRAYS

Orthogonal arrays are combinatorial arrangements introduced by Rao [30] in 1946. They have a close connection to codes, error-correcting codes, difference schemes, Latin squares, and Hadamard matrices; see [31]. The most important applications of orthogonal arrays are given in statistics and in designing experiments. An $r \times N$ array A with entries taken from a set S with d elements is said to be an *orthogonal array* with r runs, N factors, d levels, strength k , and index λ if every $r \times k$ subarray of A contains each k -tuple of symbols from S exactly λ times as a row. Here r and N denote the number of rows and columns of A , respectively, while d is the cardinality of the set S . That is, the level d is the number of different symbols appearing in A .

An excellent introduction to orthogonal arrays and their applications is provided in the book of Hedayat, Sloane, and Stufken [31]. Furthermore, extensive catalogs of orthogonal arrays can be found in the handbook [32] and in the web sites of Sloane [33] and Kuhfeld [34]. The ordering of parameters characterizing orthogonal arrays used in this work follows the standard notation introduced by Rao [30]:

$$\text{OA}(r, N, d, k).$$

However, the labels of the parameters has been conveniently adapted here to quantum theory. The same ordering is used in [31,33,34], but since some other conventions are also present in the literature, the reader is advised to check the order of the parameters before using a given resource on OAs.

One usually determines an OA by the following four independent parameters r , N , d , and k , while the index λ satisfies the relation

$$r = \lambda d^k. \tag{9}$$

Additionally, the following *Rao bounds* hold [35],

$$r \geq \sum_{i=0}^{k/2} \binom{N}{i} (d-1)^i \quad \text{if } k \text{ is even,} \tag{10}$$

$$r \geq \sum_{i=0}^{\frac{k-1}{2}} \binom{N}{i} (d-1)^i + \binom{N-1}{\frac{k-1}{2}} (d-1)^{\frac{k-1}{2}} \quad \text{if } k \text{ is odd.} \tag{11}$$

If the parameters are such that the above inequalities are saturated, then the OA is called *tight*. It is easy to check that the examples presented in Fig. 1 satisfy this property. Given an $\text{OA}(r, N, d, k)$ we can easily construct an $\text{OA}(r, N', d, k)$ for any $k \leq N' \leq N$ by removing from the array $N - N'$ columns. Therefore, it is interesting to determine the maximal factor N such that an $\text{OA}(r, N, d, k)$ exists for r , d , and k fixed. This maximal factor is defined as

$$N_{\max} = f(r, d, k). \tag{12}$$

Orthogonal arrays are very useful to design fractional factorial experiments [35]. The parameter r determines the number of *runs* of the experiments. Thus, it is important to minimize the number r without changing the rest of the parameters defining the experiment. Consequently, we define

$$r_{\min} = F(N, d, k) \tag{13}$$

0	0	1	0	0	0
1	1	0	1	0	0
		0	0	1	0
		0	0	0	1
0	0	0	0	1	1
0	1	1	1	0	1
1	0	1	1	1	0
1	1	0	1	1	1

FIG. 1. Orthogonal arrays of strength one (top left), two (bottom left), and three (right). Their symbolic expressions are $\text{OA}(2,2,2,1)$, $\text{OA}(4,3,2,2)$, and $\text{OA}(8,4,2,3)$, respectively.

as the lowest value of r such that an $\text{OA}(r, N, d, k)$ exists for N , d , and k fixed. Interestingly, the following relationships are satisfied [31]:

$$F(N, d, k) = \min\{r : f(r, d, k) \geq N\},$$

$$f(r, d, k) \leq \max\{N : F(N, d, k) \leq r\}.$$

In what follows we present one of the first important results on the existence of OAs, discovered in 1946 by Rao [30].

Theorem 1 (Rao). If d is a prime power, then an $\text{OA}(d^n, (d^n - 1)/(d - 1), d, 2)$ exists whenever $n \geq 2$.

These OAs can be constructed [35] by using Galois fields, but this is not simple to do for high values of d .

IV. k -UNIFORM QUANTUM STATES FROM ORTHOGONAL ARRAYS

In this section we study a close connection between OAs and k -uniform quantum states. Let us start by defining a pure state $|\Phi\rangle$ of a system consisting of N qudits,

$$|\Phi\rangle = \sum_{s_1, \dots, s_N} a_{s_1, \dots, s_N} |s_1, \dots, s_N\rangle, \tag{14}$$

where $a_{s_1, \dots, s_N} \in \mathbb{C}$, $s_1, \dots, s_N \in S$, and $S = \{0, \dots, d - 1\}$. The set of vectors $\{|s_1, \dots, s_N\rangle\}$ forms an orthonormal basis of \mathbb{C}^{d^N} . In fact, this is the *canonical* or *computational* basis. For simplicity, the range of the sums is omitted, but all of them go from 0 to $d - 1$. The density matrix ρ associated with this pure state reads

$$\rho = |\Phi\rangle\langle\Phi| = \sum_{\substack{s_1, \dots, s_N \\ s'_1, \dots, s'_N}} a_{s_1, \dots, s_N} a_{s'_1, \dots, s'_N}^* \times |s_1, \dots, s_N\rangle\langle s'_1, \dots, s'_N|. \tag{15}$$

Let us divide the system S into two parts S_A and S_B , each containing N_A and N_B qudits, respectively, such that $N = N_A + N_B$. In order to find the density matrix associated with S_A , we have to consider the reduced state

$$\rho_A = \text{Tr}_B(\rho_{AB}), \tag{16}$$

where ρ_{AB} denotes the state (15). Performing the partial trace we get

$$\begin{aligned} \rho_A &= \text{Tr}_B \left(\sum_{\substack{s_1, \dots, s_N \\ s'_1, \dots, s'_N}} a_{s_1, \dots, s_N} a_{s'_1, \dots, s'_N}^* |s_1, \dots, s_N\rangle \langle s'_1, \dots, s'_N| \right) \\ &= \sum_{\substack{s_1, \dots, s_N \\ s'_1, \dots, s'_N}} a_{s_1, \dots, s_N} a_{s'_1, \dots, s'_N}^* \text{Tr}_B(|s_1, \dots, s_N\rangle \langle s'_1, \dots, s'_N|) \\ &= \sum_{\substack{s_1, \dots, s_N \\ s'_1, \dots, s'_N}} a_{s_1, \dots, s_N} a_{s'_1, \dots, s'_N}^* \langle s'_{N_A+1}, \dots, s'_{N_A} | s_{N_A+1}, \dots, s_N \rangle \\ &\quad \times |s_1, \dots, s_{N_A}\rangle \langle s'_1, \dots, s'_{N_A}|. \end{aligned} \quad (17)$$

The following reasoning deserves special attention as it contains the key point of our work. First, let us assume that every coefficient a_{s_1, \dots, s_N} is *zero* or *one* for simplicity. Therefore, the state $|\Phi\rangle$ can be written as a superposition of r product states,

$$|\Phi\rangle = |s_1^1, s_2^1, \dots, s_N^1\rangle + \dots + |s_1^r, s_2^r, \dots, s_N^r\rangle, \quad (18)$$

where the upper index i on s denotes the i th term in the linear decomposition of $|\Phi\rangle$. Second, let us now arrange the symbols appearing in Eq. (18) in an array,

$$\begin{array}{cccc} s_1^1 & s_2^1 & \dots & s_N^1 \\ s_1^2 & s_2^2 & \dots & s_N^2 \\ \vdots & \vdots & \dots & \vdots \\ s_1^r & s_2^r & \dots & s_N^r \end{array}, \quad (19)$$

so every column of the array is identified with a particular qudit and every row corresponds to a linear term of the state. Here we are interested to study pure states having maximally mixed reductions. This means that the reduced system ρ_A must be proportional to the identity matrix, independently of the number N_B of qudits traced out. By imposing this requirement into Eq. (17) and considering Eqs. (18) and (19), we find two basic ingredients for constructing k -uniform states.

(A) *Uniformity*. The sequence of N_A symbols appearing in every row of every subset of N_A columns of the array given in Eq. (19) is *repeated* the same number of times. This implies that the diagonal of the reduction ρ_A is uniform, as all its elements are equal.

(B) *Diagonality*. The sequence of N_B symbols appearing in every row of a subset of N_B columns is *not repeated* along the r rows. Due to this property the reduced density matrix ρ_A becomes diagonal.

The above two conditions are sufficient to find k -uniform states of N qudits of d levels. Note that the uniformity condition (A) implies that the array defined in Eq. (19) is an OA. From a physics point of view, the first condition concerns the reduced state of subsystem A , while the second one deals with the environment B , with respect to which the partial trace is performed. Moreover, there exists a perfect match between the parameters of an $\text{OA}(r, N, d, k)$ and the parameters of a k -uniform state, as we can see in Table I.

Figure 2 presents the basic ingredients of the relation between a multipartite entangled state and an OA of N columns

TABLE I. Correspondence between parameters of OAs and quantum states.

	Orthogonal arrays	Multipartite quantum state $ \Phi\rangle$
r	Runs	No. of linear terms in the state
N	Factors	No. of qudits
d	Levels	Dimension of the subsystem ($d = 2$ for qubits)
k	Strength	Class of entanglement (k -uniform)

and r rows. It will be thus convenient to introduce the following class of arrays.

Definition. An orthogonal array $\text{OA}(r, N, d, k)$ is called *irredundant*, written IrOA , if when removing from the array any k columns all remaining r rows, containing $N - k$ symbols each, are different.

If this condition is not fulfilled, certain remaining rows are equal and carry some redundant information. Any irredundant OA satisfies thus both conditions (A) and (B) and allows us to construct a k -uniform state. However, condition (B) is not necessary, as there exist quantum states, e.g., locally equivalent to states constructed by an IrOA , which are k -uniform but do not satisfy the diagonality property.

On the other hand, we do not know whether every k -uniform state is related by a local unitary transformation to a k -uniform state constructed with an irredundant OA.

Since the seminal work of Rao [30] the theory of OAs has been developed significantly. Many important theorems concerning the existence of OAs have been found and direct connections with other combinatorial arrangements have been established. We translate here some of these relevant theorems to the construction of k -uniform quantum states. Interestingly, there is a match between basic properties of OAs and k -uniform states. For OAs the following properties hold (see Hedayat *et al.* [31], pp. 4 and 5).

- (i) The parameters of an OA satisfy equality $\lambda = r/d^k$.
- (ii) Any $\text{OA}(r, N, d, k)$ is also an $\text{OA}(r, N, d, k')$ for every $k' < k$.
- (iii) A permutation of the runs or factors in an OA results in an OA with the same parameters.

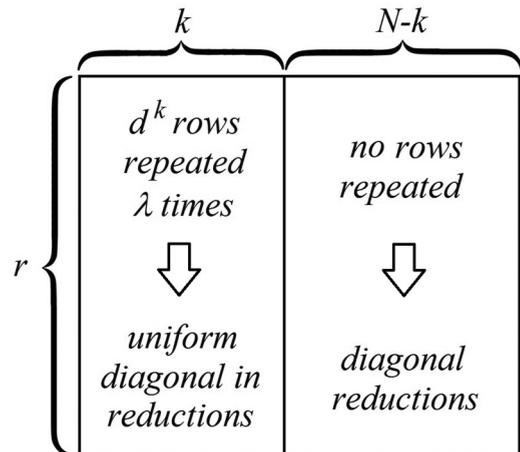


FIG. 2. Connection between OAs and k -uniform states of N qudits, where $k \leq N - k$.

(iv) A permutation of the levels of any factor in an OA results in an OA with the same parameters.

(v) Any $r \times N'$ subarray of an $\text{OA}(r, N, d, k)$ is an $\text{OA}(r, N', d, k')$, where $k' = \min\{N', k\}$ and $N' < N$.

(vi) Taking the runs in an $\text{OA}(r, N, d, k)$ that begins with a particular symbol and omitting the first column yields an $\text{OA}(r/d, N - 1, d, k - 1)$.

(vii) Take m OAs $A_i = \text{OA}(r_i, N, d, k_i)$, for $i = 0, \dots, m - 1$, and define the array A as their juxtaposition:

$$\begin{bmatrix} A_0 \\ A_1 \\ \vdots \\ A_{m-1} \end{bmatrix}. \quad (20)$$

Then A is an $\text{OA}(r, N, d, k)$, where $N = N_0 + \dots + N_{m-1}$ and the strength is k for some $k \geq \min\{k_0, \dots, k_{m-1}\}$. Furthermore, if $m = d$ and each A_i is an $\text{OA}(r, N, d, k)$, after appending a 0 to each row of A_0 , a 1 to each row of A_1 , and so on, we obtain an $\text{OA}(dr, N + 1, d, k)$.

Now let us translate the above properties to the setup of k -uniform quantum states.

(i') All the reduced density matrices of a k -uniform state satisfy $\text{Tr}(\rho) = 1$.

(ii') A k -uniform state is also a k' -uniform for every $k' < k$.

(iii') A permutation of terms or qudits in a k -uniform state leads us to a k -uniform state.

(iv') Any permutation of the d symbols defining a qudit (e.g., caused by the Pauli X gate or NOT gate), does not change the k -uniform property.

(v') Any k -uniform state of N qudits can be reduced to a k' -uniform state of $N' < N$ qudits, where $k' = \min\{N', k\}$ [whenever the property (B) holds]. For example, it fails for $k' \leq N'/2$.

(vi') We can always decompose a k -uniform state as a function of k' -uniform states with $k' \leq k$. For example, a 2-uniform state of five qubits can be decomposed in the following way:

$$|\Psi_5\rangle = |0\rangle|\Psi_{4a}\rangle + |1\rangle|\Psi_{4b}\rangle, \quad (21)$$

where

$$|\Psi_{4a}\rangle = |0000\rangle + |1010\rangle + |0101\rangle + |1111\rangle, \quad (22)$$

and

$$|\Psi_{4b}\rangle = |0011\rangle + |1001\rangle + |0110\rangle + |1100\rangle \quad (23)$$

are orthogonal 1-uniform states of four qubits.

(vii') If the states $|\Phi_i\rangle, i = 0, \dots, m - 1$ are k_i -uniform and come from $\text{OA}(r_i, N, d, k_i)$, then the state $|\Psi\rangle = \sum_{i=0}^{m-1} |\Phi_i\rangle$ is a k -uniform state for some $k \geq \min\{k_0, \dots, k_{m-1}\}$. For instance, the sum of the four qubits states $|\Psi_{4a}\rangle$ and $|\Psi_{4b}\rangle$ given in Eqs. (22) and (23), respectively, forms a 1-uniform state. Furthermore, if $m = d$ and each state $|\Phi_i\rangle$ of N qudits is k -uniform, then the state

$$|\Phi\rangle = \sum_{i=0}^{m-1} |i\rangle|\Phi_i\rangle \quad (24)$$

is a k -uniform state of $N + 1$ qudits whenever the diagonality property (B) holds.

The latter condition is crucial, as shown with an example of the following state of six qubits,

$$\begin{aligned} |\Psi_6\rangle &= |0\rangle|\Phi_5\rangle + |1\rangle\overline{|\Phi_5\rangle} \\ &= -|000000\rangle + |001111\rangle - |010011\rangle + |011100\rangle \\ &\quad + |000110\rangle + |001001\rangle + |010101\rangle \\ &\quad + |011010\rangle - |100000\rangle + |101111\rangle \\ &\quad - |110011\rangle + |111100\rangle + |100110\rangle \\ &\quad + |101001\rangle + |110101\rangle + |111010\rangle, \end{aligned} \quad (25)$$

where the overbar denotes the flip operation performed on every qubit. In this case the property (B) does not hold, so the state $|\Psi_6\rangle$ is not 2-uniform nor even 1-uniform. However, 3-uniform states of six qubits can be obtained in this way for an appropriate choice of states; see Eq. (5) in [20].

In decomposition (24) every k_i -uniform state $|\Phi_i\rangle$ must come from an $\text{OA}(r_i, N, d, k_i)$ in order to get a k uniform state with $k \geq \min\{k_0, \dots, k_{m-1}\}$. For instance, the seven-qubit state

$$\begin{aligned} |\Phi_7\rangle &= |0\rangle|\Phi_6\rangle + |1\rangle\overline{|\Phi_6\rangle} \\ &= -|0000000\rangle + |0001111\rangle - |0010011\rangle + |0011100\rangle + |0000110\rangle + |0001001\rangle + |0010101\rangle + |0011010\rangle \\ &\quad - |0111111\rangle + |0110000\rangle + |0101100\rangle - |0100011\rangle + |0111001\rangle + |0110110\rangle - |0101010\rangle - |0100101\rangle \\ &\quad - |1111111\rangle + |1110000\rangle - |1101100\rangle + |1100011\rangle + |1111001\rangle + |1110110\rangle + |1101010\rangle + |1100101\rangle \\ &\quad - |1000000\rangle + |1001111\rangle + |1010011\rangle - |1011100\rangle + |1000110\rangle + |1001001\rangle - |1010101\rangle - |1011010\rangle \end{aligned} \quad (26)$$

is constructed from the 3-uniform state of six-qubit $|\Phi_6\rangle$ defined in Eq. (8). The state (26) is an *almost* 3-uniform state, as only 3 out of its $\binom{7}{3} = 35$ reductions to three qubits are not maximally mixed. These three reductions are identically equal and have four nonzero eigenvalues equal to $1/4$. Evidently,

three sets of four columns of the OAs forming this state have repeated rows and, consequently, the diagonality property (B) does not hold. Therefore, the question of whether a 3-uniform state of seven qubits exists remains open. Additionally, a 2-uniform state of seven qubits having 8 terms appears in

TABLE II. Existence of k -uniform states for qubits. Here “p” and “n” denote that a state constructed as a superposition of product states with all positive and some negative coefficients, respectively, is known in the literature. Symbol “-” denotes that such a state cannot exist, as the necessary condition $k \leq N/2$ is not fulfilled, while 0 means that such states do not exist, although the necessary condition is satisfied.

$k \setminus N$	2	3	4	5	6	7	8
1	p	p	p	p	p	p	p
2	-	-	0	n	p	p	p
3	-	-	-	-	n	?	p
4	-	-	-	-	-	-	0

Appendix A and another one having 64 terms in Appendix B. In Table II we resume the existence of k -uniform states for a few number of qubits. Interestingly, some k -uniform states seem to require negative terms.

We have shown that OAs are useful to construct k -uniform states. However, the relation between these objects is not injective. The following proposition connects OAs with a subset of multipartite quantum states.

Proposition 1. For every orthogonal array $OA(r, N, d, k)$ there exists a quantum state of N qudits such that every reduction to k qudits has its d^k diagonal entries equal to d^{-k} .

Observe that these pure states are not necessarily entangled. Moreover, the reductions to k qudits are not necessarily diagonal. This proposition can be inferred directly from the scheme presented in Fig. 2. Note that the entries of such states must have the same amplitude, which follows from Eq. (17).

In order to find a k -uniform state, we require an irredundant OA satisfying the diagonality property (B). Interestingly, this condition holds for any OA with index unity ($\lambda = 1$).

Theorem 2. Every $OA(r, N, d, k)$ of index unity is irredundant so it is equivalent to a k -uniform state of N qudits whenever $k \leq N/2$.

Proof. Given that $\lambda = 1$ every k -tuple of symbols appears only one time along the rows of the OA. Moreover, all the possible combinations of the d different symbols appear along the $r = d^k$ rows. This means that any set of $k' > k$ columns has repeated rows. Thus, to satisfy the irredundancy property it is enough to assume that $k \leq N - k$, which means that $k \leq N/2$; see Fig. 2. The reciprocal implication is straightforward from the definition of k -uniform states and OAs. ■

We note that OA of index unity contains all possible combinations of symbols in every subset of k columns ($r = d^k$). The bound $k \leq N/2$, first discussed by Scott [21] for qubits, holds in general for subsystems with an arbitrary number of d levels. As we show later, it is also predicted by the quantum Singleton bound. In Appendix B we present a list of OAs of index unity obtained from the catalog of Sloane [33]. In general, these OAs are not easy to find as one needs to use tables of Galois fields. An efficient algorithm to construct these tables is presented in Chapter 4 of the book [36]. Given that OAs with index unity are relevant to construct k -uniform states, let us mention two important theorems [37].

Theorem 3 (Bush). If $d \geq 2$ is a prime-power number, then an $OA(d^k, d + 1, d, k)$ of index unity exists whenever $d \geq k - 1 \geq 0$.

TABLE III. Existence of k -uniform states of d levels for the highest strength $k = N/2$. The first row corresponds to bipartite systems and it includes the GHZ states (i.e., $k = 1$). Here the symbol \checkmark denotes existing states, “-” that they do not exist, and ? open existence.

$N \setminus d$	2	3	4	5	6	7	8
2	\checkmark						
4	-	\checkmark	\checkmark	\checkmark	?	\checkmark	\checkmark
6	\checkmark						
8	-	?	?	?	?	\checkmark	\checkmark

Theorem 4 (Bush). If $d = 2^m$ and $m \geq 1$, then there exist an $OA(d^3, d + 2, d, 3)$.

Also, the following corollary of Theorem 3.7 (see [31], p. 41) is remarkably important here.

Corollary 1 (Hedayat). If $d = 2^m$ and $m \geq 1$, then there exists an $OA(d^{d-1}, d + 2, d, d - 1)$.

The translation of these three results to the theory of quantum entanglement is resumed in the following proposition.

Proposition 1. The following k -uniform states exist:

- (i) k -uniform states of $d + 1$ qudits with d levels, where $d \geq 2$ and $k \leq \frac{d+1}{2}$;
- (ii) 3-uniform states of $2^m + 2$ qudits with 2^m levels, where $m \geq 2$;
- (iii) $2^m - 1$ -uniform states of $2^m + 2$ qudits with 2^m levels, where $m = 2, 4$.

A k -uniform state has the maximal attainable value $k = N/2$ for N even. States attaining this bound are known as *absolutely maximally entangled states* [38] and their existence is open in general. Such states are remarkably important in quantum information theory; for example, they are equivalent to pure state threshold quantum secret sharing scheme [39]. As we mentioned, for qubit states the unique solutions of this kind correspond to 1-uniform states of two qubits (Bell states) and 3-uniform states of six qubits where, curiously, a 2-uniform state of four qubits does not exist. Interestingly, property (i) of the above proposition provides us the existence of an infinite set of absolutely maximally entangled states.

Corollary 2. For every $d \geq 3$ odd there are $(d + 1)/2$ -uniform states of $d + 1$ qudits with d levels.

Also, Properties (ii) and (iii) lead us to the existence of 3-uniform states of six qudits with four levels. In Table III we resume the existence of $N/2$ -uniform states of N qudits with d levels. The existence of the 3-uniform state of six ququarts shown in Table III is based on the irredundant orthogonal array $OA(64, 6, 4, 3)$ [33].

A. k -uniform states and QECC

Quantum error-correcting codes (QECCs) theory deals with the problem of encoding quantum states into qudits such that a small number of errors can be detected, measured, and efficiently corrected. QECCs are denoted as $((N, K, D))_d$, where N is the length of the code, K is the dimension of the encoding state, D is the Hamming minimum distance, and d is the levels number of the qudit system. An introduction to QECCs can be found in the recent book of Lidar and Brun [40]. The standard notation used here for QECCs (double

parentheses) is in order to avoid confusion with classical codes. A code having a minimum distance D makes it possible to correct an arbitrary number of errors affecting up to $(D - 1)/2$ qudits. It is known that a $((N, K, D))_2$ QECC exists when the quantum Gilbert-Varshamov bound [41] is satisfied:

$$\sum_{j=0}^{D-1} 3^j \binom{N}{j} \leq 2^N / K. \tag{27}$$

Note that this inequality seems to be closely related to the Rao bounds given in Eq. (10). However, Rao bounds cannot be used to predict the existence of OAs. Very interestingly, $((N, 1, k + 1))_d$ QECCs are one-to-one connected to k -uniform states of N qubits. Thus, following Eq. (27), we have

$$\sum_{j=0}^k 3^j \binom{N}{j} \leq 2^{N-1}. \tag{28}$$

From this equation we show that k -uniform states of N qubits exist for every $k \in \mathbb{N}$ if N is sufficiently large. However, this inequality does not allow us to find the minimal number N for which k -uniform states of N qubits do exist. For instance, (28) predicts the existence of 3-uniform states for $N \geq 14$ qubits and we know that such class of states exists for six-qubit systems. Another important inequality for QECCs is the quantum Singleton bound [42]:

$$N - \log K \geq 2(D - 1). \tag{29}$$

From here we immediately get the upper bound $k \leq N/2$ for any k -uniform state of N qudits. The codes achieving this bound are called *maximal distance separable* (MDS) and their existence is open in general. Interestingly, any violation of this bound would allow us to get perfect copies of quantum states, which is forbidden by the *no-cloning theorem*.

We have shown above that QECCs are partially related to OAs through k -uniform states. Furthermore, classical error-correcting codes (CECCs) are one-to-one connected to OAs (see Sec. 4.3 of [31]):

$$(N, K, D)_d \Leftrightarrow OA(K, N, d, k) \text{ for some } k \geq 1, \tag{30}$$

where single parentheses are used to denote CECCs. For classical codes $(N, K, D)_d$ we also have a version of the Singleton bound:

$$N \leq d^{K-D+1}. \tag{31}$$

The codes achieving this bound are the classical MDS codes. An interesting result arises here (see Theorem 4.21, p. 79 in [31]).

Proposition 3. Classical MDS codes are OAs of index unity.

We have shown, therefore, that CECCs are also useful to construct k -uniform states. In particular, MDS-CECCs are connected with k -uniform states if $N = d^{K-D+1}$. The above proposition also establishes the following connection between classical and quantum error-correction codes.

Proposition 4. An MDS-CECC $(N, K, D')_d$ is also a QECC $(N, 1, D)_d$, for any K, D' whenever $D - 1 \leq N/2$.

The proof is straightforward from the above discussions. Although classical and quantum error-correction codes have been previously related [43], our results show that some

CECCs are useful to construct k -uniform states and that an arrangement can define both a CECC and a QECC.

B. Graph representation of k -uniform states

Graph theory is useful to represent a special kind of multipartite maximally entangled pure states known as *graph states* [44], construct QECCs associated with graphs [45,46] or to define ensembles of random states with interaction between subsystems specified to a graph [47]. Here we show that OAs theory provides a natural graph representation of k -uniform states which is not related to the graph states. Observe that the graph states correspond to quantum systems composed of subsystems with the same number of levels. On the other hand, the graph representation applied here is suitable for any number of multilevel qudit systems. For simplicity, we concentrate in this paper on homogeneous subsystems and postpone the more general case of heterogeneous subsystems with different numbers of levels for a subsequent publication. In the following lines we assume basic definitions and properties of graph theory. Further details can be found in the book [48] of Bondy and Murty.

An orthogonal array $OA(r, N, d, k)$ corresponds to a naturally associated graph composed of $d^k + d^{N-k}$ vertices and r edges. The vertices define two regular polygons \mathcal{P}_A and \mathcal{P}_B , where \mathcal{P}_A is inscribed into \mathcal{P}_B , as shown in Fig. 3.

The number of vertices of \mathcal{P}_A and \mathcal{P}_B is d^k and d^{N-k} , respectively. This graph is a sort of graphical representation of a bipartition of the system AB , where \mathcal{P}_A and \mathcal{P}_B are associated with the subsystems A and B . One has $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\text{Dim}(\mathcal{H}_A) = \#\mathcal{P}_A = d^k$ and $\text{Dim}(\mathcal{H}_B) = \#\mathcal{P}_B = d^{N-k}$. Note that $\mathcal{P}_A = \mathcal{P}_B$ if and only if the state is absolutely maximally entangled (i.e., $k = N/2$) [38]. As we have shown, for N -qubit states this is only possible for $N = 2$ and $N = 6$.

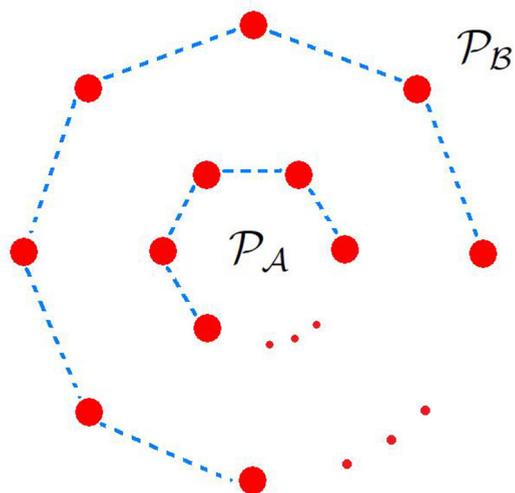


FIG. 3. (Color online) Graphic representation of k -uniform states. The regular polygons \mathcal{P}_A and \mathcal{P}_B represent the subsystems A and B , respectively. The number of vertices of the polygons depend on the number N of d -level subsystems and on the number $k \leq k_{\max} = \lfloor N/2 \rfloor$, whereas the edges characterize the degree of entanglement of the state.

In the following lines we describe our graph construction. Let us define $S = \{0, \dots, d - 1\}$ and consider the partition

$$S^{\otimes N} = S_A \oplus S_B, \quad (32)$$

such that an OA and its associated pure state are given by

$$\text{OA} = \begin{pmatrix} s_A^0 & s_B^0 \\ s_A^1 & s_B^1 \\ \vdots & \vdots \\ s_A^{r-1} & s_B^{r-1} \end{pmatrix} \quad |\Phi\rangle = \sum_{i=0}^{r-1} |s_A^i, s_B^i\rangle, \quad (33)$$

where $s_A^i \in S_A$ and $s_B^i \in S_B$ for every $i = 0, \dots, r - 1$, whereas $\text{Dim}(S_A) = k$ and $\text{Dim}(S_B) = N - k$. We assume that every entry of $|\Phi\rangle$ is zero or one without taking the normalization into account. Therefore, a simple rule for constructing graphs arises.

(i) Associate the value $s_A^i \in S_A$ with every vertex of the polygon \mathcal{P}_A and $s_B^i \in S_B$ with every vertex of the polygon \mathcal{P}_B , for every $i = 0, \dots, r - 1$.

(ii) Connect the vertex s_A^i to s_B^i , for every $i = 0, \dots, r - 1$.

From here, we note the following interesting properties.

(1) Polygons \mathcal{P}_A and \mathcal{P}_B are connected by, at least, an edge for entangled states (i.e., if \mathcal{P}_A and \mathcal{P}_B are disconnected then the state is separable).

(2) Two vertices of the same polygon are never connected.

(3) If only one vertex of \mathcal{P}_A (or \mathcal{P}_B) is connected, then the qudit associated with the corresponding subsystem is not entangled to the rest.

(4) A quantum state is k -uniform if and only if the corresponding graph satisfies the following properties equivalent to (A) and (B):

(A') *Diagonality*. Every vertex of \mathcal{P}_B is connected, at most, to one edge.

(B') *Uniformity*. Every vertex of \mathcal{P}_A is connected to the same number of edges.

These conditions must hold for every bipartition of the state. In Fig. 4 we show some graphs associated with known pure states: the Bell state $|\Phi_2^+\rangle$ of (1), the GHZ state (2), and the three-qubit W state (4). Given that edges define the degree of entanglement of a state, one could speculate that the entanglement increases with the number of edges. However, it is not so, as some separable states correspond to the graphs with several edges. For example, the graph shown in Fig. 4(d) represents a separable state,

$$\begin{aligned} |\Phi_{\text{sep}}\rangle &= |100\rangle + |101\rangle + |110\rangle + |111\rangle \\ &= |1\rangle(|00\rangle + |01\rangle + |10\rangle + |11\rangle), \end{aligned} \quad (34)$$

where the first qubit is not entangled to the rest. This state reflects property (3) mentioned above. By construction, every graph uniquely determines a pure state. However, the graph representation depends on the bipartition considered, in general. Therefore, for constructing k -uniform states from graphs we should follow rules (A') and (B') above for every bipartition, which is a complicated task for a high number of qudits. In fact, there are $\binom{N}{k}$ graphs associated with every k -uniform state of N qudits. Interestingly, k -uniform states having entries from the set $\{0, 1\}$ have all its graphs *isomorphic*. We recall that two graphs G and H are isomorphic if there exist

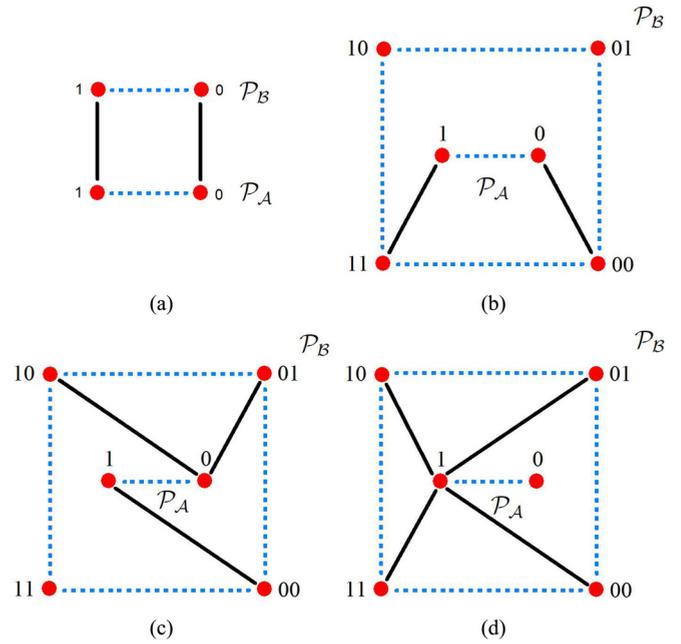


FIG. 4. (Color online) Graphic representation of certain states of systems composed of (a) two and (b)–(d) three qubits, where (a) is a Bell state, (b) is the GHZ state, (c) is the W state, and (d) is the separable state given in Eq. (34). Auxiliary dashed lines do not belong to the graph as they represent polygons \mathcal{P}_A and \mathcal{P}_B . States (a) and (b) represent 1-uniform states, while states (c) and (d) do not, as they do not satisfy the diagonality (A') and uniformity (B') conditions.

an isomorphism f such that every pair of vertices $u, v \in G$ are adjacent if and only if $f(u), f(v) \in H$ are adjacent. This property holds because the subset of k -uniform states considered is one-to-one connected with OAs. In particular, if a k -uniform state comes from an OA of index unity then all its graphs are identical. This property has a straightforward proof from the definition of OAs.

A graph G associated with a k -uniform state has a simple representation as a function of its adjacency matrix M^G . Here $M_{ij}^G = 1$ if the vertex $V_i \in \mathcal{P}_A$ is connected to the vertex $V_j \in \mathcal{P}_B$ and it is zero otherwise. As we have shown in Fig. 4, the indices i and j are labeled by d -inary numbers. Therefore, the state associated with a matrix M^G is given by

$$|\Phi\rangle = \sum_{i,j} M_{ij}^G |ij\rangle, \quad (35)$$

where i and j take all the values such that $V_i \in \mathcal{P}_A$ and $V_j \in \mathcal{P}_B$.

A final comment deserves our attention here. The graph representation used here allows us to distinguish completely disentangled states of d -level systems [see Fig. 4(d)] from genuine multipartite entangled states. Furthermore, the structure of the graphs associated with pure quantum states suggests to divide them into following classes:

- (i) fully separable;
- (ii) partially entangled;
- (iii) genuinely entangled;
- (iv) k -uniform.

In the last case, the strength k is also encoded in the graph. We encourage to the reader to verify that the rules (A') and (B') are satisfied for any GHZ state of N qudits.

C. Quantum gate allowing one to generate a k -uniform state

We describe here how to generate k -uniform states by applying quantum gates to a selected initial *blank* state:

$$|\eta_0\rangle = |0\rangle^{\otimes N}. \quad (36)$$

A general recipe to generate a k -uniform states reads as follows.

(1) Apply the local operation $H^{\otimes(k)} \otimes \mathbb{I}^{\otimes(N-k)}$, where H is the single-qubit Hadamard gate and expand the terms involving the first k qudits. That is, create the state

$$|\eta_1\rangle = \left(\sum_{s_1, \dots, s_k=0}^d |s_1, \dots, s_k\rangle \right) |0\rangle^{\otimes(N-k)}, \quad (37)$$

with $r = d^k$ terms.

(2) Apply a unitary transformation satisfying the restriction

$$U(|i\rangle \otimes |0\rangle^{\otimes(N-k)}) = |i\rangle \otimes \sum_{j=0}^{\lambda-1} |f_j(i)\rangle \quad (38)$$

for every i such that $V_i \in \mathcal{P}_A$. A pair $\{i, f_j(i)\}$ denotes two connected vertices $V_i \in \mathcal{P}_A$ and $V_{f_j(i)} \in \mathcal{P}_B$, where $j = 0, \dots, \lambda - 1$. Here we require λ functions f because every vertex of the polygon \mathcal{P}_A has exactly λ connections to a vertex of \mathcal{P}_B , where λ is the index of the OA associated with the k -uniform state. Note that U is well defined because the inner product between vectors in the domain is preserved in the image. Additionally, this operator clones partial information stored in orthogonal states, so it does not contain a universal cloning machine for any pair of qudits. In general, U can be always decomposed as simple nonlocal unitaries. For instance, to prepare the GHZ state of N qubits we use a generalized control-NOT gate, with a single control qubit and $N - 1$ target qubits. That is,

$$U = \mathbb{1}_2^{\otimes(N-1)} \oplus \sigma_x^{\otimes(N-1)}. \quad (39)$$

Finally, step (1) of our recipe has only an illustrative meaning, in order to have a smooth connection with our scheme. Thus, it can be removed in practice because it only involves local operations.

D. Mutually unbiased bases and k -uniform states

The construction of mutually unbiased bases (MUBs) has become an intriguing problem in pure mathematics and it has some important applications to quantum tomography. MUBs are naturally connected with other interesting hard problems: affine planes, mutually orthogonal Latin squares, and complex Hadamard matrices [49]. Here our intention is to connect MUBs with k -uniform states. Specifically, we relate *single* multipartite quantum states with maximal sets of MUBs. Two orthonormal bases $\{|\varphi_s\rangle\}$ and $\{|\phi_t\rangle\}$ defined on a d -dimensional

Hilbert space are *mutually unbiased* if

$$|\langle \varphi_s | \phi_t \rangle|^2 = \frac{1}{d}, \quad (40)$$

for every $s, t = 0, \dots, d - 1$. A maximal set of $(d + 1)$ MUBs exists in every prime-power dimension [50], and in spite of many efforts its existence remains open for every composite dimension. In order to connect MUBs with k -uniform states, we consider a reduced version of Theorem 8.43 (see p. 192, [31]).

Theorem 5 (Hedayat). An $OA(d^2, d + 1, d, 2)$ exists if and only if a projective plane of order d exists.

On one hand, the OAs considered in this theorem have index unity (leading us to k -uniform states). On the other hand, it is conjectured that the existence of a projective plane of order d determines a maximal set of $(d + 1)$ MUBs in dimension d [51]. A confirmation of this result together with Theorems 2 and 5 would imply the following interesting result: If a 2-uniform state of $d + 1$ qudits having d^2 terms exists, then a maximal set of $d + 1$ MUBs exist in dimension d . Note that 2-uniform states of seven subsystems of six levels each composed of 36 terms do not exist because of projective planes of order 6 do not exist. A similar conclusion can be done from the nonexistence of a projective plane of order 10.

In the subsequent section we present illustrative examples of our construction of k -uniform states.

V. EXAMPLES OF k -UNIFORM MULTIPARTITE STATES

To show our construction in action we present in this section some simple examples of k -uniform states connected to OAs. Working with quantum states we use the standard notation, but for brevity we work with not normalized pure states. Let us start with the simplest OA:

$$OA(2,2,2,1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \quad (41)$$

In a combinatorial context we deal with a 2×2 OA of $d = 2$ levels and strength $k = 1$. As defined in Sec. III, the strength denotes the number of columns that should be considered in order to have $\lambda = 1$ times repeated a sequence of $k = 1$ symbols along the rows. In the physical context, $OA(2,2,2,1)$ represents a quantum superposition of qubits containing two terms, each representing a separable pure state of two qubits, $d = 2$, which is a 1-uniform state. The strength k of the array determines the class of the k -uniform state. Thus, the state connected with the array (41) is the maximally entangled Bell state,

$$|\Psi_2^+\rangle = |01\rangle + |10\rangle. \quad (42)$$

In this section we use OAs listed in the resource of Sloane [33]. In this catalog, we find the following OA:

$$OA(4,3,2,2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}. \quad (43)$$

It cannot be used to construct a 2-uniform state of $N = 3$ qubits since the inequality $k \leq N/2$ does not hold (see Theorem 2). However, an $OA(r, N, d, k)$ is always an $OA(r, N, d, k - 1)$, so

we can consider $OA(4,3,2,2)$ as $OA(4,3,2,1)$, which is an IrOA. Then inequality $k \leq N/2$ holds and the following 1-uniform, balanced state of three qubits arises:

$$|\Phi_3\rangle = |000\rangle + |011\rangle + |101\rangle + |110\rangle. \quad (44)$$

Acting on this state with a tensor product of Hadamard matrices $H_2 \otimes H_2 \otimes H_2$, one can verify that it is locally equivalent to the standard GHZ state (2). Interestingly, this three-qubit GHZ state defines the OA,

$$OA(2,3,2,1) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}. \quad (45)$$

Orthogonal arrays (43) and (45) do not have anything in common in the theory OAs. Thus, a natural question arises: Which OAs are related by the fact that the corresponding quantum states are equivalent with respect to local unitary transformations?

Let us now consider the array

$$OA(8,4,2,3) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix}, \quad (46)$$

which is also an $OA(8,4,2,1)$. This array leads to the following 1-uniform state of four qubits,

$$|\Phi_4\rangle = |0000\rangle + |0011\rangle + |0101\rangle + |0110\rangle \\ + |1001\rangle + |1010\rangle + |1100\rangle + |1111\rangle, \quad (47)$$

well known in the literature [3]. It is worth adding that a generalization of the above state with complex coefficients expressed by the third root of unity is known in the literature as the L state, for which certain measures of quantum entanglement achieve its maximum [25]. Recently, it was proved [52] that this state yields the maximum of the absolute value of the hyperdeterminant for four qubits.

Observe that the existence of an $OA(4,4,2,2)$ would immediately lead us to a 2-uniform state of four qubits. However, the Rao bound tells us that an $OA(r,4,2,2)$ must satisfy $r \geq 5$. Note that for arrays with $\lambda \geq 2$, the diagonality condition (B) is not always satisfied, so not every such array allows us to construct the corresponding k -uniform state. For example, from the array with the index $\lambda = 2$,

$$OA(8,5,2,2) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}, \quad (48)$$

we obtain the following 1-uniform state of five qubits:

$$|\Psi_5\rangle = |00000\rangle + |10011\rangle + |01010\rangle + |00101\rangle \\ + |11001\rangle + |10110\rangle + |01111\rangle + |11100\rangle. \quad (49)$$

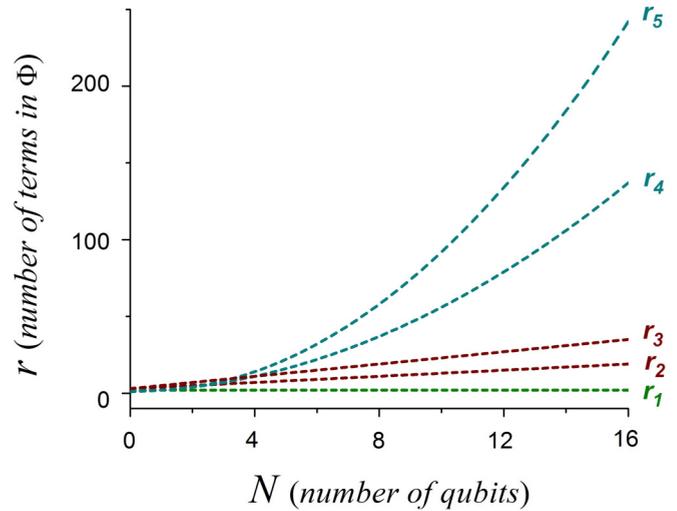


FIG. 5. (Color online) Lower bound for the number of terms r_k of a k -uniform state of N qubits imposed by the Rao bounds (10): $r_1 = 2$, $r_2 = N + 1$, $r_3 = 2N$, $r_4 = N^2/2 + N/2 + 1$, and $r_5 = N^2 - N + 2$.

It is easy to verify that this state is not 2-uniform. The only nonmaximally mixed reductions to two qubits are ρ_{24} and ρ_{35} . Curiously, we could not find a 2-uniform state of five qubits having $\{0, 1\}$ entries and it is likely they do not exist. In Appendix C we derive a 2-uniform state of five qubits having $\{0, \pm 1\}$ entries. Remarkably, for any number of qubits $N > 5$ it is possible to find a 2-uniform state having $\{0, 1\}$ entries, as we show in Theorem 7 (see also Appendix A).

In the case of a higher number of qubits the number of terms r in a k -uniform state becomes large according to the Rao bound (10). In Fig. 5 we show how the minimal number of terms r required to produce a k -uniform pure state of N qubits depends on the number N . The following remarks arise.

(1) For 1-uniform states we have $r \geq 2$, as there exist a family of generalized GHZ states of N qubits, which consist of two terms. From a combinatorial point of view, the arrays corresponding to multiqubit GHZ states were discussed by Hedayat *et al.* [31] (Example 1.4, p. 3) as a trivial way to construct $OA(r, N, 2, 1)$.

(2) Rao bounds tells us that 1-uniform states are the only kind of k -uniform states allowing a constant number of terms, $r_1 = 2$.

(3) The minimum number of terms r_2 and r_3 required to have a 2-uniform or 3-uniform state, respectively, is a growing *linear function* of the number of qubits N , with slopes 1 and 2.

(4) The minimum number of terms r required to have a k -uniform state for $k > 3$ is a growing *nonlinear function* of the number of qubits N . For example, in the case of $k = 4$ we have $r_4 \geq 1 + N/2 + N^2/2$ and $r_5 \geq 2 - N + N^2$.

VI. CONSTRUCTION OF 2-UNIFORM STATES FOR EVERY $N > 4$

Generalized GHZ states provide a simple example of 1-uniform states for an arbitrary number of N qubits. Construction of 2-uniform states usually requires computing simulations and numerical approximations; see [12] and

references therein. In fact, 2-uniform states are explicitly known for five and six qubits only [20].

In this section we solve the problem of constructing a kind of 2-uniform states for an arbitrary number of N qubits. As is often the case [53], the theory of quantum information can benefit from combinatorics and Hadamard matrices. A Hadamard matrix is up to a prefactor an orthogonal matrix with entries ± 1 . Some OAs of strength 2 are connected with the famous Hadamard conjecture; see Theorem 7.5, p. 148 [31].

Theorem 6 (Hedayat). Orthogonal arrays $\text{OA}(4\lambda, 4\lambda - 1, 2, 2)$ exists if and only if there exists a Hadamard matrix of order 4λ .

This theorem combined with a partial classification of Hadamard matrices allows us to find a kind of 2-uniform states of an arbitrarily large number of qubits. Consider a Hadamard matrix of order κ in normalized form so that the entries appearing in the first row and the first column are equal to unity.

Making use of Theorem 6 we realize that Hadamard matrices could be useful to construct 2-uniform states of $\kappa - 1$ (or less) qubits. That is, we will be able to find an OA which satisfies the diagonality property (B). In what follows we derive for which values of N we can construct a 2-uniform state from a given Hadamard matrix of size κ . On one hand, every entry of the first row of the OA contains ones. Thus, we must avoid having another row containing $N - 2$ ones since, according to (B) we need to have different rows for every subset of $N - 2$ columns. Given that we work with normalized Hadamard matrices, zeros and ones appear at most $\kappa/2 - 1$ times in every row, excluding the first one. Therefore, if the number of qubits to be reduced ($N - 2$) is greater than $\kappa/2 - 1$ it is not possible to obtain a second row of ones. Additionally, if the first row is not repeated then the remaining rows are not repeated either, as in OAs every repeated pair of rows is repeated the same number of times. Thus, the $N - 2$ columns to be reduced are different if and only if $\kappa/2 - 1 < N - 2$ and thus property (B) holds. Given that $N \leq \kappa - 1$ we arrive at the following statement.

Theorem 7. A Hadamard matrix of order κ allows one to find 2-uniform states of N qubits having entries from the set $\{0, 1\}$ if and only if

$$\kappa/2 + 2 \leq N \leq \kappa - 1. \quad (50)$$

In Appendix A we exemplify this theorem by constructing 2-uniform states for 6 to 15 qubits.

Combinatorial Theorem 6 can be thus adopted to the theory of multipartite entanglement. Together with the above explanations it leads us to the following result.

Theorem 8. The problem of classifying 2-uniform states for qubits contains the Hadamard conjecture.

Precisely, the 2-uniform states of $N = 4\lambda - 1$ qubits connected with the Hadamard conjecture are those having $r = 4\lambda$ terms ($\lambda \in \mathbb{N}$). The Hadamard conjecture states that a $\kappa \times \kappa$ Hadamard matrix exists for $\kappa = 2$ and for every $\kappa = 4n$, $n \in \mathbb{N}$. It has been open since 1893 and it represents one of the most important problems in combinatorics. Not knowing whether the Hadamard conjecture holds, we show that Theorem 7 allows us to construct another kind of 2-uniform states for an arbitrary number of $N > 4$ qubits.

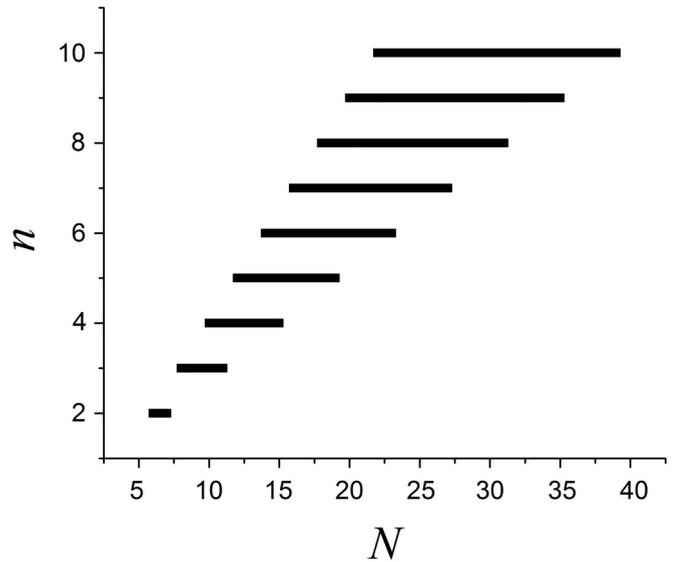


FIG. 6. Overlap of the bounds $3 \times 2^{n-2} + 2 \leq N \leq 3 \times 2^{n-1}$ [see Eq. (51)].

First, we note that $\kappa = 2^n$ ($n \in \mathbb{N}$) is not a solution for every N . That is, the intervals given in Eq. (50) have gaps for some values of N . Specifically, the gaps are given for $N = 2^n$ and $N = 2^n + 1$ (for every $n > 2$). In order to cover these gaps it is enough to consider $\kappa = 12 \times 2^{n-3}$, where the new interval is given by

$$3 \times 2^{n-2} + 2 \leq N \leq 3 \times 2^{n-1}. \quad (51)$$

Figure 6 shows that many pairs (n, N) satisfying Eq. (51) exist for every $N > 5$. As Hadamard matrices of the size $\kappa = 12 \times 2^{n-3}$ can be written explicitly as a tensor product ($H_{12} \otimes H_2^{\otimes n-3}$), we are able to construct 2-uniform states of N qubits for $N > 5$. Thus, the following statement holds.

Theorem 9. For every $N > 5$ there exists 2-uniform quantum states of N qubits and they can be constructed from known Hadamard matrices. Moreover, they have entries from the set $\{0, 1\}$.

The proof of the above theorem is constructive.

(i) Consider a $\kappa \times \kappa$ Hadamard matrix H in the normalized form, so that its first column and first row consist of +1 only. If $N \neq 2^n$ and $N \neq 2^n + 1$ for every $n > 2$ then consider $\kappa = 2^v$, where v is chosen such that $\kappa - N$ is positive and, for simplicity, is as small as possible. Otherwise, consider $\kappa = 12 \times 2^{v-3}$ in the same way.

(ii) Discard the first column of H , keep any subset of N columns and discard the rest to obtain a rectangular matrix with κ rows and N columns.

(iii) Replace all elements equal to -1 s with 0. This leads us to an OA according to Theorem 6 and property (v) from Sec. IV.

(iv) Every row of the OA obtained should be put in kets, summed, and normalized to produce the desired 2-uniform state of N qubits, which completes the construction. ■

Note that for $\lambda = 1$ in Theorem 6 we have an $\text{OA}(4, 3, 2, 2)$ which cannot be used to construct a 2-uniform state of three qubits as condition $k \leq N/2$ is not satisfied. Instead, we have

shown in Eq. (43) that this array allows one to build a 1-uniform state of three qubits.

As final comments we realize that the complete classification of 2-uniform states is currently out of reach, as it includes the Hadamard conjecture. Moreover, already for $\kappa = 32$ there are more than 13×10^6 nonequivalent Hadamard matrices [54], so the number of ways to generate 2-uniform states for 31 qubits is huge. We do not know how many of these states are locally equivalent.

Another observation is worth making. We realize that a k -uniform state constructed from IrOA according to the scheme shown in Fig. 2 allows us to generate entire classes of k -uniform states. For instance, taking

$$|\Phi_3\rangle = |000\rangle + |011\rangle + |101\rangle + |110\rangle, \quad (52)$$

we generate the three-parameter class of states

$$|\Phi_3\rangle(\alpha_1, \alpha_2, \alpha_3) = |000\rangle + e^{i\alpha_1}|011\rangle + e^{i\alpha_2}|101\rangle + e^{i\alpha_3}|110\rangle, \quad (53)$$

as $\alpha_1, \alpha_2, \alpha_3 \in [0, 2\pi)$. In general, from any k -uniform state of N qudits generated by an OA of index $\lambda = 1$, i.e.,

$$|\Phi\rangle = |s_1^1, s_2^1, \dots, s_N^1\rangle + |s_1^2, s_2^2, \dots, s_N^2\rangle + \dots + |s_1^r, s_2^r, \dots, s_N^r\rangle, \quad (54)$$

we can generate the entire $d^N - 1$ -dimensional orbit of k -uniform quantum states,

$$|\Phi\rangle = |s_1^1, s_2^1, \dots, s_N^1\rangle + e^{i\alpha_1} |s_1^2, s_2^2, \dots, s_N^2\rangle + \dots + e^{i\alpha_{d^N-1}} |s_1^r, s_2^r, \dots, s_N^r\rangle. \quad (55)$$

Therefore, our approach is not only useful to generate a k -uniform state but also to generate entire orbits of maximally entangled states. It is easy to see that an orbit of k -uniform states can be considered as a starting point for a search for a single $(k + 1)$ -uniform state.

VII. CONCLUDING REMARKS

We presented a combinatorial tool to systematically generate genuine multipartite entangled pure states of N qudits: the orthogonal arrays. Our construction of k -uniform states works if an OA of strength k is irredundant (i.e., after removing from the array any k columns all remaining rows are distinct). This class of OAs differs, in general, from the supersimple OAs, recently discussed in [55]. However, these classes coincide when $k + 1 = N - 1$, which is only possible for 1-uniform states of three subsystems and 2-uniform states of four subsystems of qudits.

Our approach allows us to find new entangled states of several qudits and to establish their properties. In Sec. IV we derived two conditions sufficient to construct k -uniform states and realized that the first one, uniformity condition (A), is fulfilled by any OA. Having at hand any OA one needs to verify that it is irredundant, so the diagonality condition (B) is satisfied. In practice, it is easy to find examples of arrays for which both conditions are satisfied, so they lead to k -uniform states.

Results obtained in this paper include the following.

(1) Every OA of index unity OA(d^k, N, d, k) allows us to generate a k -uniform state of N qudits of d levels if and

only if $k \leq N/2$; see Theorem 2. In Appendix B we present several examples of k -uniform states for subsystems with $d > 2$ levels each.

(2) We demonstrated that 2-uniform states of N qubits exist for every $N > 4$; see Sec. VI. An explicit construction of these states is presented in Theorem 9, which involves known Hadamard matrices. Such states are listed in Appendix A for $N = 6, \dots, 15$.

(3) We revisited the connection between k -uniform states and quantum error correction codes. In particular, we find that minimal distance separable *classical* codes are equivalent to some k -uniform states (see Proposition 3) and also established connections between classical and quantum codes; see Proposition 4.

(4) A certain graph representation of k -uniform states is proposed. It arises from OAs and it allows us to identify the k -uniform states; see Sec. IV B. Additionally, the key conditions of uniformity (A) and diagonality (B) stated above receive a simple graphical interpretation.

(5) The existence of a *single* 2-uniform state of $d + 1$ qudits is connected with the existence of a maximal set of $d + 1$ mutually unbiased bases in prime-power dimensions (see Theorem 5). If Saniga *et al.*'s conjecture [51] is true, then our connection is valid for every dimension d .

(6) We proved that the existence of a particular kind of 2-uniform states, those considering $N = \kappa - 1$ qubits and having κ terms are equivalent to construct $\kappa \times \kappa$ Hadamard matrices. Consequently, for $\kappa \neq 4n$ they do not exist and for $\kappa = 4n$ this is equivalent to the Hadamard conjecture, for every $n \in \mathbb{N}$ (see Theorem 8). This suggests that the complete classification of 2-uniform states of N -qubit states could be temporarily out of reach.

(7) An entire orbit of maximally entangled states can be constructed from every k -uniform state generated from an OA; see Sec. VI.

Additionally, in Appendix C we explain how to construct k -uniform states from OAs when the diagonality assumption (B) is not satisfied. In Fig. 7 we show the most important connections made in this work.

Let us conclude this work by listing some open issues.

(A) Check if every k -uniform state is equivalent under stochastic local operations and classical communication to a

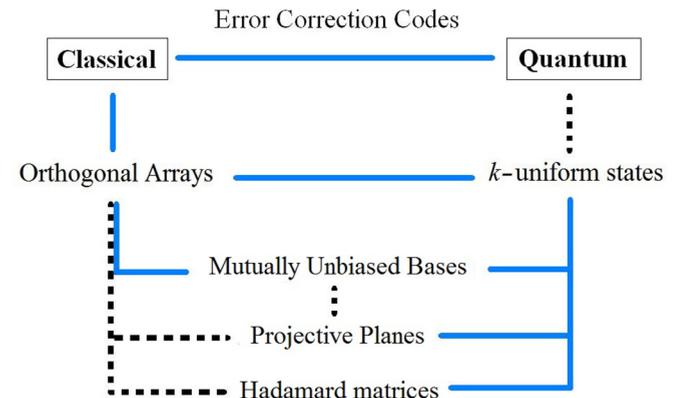


FIG. 7. (Color online) Relationship between k -uniform states, QECCs, CECCs, and other relevant mathematical notions. Dashed (black) lines represent known connections, while solid (blue) lines denote the relations discussed in this paper.

state generated from an OA. This property holds for systems containing $N = 2, 3, 4$ qubits.

(B) Find for what N there are 3-uniform states of N qubits and 2-uniform states of N qutrits. Also, provide an explicit construction of 3-uniform states of qutrits for high values of N .

(C) Solve the existence problem of 3-uniform states of seven qubits.

(D) Find how the maximal value k_{\max} , for which k_{\max} -uniform states of N -qubit exist, depends on N . Analyze the dependence $k_{\max}(N)$ for qutrits and higher, d -dimensional systems.

(E) Investigate the existence of the approximate (ϵ, k) -uniform states, for which all reductions are maximally mixed up to ϵ corrections. Analyze generic random pure states of N qutrits, distributed with respect to a Haar measure [56], to find for which values of both parameters they are (ϵ, k) -uniform.

(F) Extend the method developed here to heterogeneous systems composed of subsystems with different numbers of levels, for instance, the qubit-qutrit systems.

(G) Extend Theorem 9 to the case of 2-uniform states of qutrits by using Butson-type complex Hadamard matrices [57].

(H) Find which OAs are *locally equivalent*, $\text{OA}(r, N, d, k) \sim_{\text{loc}} \text{OA}(r', N, d, k')$, in a sense that they lead to locally equivalent quantum states. Note that neither the number r of the terms in each state nor the strength k needs to be preserved by the local equivalence relation.

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APPENDIX A: EXPLICIT CONSTRUCTION OF 2-UNIFORM STATES

In this section, we use known Hadamard matrices to exemplify the construction of 2-uniform states defined in Theorem 9. Let us consider the 8×8 Hadamard matrix H_8

and its associated OA obtained from the last seven columns of the matrix,

$$H_8 = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix}, \quad (\text{A1})$$

$$\rightarrow \text{OA} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}. \quad (\text{A2})$$

Applying Theorem 9 we immediately get the following 2-uniform state of seven qubits:

$$|\Phi_7\rangle = |1111111\rangle + |0101010\rangle + |1001100\rangle \\ + |0011001\rangle + |1110000\rangle + |0100101\rangle \\ + |1000011\rangle + |0010110\rangle. \quad (\text{A3})$$

This state can be called a *simplex state*, since its eight terms, interpreted as coordinates of points in \mathbb{R}^7 , form a regular 7-simplex after changing 0s to -1 s [58]. Notice that the scalar product between any two such vectors is constant, so the collection of these vectors forms equiangular lines. The same property holds also for the states $|\Phi_{2^m-1}\rangle$, e.g., the state $|\Phi_{15}\rangle$ given in (A9).

Removing the first column of the OA given in Eq. (A1) we have the following 2-uniform state of six qubits:

$$|\Phi_6\rangle = |111111\rangle + |101010\rangle + |001100\rangle + |011001\rangle \\ + |110000\rangle + |100101\rangle + |000011\rangle + |010110\rangle. \quad (\text{A4})$$

Note that we attained the lower bound given in Eq. (51) for the 2-uniform states that can be directly obtained from H_8 . However, a 2-uniform state of five qubits can be obtained from H_8 in a different way (see Appendix C). In the following examples we systematically eliminate the first (left) qubit. From H_{12} [59] and Theorem 9 we find the following 2-uniform states:

$$|\Phi_{11}\rangle = |0000000000\rangle + |10100011101\rangle + |11010001110\rangle + |01101000111\rangle \\ + |10110100011\rangle + |11011010001\rangle + |11101101000\rangle + |01110110100\rangle \\ + |00111011010\rangle + |00011101101\rangle + |10001110110\rangle + |01000111011\rangle, \quad (\text{A5})$$

$$|\Phi_{10}\rangle = |0000000000\rangle + |0100011101\rangle + |1010001110\rangle + |1101000111\rangle \\ + |0110100011\rangle + |1011010001\rangle + |1101101000\rangle + |1110110100\rangle \\ + |0111011010\rangle + |0011101101\rangle + |0001110110\rangle + |1000111011\rangle, \quad (\text{A6})$$

$$\begin{aligned}
|\Phi_9\rangle = & |00000000\rangle + |100011101\rangle + |010001110\rangle + |101000111\rangle \\
& + |110100011\rangle + |011010001\rangle + |101101000\rangle + |110110100\rangle \\
& + |111011010\rangle + |011101101\rangle + |001110110\rangle + |000111011\rangle,
\end{aligned} \tag{A7}$$

and

$$\begin{aligned}
|\Phi_8\rangle = & |00000000\rangle + |00011101\rangle + |10001110\rangle + |01000111\rangle \\
& + |10100011\rangle + |11010001\rangle + |01101000\rangle + |10110100\rangle \\
& + |11011010\rangle + |11101101\rangle + |01110110\rangle + |00111011\rangle.
\end{aligned} \tag{A8}$$

Now, from the Hadamard matrix $H_{16} = H_2^{\otimes 4}$ we construct the following 2-uniform states:

$$\begin{aligned}
|\Phi_{15}\rangle = & |11111111111111\rangle + |01010101010101\rangle + |100110011001100\rangle + |001100110011001\rangle \\
& + |111000011110000\rangle + |010010110100101\rangle + |100001111000011\rangle + |001011010010110\rangle \\
& + |11111100000000\rangle + |010101001010101\rangle + |100110000110011\rangle + |001100101100110\rangle \\
& + |111000000001111\rangle + |010010101011010\rangle + |100001100111100\rangle + |001011001101001\rangle,
\end{aligned} \tag{A9}$$

$$\begin{aligned}
|\Phi_{14}\rangle = & |11111111111111\rangle + |10101010101010\rangle + |00110011001100\rangle + |01100110011001\rangle \\
& + |11000011110000\rangle + |10010110100101\rangle + |00001111000011\rangle + |01011010010110\rangle \\
& + |11111100000000\rangle + |10101001010101\rangle + |00110000110011\rangle + |01100101100110\rangle \\
& + |11000000001111\rangle + |10010101011010\rangle + |00001100111100\rangle + |01011001101001\rangle,
\end{aligned} \tag{A10}$$

$$\begin{aligned}
|\Phi_{13}\rangle = & |11111111111111\rangle + |01010101010101\rangle + |0110011001100\rangle + |1100110011001\rangle \\
& + |1000011110000\rangle + |0010110100101\rangle + |0001111000011\rangle + |1011010010110\rangle \\
& + |1111100000000\rangle + |0101001010101\rangle + |0110000110011\rangle + |1100101100110\rangle \\
& + |1000000001111\rangle + |0010101011010\rangle + |0001100111100\rangle + |1011001101001\rangle,
\end{aligned} \tag{A11}$$

$$\begin{aligned}
|\Phi_{12}\rangle = & |11111111111111\rangle + |10101010101010\rangle + |110011001100\rangle + |100110011001\rangle \\
& + |000011110000\rangle + |010110100101\rangle + |001111000011\rangle + |011010010110\rangle \\
& + |111100000000\rangle + |101001010101\rangle + |110000110011\rangle + |100101100110\rangle \\
& + |000000001111\rangle + |010101011010\rangle + |001100111100\rangle + |011001101001\rangle,
\end{aligned} \tag{A12}$$

$$\begin{aligned}
|\tilde{\Phi}_{11}\rangle = & |111111111111\rangle + |010101010101\rangle + |10011001100\rangle + |00110011001\rangle \\
& + |00011110000\rangle + |10110100101\rangle + |01111000011\rangle + |11010010110\rangle \\
& + |11100000000\rangle + |01001010101\rangle + |10000110011\rangle + |00101100110\rangle \\
& + |00000001111\rangle + |10101011010\rangle + |01100111100\rangle + |11001101001\rangle,
\end{aligned} \tag{A13}$$

$$\begin{aligned}
|\tilde{\Phi}_{10}\rangle = & |111111111111\rangle + |101010101010\rangle + |0011001100\rangle + |0110011001\rangle \\
& + |0011110000\rangle + |0110100101\rangle + |1111000011\rangle + |1010010110\rangle \\
& + |1100000000\rangle + |1001010101\rangle + |0000110011\rangle + |0101100110\rangle \\
& + |0000001111\rangle + |0101011010\rangle + |1100111100\rangle + |1001101001\rangle,
\end{aligned} \tag{A14}$$

$$\begin{aligned}
|\tilde{\Phi}_9\rangle = & |111111111111\rangle + |010101010101\rangle + |011001100\rangle + |110011001\rangle \\
& + |011110000\rangle + |110100101\rangle + |111000011\rangle + |010010110\rangle \\
& + |100000000\rangle + |001010101\rangle + |000110011\rangle + |101100110\rangle \\
& + |000001111\rangle + |101011010\rangle + |100111100\rangle + |001101001\rangle,
\end{aligned} \tag{A15}$$

and

$$\begin{aligned}
|\tilde{\Phi}_8\rangle = & |1111111111\rangle + |1010101010\rangle + |11001100\rangle + |10011001\rangle + |11110000\rangle + |10100101\rangle + |11000011\rangle + |10010110\rangle \\
& + |00000000\rangle + |01010101\rangle + |00110011\rangle + |01100110\rangle + |00001111\rangle + |01011010\rangle + |00111100\rangle + |01101001\rangle.
\end{aligned}$$

In the case of $N = 8, 9, 10, 11$ we marked the states with a tilde in order to emphasize that the states obtained from the Hadamard matrix H_{16} need not coincide with these constructed from H_8 . Furthermore, for $N = 16$ there exists five different

classes of nonequivalent Hadamard matrices [59], and all of them allow us to generate 2-uniform states for $N \leq 15$ qubits. Finally, we remark that each of the 2-uniform states of k qubits generated from our method leads us to the orbits of 2-uniform states with $2^k - 1$ real parameters constructed according to Eq. (55).

APPENDIX B: k -UNIFORM STATES FOR d -LEVEL SUBSYSTEMS

In this section we construct k -uniform states from OAs for qudits having $d > 2$. Let us present some k -uniform

states straightforwardly constructed from OAs of index $\lambda = 1$. The following OAs have been taken from the catalog of Sloane [33]. From OA(9,4,3,2) we have the 2-uniform of four qutrits:

$$|\Psi_3^4\rangle = |0000\rangle + |0112\rangle + |0221\rangle + |1011\rangle + |1120\rangle \\ + |1202\rangle + |2022\rangle + |2101\rangle + |2210\rangle. \quad (\text{B1})$$

Interestingly, a 2-uniform state of four qutrits exists, whereas there is no 2-uniform state of four qubits. From OA(16,5,4,2) we get the 2-uniform state of five ququarts, related to the Reed-Solomon code [33] of length 5,

$$|\Psi_4^5\rangle = |00000\rangle + |01111\rangle + |02222\rangle + |03333\rangle + |10123\rangle + |11032\rangle + |12301\rangle + |13210\rangle \\ + |20231\rangle + |21320\rangle + |22013\rangle + |23102\rangle + |30312\rangle + |31203\rangle + |32130\rangle + |33021\rangle. \quad (\text{B2})$$

From OA(64,6,4,3) we obtain a 3-uniform state of six ququarts:

$$|\Psi_4^6\rangle = |000000\rangle + |001111\rangle + |002222\rangle + |003333\rangle + |010123\rangle + |011032\rangle + |012301\rangle + |013210\rangle + |020231\rangle + |021320\rangle \\ + |022013\rangle + |023102\rangle + |030312\rangle + |031203\rangle + |032130\rangle + |033021\rangle + |100132\rangle + |101023\rangle + |102310\rangle \\ + |103201\rangle + |110011\rangle + |111100\rangle + |112233\rangle + |113322\rangle + |120303\rangle + |121212\rangle + |122121\rangle + |123030\rangle \\ + |130220\rangle + |131331\rangle + |132002\rangle + |133113\rangle + |200213\rangle + |201302\rangle + |202031\rangle + |203120\rangle + |210330\rangle \\ + |211221\rangle + |212112\rangle + |213003\rangle + |220022\rangle + |221133\rangle + |222200\rangle + |223311\rangle + |230101\rangle + |231010\rangle \\ + |232323\rangle + |233232\rangle + |300321\rangle + |301230\rangle + |302103\rangle + |303012\rangle + |310202\rangle + |311313\rangle + |312020\rangle \\ + |313131\rangle + |320110\rangle + |321001\rangle + |322332\rangle + |323223\rangle + |330033\rangle + |331122\rangle + |332211\rangle + |333300\rangle. \quad (\text{B3})$$

From OA(25,6,5,2) we get a 2-uniform state consisting of six five-level systems:

$$|\Psi_5^6\rangle = |000000\rangle + |011234\rangle + |022341\rangle + |033412\rangle + |044123\rangle + |101111\rangle + |112403\rangle + |124032\rangle + |130324\rangle \\ + |143240\rangle + |202222\rangle + |214310\rangle + |223104\rangle + |231043\rangle + |240431\rangle + |303333\rangle + |310142\rangle + |321420\rangle \\ + |334201\rangle + |342014\rangle + |404444\rangle + |413021\rangle + |420213\rangle + |432130\rangle + |441302\rangle. \quad (\text{B4})$$

Interestingly, the 1-uniform state of seven qubits obtained from OA(64,7,2,6),

$$|\Psi_2^7\rangle = |0000000\rangle + |1000001\rangle + |0100001\rangle + |0010001\rangle + |0001001\rangle + |0000101\rangle + |0000011\rangle + |1100000\rangle \\ + |1010000\rangle + |1001000\rangle + |1000100\rangle + |1000010\rangle + |0110000\rangle + |0101000\rangle + |0100100\rangle + |0100010\rangle \\ + |0011000\rangle + |0010100\rangle + |0010010\rangle + |0001100\rangle + |0001010\rangle + |0000110\rangle + |1110001\rangle + |1101001\rangle \\ + |1100101\rangle + |1100011\rangle + |1011001\rangle + |1010101\rangle + |1010011\rangle + |1001101\rangle + |1001011\rangle + |1000111\rangle \\ + |0111001\rangle + |0110101\rangle + |0110011\rangle + |0101101\rangle + |0101011\rangle + |0100111\rangle + |0011101\rangle + |0011011\rangle \\ + |0010111\rangle + |0001111\rangle + |1111000\rangle + |1110100\rangle + |1110010\rangle + |1101100\rangle + |1101010\rangle + |1100110\rangle \\ + |1011100\rangle + |1011010\rangle + |1010110\rangle + |1001110\rangle + |0111100\rangle + |0111010\rangle + |0110110\rangle + |0101110\rangle \\ + |0011110\rangle + |1111101\rangle + |1111011\rangle + |1110111\rangle + |1101111\rangle + |1011111\rangle + |0111111\rangle + |1111110\rangle, \quad (\text{B5})$$

is symmetric with respect to permutations and it has all its two-, three-, four-, five-, and six-qubit reductions identically equal. That is, every subsystem of $N' < 7$ qubits contains the same physical information. Moreover, these reductions have rank 2 independently of the size of the reduction. Thus, the above seven-qubit state is a genuine multipartite entangled state but the entanglement of its parties is weak. The strong regularity observed in the reductions of this state is due to the high strength $k = 6$ of the orthogonal array OA(64,7,2,6). That is, every subset of six columns of the OA contains the 2^6 possible combinations of the symbols along the 64 rows.

Note that most of the 1-uniform states presented are symmetric under permutation of qubits. It has been proven that a k -uniform state is symmetric if $k \leq 1$ [20]. Therefore, k -uniform states cannot be symmetric for $k \geq 2$. This means that neither 2-uniform the states constructed from Table IV nor those appearing in Appendix B are symmetric. Interestingly, this property can be also applied to OAs of index unity through Theorem 2: An OA of index unity, N runs, and two levels is invariant under permutation of columns if it has strength $k = 1$. Indeed, given that $r = \lambda 2^k$, an OA of index $\lambda = 1$ and $k = 1$ have $r = 2$ runs. Thus, the only symmetric

TABLE IV. Orthogonal arrays of index $\lambda = 1$ allowing for a direct construction of 2-uniform states.

Orthogonal array	r	N	d
OA(9,4,3,2)	9	4	3
OA(16,5,4,2)	16	5	4
OA(25,6,5,2)	25	6	5
OA(49,8,7,2)	49	8	7
OA(64,9,8,2)	64	9	8
OA(81,10,9,2)	81	10	9
OA(100,4,10,2)	100	4	10
OA(121,12,11,2)	121	12	11
OA(144,7,12,2)	144	7	12
OA(169,14,13,2)	169	14	13
OA(256,17,16,2)	256	17	16
OA(289,18,17,2)	289	18	17

OA of index unity are one-to-one connected to the GHZ states.

Table IV provides a list of some OAs of index $\lambda = 1$ [33]. As we have shown along the work, they can be used for a direct construction of 2-uniform quantum states of N subsystems, with d levels each: Note that in most cases the values of the parameter r in the above OA is a power of a prime, as these arrays were obtained with Theorem 1. However, the cases of OA(64,6,4,3), OA(100,4,10,2), and OA(144,7,12,2) have been probably constructed by computing simulations. We remark here that there is no simple way to generate OAs of index $\lambda = 1$ for non-prime-power values d . However, they might exist. Construction of OAs of strength $k = 3$ and index $\lambda = 1$ is implied by the following theorem [60].

Theorem 10 (Kounias). For any even d every orthogonal array OA($d^3, d + 1, d, 3$) can be extended to OA($d^3, d + 2, d, 3$).

This theorem can be restated in the context of multipartite quantum states.

Proposition 5. For any even d every 3-uniform state of $d + 1$ qudits generated by OA($d^3, d + 1, d, 3$) can be extended to a 3-uniform state of $d + 2$ qudits.

Unfortunately, in current libraries of OAs we could not find a suitable example to illustrate this proposition.

APPENDIX C: CONSTRUCTION OF k -UNIFORM STATES WITH NONPOSITIVE TERMS

Some orthogonal arrays OA(r, N, d, k) satisfying $k \leq N/2$ do not lead us to a k -uniform state of positive terms. This is because some rows within sets of $N - k$ columns are repeated and, consequently, the reductions to k qudits are not diagonal. However, sometimes it is possible to introduce *minus signs* in some terms of the state such that it becomes a k -uniform state. Here we carefully illustrate this procedure for a 2-uniform state of five qubits obtained from the Hadamard matrix H_8 . Despite that this *patch* can be applied in many cases we cannot prove the existence of k -uniform states of N qudits for every k and N such that $k \leq N/2$. For example, we could find neither 3-uniform states of seven qubits nor 4-uniform states of eight qubits. As already mentioned, the problem relies on the violation of the diagonality condition (B).

Let us consider the last five columns of the Hadamard matrix H_8 given in Eq. (A1). Here, $N = 5$ and $\kappa = 8$, so inequality (51) does not hold. Let us define the state

$$|\Phi'_5\rangle = |11111\rangle + |01010\rangle + |01100\rangle + |11001\rangle + |10000\rangle \\ + |00101\rangle + |00011\rangle + |10110\rangle, \quad (C1)$$

which is not a 2-uniform state. In order to find a 2-uniform state we start by adding some unimodular complex numbers in the terms of $|\Phi'_5\rangle$:

$$|\Phi''_5\rangle(\vec{\alpha}) = (-1)^{\alpha_0}|11111\rangle + (-1)^{\alpha_1}|01010\rangle + (-1)^{\alpha_2}|01100\rangle \\ + (-1)^{\alpha_3}|11001\rangle + (-1)^{\alpha_4}|10000\rangle \\ + (-1)^{\alpha_5}|00101\rangle + (-1)^{\alpha_6}|00011\rangle \\ + (-1)^{\alpha_7}|10110\rangle. \quad (C2)$$

So, from every nonmaximally mixed reduction we obtain a linear system of equations for the phases $\{\alpha_r\}$. If this system is compatible, then there exists states of the form given in Eq. (C2) such that all its reductions are diagonal. Additionally, as we consider OAs the reductions are maximally mixed. On the other hand, the linear system of equations is incompatible when $k > N/2$.

For $|\Phi'_5\rangle$ the only nonmaximally mixed reductions are $\rho_{34} = \text{Tr}_{125}(\rho_{12345})$ and $\rho_{25} = \text{Tr}_{134}(\rho_{12345})$. In the three tables

4	5	1	2	3	3	4	1	2	5	2	5	1	3	4
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1	0	0	1	0	0	1	0	1	0	1	0	0	0	1
0	0	0	1	1	1	0	0	1	0	1	0	0	1	0
0	1	1	1	0	0	0	1	1	1	1	1	1	0	0
0	0	1	0	0	0	0	1	0	0	0	0	1	0	0
0	1	0	0	1	1	0	0	0	1	0	1	0	1	0
1	1	0	0	0	0	1	0	0	1	0	1	0	0	1
1	0	1	0	1	1	1	1	0	0	0	0	1	1	1

we can see the OA version of the process to obtain the reductions $\rho_{45} = \text{Tr}_{123}(\rho_{12345})$, $\rho_{34} = \text{Tr}_{125}(\rho_{12345})$, and $\rho_{25} = \text{Tr}_{134}(\rho_{12345})$. The first table corresponds to the analysis of the reduction $\rho_{45} = \mathbb{I}/4$. Note that the binary numbers determined by columns **4** and **5** cover exactly 2 times the four pairs $\{(0,0); (0,1); (1,0); (1,1)\}$. This happens in every pair of columns because the OA has strength $k = 2$. Also, the frequency of appearance of every pair is 2 given that the OA has index $\lambda = 2$. Therefore, the diagonal of the reduced state ρ_{45} is uniform. Additionally, this reduction is diagonal because the rows determined by columns **1**, **2**, and **3** are different. In the second table we analyze the reduction ρ_{34} . Here columns **3** and **4** determine a uniform diagonal of the reduction. However, columns **1**, **2**, and **5** are repeated, so ρ_{34} is not a diagonal matrix. So is the case for the third table corresponding to ρ_{25} . These are the reasons why $|\Phi'_5\rangle$ is not a 2-uniform state. In what follows, we explain how to modify this construction allowing for negative terms in $|\Phi'_5\rangle$ to satisfy the 2-uniform property. In higher dimensions, possibly complex phases are required as k -uniform states containing all terms with real weights may not exist. Basically, the nondiagonal terms in the reductions arise from an even number of terms. If we consider OAs, all these contributions are positive. Therefore, one can change

the sign of some terms of the state $|\Phi\rangle$ in order to achieve that all the reduced states are diagonal. Let us concentrate in the middle array in the above table. Making use of the definition

$$\begin{aligned}\rho_A &= \text{Tr}_B(\rho_{AB}) \\ &= \sum_{i,j,k,l} a_{i,j,k,l} \text{Tr}_B(|i\rangle_A \langle j|_{BA} |k\rangle_B \langle l|) \\ &= \sum_{i,j,k,l} a_{i,j,k,l} |l\rangle \langle j|_A |i\rangle \langle k| \quad (\text{C3})\end{aligned}$$

and the data from the table above one can construct the following reduction:

$$\rho_{34} = \begin{array}{cc} & \begin{array}{cccc} 00 & 01 & 10 & 11 \end{array} \\ \begin{array}{c} 00 \\ 01 \\ 10 \\ 11 \end{array} & \begin{pmatrix} 1/4 & 0 & 0 & 1/4 \\ 0 & 1/4 & 1/4 & 0 \\ 0 & 1/4 & 1/4 & 0 \\ 1/4 & 0 & 0 & 1/4 \end{pmatrix}. \quad (\text{C4}) \end{array}$$

Therefore, assuming that

$$\begin{aligned}|\Phi_5''\rangle &= |11111\rangle + (-1)^{\alpha_1}|01010\rangle + (-1)^{\alpha_2}|01100\rangle \\ &+ (-1)^{\alpha_3}|11001\rangle + (-1)^{\alpha_4}|10000\rangle \\ &+ (-1)^{\alpha_5}|00101\rangle + (-1)^{\alpha_6}|00011\rangle \\ &+ (-1)^{\alpha_7}|10110\rangle, \quad (\text{C5})\end{aligned}$$

we have the following linear system of equations:

$$\begin{aligned}\alpha_0 + \alpha_3 + \alpha_4 + \alpha_7 + 1 &= 0, \\ \alpha_1 + \alpha_2 + \alpha_5 + \alpha_6 + 1 &= 0.\end{aligned} \quad (\text{C6})$$

Interestingly, the same linear equations appear for the reduction ρ_{25} . For example, a solution arises from considering that the only non-null phases are $\alpha_6 = \alpha_7 = 1$. Consequently, the state

$$\begin{aligned}|\Phi_5''\rangle &= |11111\rangle + |01010\rangle + |01100\rangle + |11001\rangle \\ &+ |10000\rangle + |00101\rangle - |00011\rangle - |10110\rangle \quad (\text{C7})\end{aligned}$$

is a 2-uniform state of $N = 5$ qubits consisting of eight terms, of which two are negative.

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