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Joint numerical ranges, quantum maps, and joint numerical shadows

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ABSTRACT

We associate with a k -tuple of hermitian $N \times N$ matrices a probability measure on \mathbb{R}^k supported on their joint numerical range: The joint numerical shadow of these matrices. When $k = 2$ we recover the numerical range and the numerical shadow of the complex matrix corresponding to a pair of hermitian matrices. We apply this material to the theory of quantum information. Thus, we show that quantum maps on the set of quantum states defined by Kraus operators satisfying the identity resolution assumption shrink joint numerical ranges.

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1. Introduction

Let \mathcal{H}_N be the N -dimensional Hilbert space with the scalar product $\langle \varphi | \psi \rangle$, and let A be an operator on \mathcal{H}_N . Its *numerical range* (also called the *field of values*) $W(A) \subset \mathbb{C}$ is the set of numbers $z = \langle \psi | A | \psi \rangle$, where $\langle \psi | \psi \rangle = 1$ [1,2]. A crucial fact, conjectured by O. Toeplitz in 1918 and proved by F. Hausdorff

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in 1919 [3,4] is that $W(A)$ is convex. See [5] for an exposition of [3,4] from a modern perspective. If A is a normal operator,¹ then $W(A)$ is the convex hull of the spectrum of A , hence a convex polygon. For generic A the boundary $\partial W(A)$ is smooth [6]. If $N = 2$, $\partial W(A)$ is a (possibly degenerate) ellipse [3,7]. See [8] for $W(A)$ when $N = 3$.

The present work is motivated by the applications of numerical range and related material in quantum mechanics, especially in the theory of quantum information [9–11]. We will freely use the relevant physics terminology in what follows. The set Ω_N of *pure quantum states*² is naturally isomorphic to the complex projective space $\mathbb{C}P^{N-1}$. *Quantum states*³ are the operators $\rho \geq 0$ on \mathcal{H}_N satisfying $\text{Tr} \rho = 1$. The set \mathcal{Q}_N of quantum states is convex, and $\Omega_N \subset \mathcal{Q}_N$ is the set of extremal points of \mathcal{Q}_N . The elements of $\mathcal{Q}_N \setminus \Omega_N$ are *mixed quantum states*.

For applications to quantum information it is crucial that $W(A)$ is a plane projection of Ω_N [12]. If $N = 2$ (i.e., the *one qubit case*), $\Omega_2 = \mathbb{C}P^1$ is the *Bloch sphere* and $\mathcal{Q}_2 \subset \mathbb{R}^3$ is the *Bloch ball*. The fact that a projection of the two-sphere is a (possibly degenerate) ellipse underlies the well known claims about numerical ranges of 2×2 matrices [7]. The *numerical shadow* of an operator A on \mathcal{H}_N is a probability distribution $P_A(z)$ supported on $W(A)$ [15,13,14]. Let $|\psi\rangle$ be distributed on the unit sphere $S(\mathcal{H}_N)$ according to the *Haar measure*. Then $P_A(z)$ is the probability density of $z = \langle \psi | A | \psi \rangle$. In the one qubit case, $P_A(z)$ is the density of the plane shadow of the Bloch sphere under a light beam [15].

Let A_1, \dots, A_m be hermitian operators on \mathcal{H}_N . Their *joint numerical range* (JNR) [17] is the set in \mathbb{R}^m defined by

$$W(A_1, \dots, A_m) = \{(\langle \psi | A_1 | \psi \rangle, \dots, \langle \psi | A_m | \psi \rangle) : \langle \psi | \psi \rangle = 1\}. \quad (1)$$

Since $W(A_1, A_2) = W(A_1 + iA_2)$, Eq. (1) generalizes the notion of the numerical range of a complex operator. We note that $W(A_1, \dots, A_m)$ is not necessarily convex for $m > 2$ [17]. For instance, the JNR of *Pauli matrices* is the Bloch sphere; see Section 2.

We study the above notions and the relationships between them and quantum maps. Theorem 1 in Section 2 shows that the JNR of an m -tuple of hermitian operators is a linear projection of the set of pure quantum states to \mathbb{R}^m . In Section 4 we associate with any m -tuple of hermitian operators a probability measure on \mathbb{R}^m . This is the *joint numerical shadow* of the m -tuple of hermitian operators; it extends the concept of numerical shadow of a complex operator [15,13,14,16]. We point out a few basic properties of joint numerical shadows, deferring a deeper study to a separate publication.

Let Φ be the *quantum map* on the set of quantum states defined by a k -tuple of *Kraus operators* satisfying the *identity resolution* (13). In Section 3 we study the effects of Φ on m -tuples of hermitian operators. As Corollaries 3 and 4 show, Φ shrinks the joint numerical ranges.

Throughout the paper we emphasize the applications of our material in the theory of quantum information. Examples 1–6 illustrate these applications. For instance, Examples 3 and 4 show the shrinking of numerical ranges under particularly well known quantum maps in the qubit and the qutrit cases.

2. Joint numerical ranges

Let \mathcal{L}_N (resp. $\mathcal{M}_N, \mathcal{P}_N, \mathcal{Q}_N$) denote the space of all (resp. hermitian, positive definite, positive definite with trace 1) linear operators on \mathcal{H}_N . Let $\Omega_N \subset \mathcal{Q}_N$ be the set of rank one projections. As vector spaces, $\mathcal{L}_N = \mathbb{C}^{N^2}$, $\mathcal{M}_N = \mathbb{R}^{N^2}$. The scalar product

$$(A, B) = \text{tr}(AB^*), \quad (2)$$

makes \mathcal{L}_N (resp. $\mathcal{M}_N \subset \mathcal{L}_N$) a Hilbert space (resp. Euclidean space). The set $\mathcal{P}_N \subset \mathcal{M}_N$ is a closed convex cone. Its interior consists of strictly positive operators, $\rho > 0$, and its apex is the zero operator.

¹ Recall that an operator A is normal if $AA^* = A^*A$.

² These are the hermitian projections of \mathcal{H}_N onto one-dimensional subspaces $\mathbb{C}|\psi\rangle$.

³ Also called *density matrices*.

The set \mathcal{Q}_N is the intersection of \mathcal{P}_N and the hyperplane $\{\text{Tr}\rho = 1\}$. It is a bounded convex region (i.e., has nonempty interior) in the $(N^2 - 1)$ -dimensional affine space, and $\Omega_N \subset \mathcal{Q}_N$ is the set of its extremal points. For a unit vector $\psi \in \mathcal{H}_N$ we set $\rho_\psi = |\psi\rangle\langle\psi| \in \Omega_N$. The manifold $\mathbb{C}P^{N-1}$ is the quotient of the unit sphere $S(\mathcal{H}_N)$ by the natural linear action of the unit circle $S^1 \subset \mathbb{C}$. The map $\psi \mapsto \rho_\psi$ yields an isomorphism of $\mathbb{C}P^{N-1}$ and Ω_N . In what follows we will identify $\mathbb{C}P^{N-1}$ and Ω_N via this isomorphism.

Let $A_1, \dots, A_m \in \mathcal{M}_N$ be arbitrary. The mapping from $\mathbb{C}P^{N-1}$ to \mathbb{R}^m given by

$$\text{jnr}_{(A_1, \dots, A_m)} : |\psi\rangle \mapsto (\langle\psi|A_1|\psi\rangle, \dots, \langle\psi|A_m|\psi\rangle). \tag{3}$$

is the *joint numerical range map*. The range $W(A_1, \dots, A_m)$ of this map is the *joint numerical range* (JNR) of operators A_1, \dots, A_m . By the isomorphism $\mathbb{C}P^{N-1} \simeq \Omega_N$, we have $\text{jnr}_{(A_1, \dots, A_m)} : \Omega_N \rightarrow \mathbb{R}^m$.

We will recall a few basic notions in affine geometry. By a vector space we will mean a finite dimensional real vector space. Let V be a vector space. A set $H \subset V$ is an *affine subspace* if there is a linear subspace $H_0 \subset V$ and a vector $h_0 \in H$ such that $H = H_0 + h_0$. Let $G \subset U, H \subset V$ be affine subspaces. Let $G_0 \subset U, H_0 \subset V$ be the corresponding linear subspaces. A map $Af : G \rightarrow H$ is an *affine isomorphism* if there is a linear isomorphism $Af_0 : G_0 \rightarrow H_0$ and vectors $h_0 \in H, g_0 \in G$ so that

$$Af(g) = Af_0(g - g_0) + h_0.$$

Definition 1. Let U, V be vector spaces, and let $X \subset U, Y \subset V$ be arbitrary sets. They are *affinely isomorphic* if there exist affine subspaces $H \subset U, G \subset V$ such that $X \subset H, Y \subset G$, and an affine isomorphism $A : G \rightarrow H$ such that $A(X) = Y$. The induced map $A : X \rightarrow Y$ is an affine isomorphism of X onto Y .

Note that linear isomorphism of sets are the special cases in the above setting when the subspaces and the maps in question are, actually, linear. We will not distinguish between the linear and affine situations in what follows.

We will now expose a general topic in linear algebra. Let U, V be vector spaces. We assume that U is a Euclidean space with the scalar product (\cdot, \cdot) . Let $m \geq 1$. With any nonzero vectors $u_1, \dots, u_m \in U$ and $v_1, \dots, v_m \in V$ we associate a linear operator $L : U \rightarrow V$ by

$$L(u) = \sum_{i=1}^m (u, u_i) v_i. \tag{4}$$

Let $\mathcal{A} \subset U$ (resp. $\mathcal{B} \subset V$) be the subspace spanned by u_1, \dots, u_m (resp. v_1, \dots, v_m). We will need a simple lemma about the operator in Eq. (4).⁴

Lemma 1

1. Let $\dim \mathcal{A} = p \leq m$. Assume without loss of generality that the vectors u_1, \dots, u_p span \mathcal{A} . Then there are vectors $\tilde{v}_1, \dots, \tilde{v}_p \in \mathcal{B}$ such that the linear operator in Eq. (4) satisfies

$$L(u) = \sum_{i=1}^p (u, u_i) \tilde{v}_i. \tag{5}$$

2. Let $\dim \mathcal{B} = q \leq m$. Assume without loss of generality that the vectors v_1, \dots, v_q span \mathcal{B} . Then there are vectors $\tilde{u}_1, \dots, \tilde{u}_q \in \mathcal{A}$ such that the linear operator in Eq. (4) satisfies

$$L(u) = \sum_{i=1}^q (u, \tilde{u}_i) v_i. \tag{6}$$

3. If $p = m$ (resp. $q = m$) then $\tilde{v}_i = v_i$ (resp. $\tilde{u}_i = u_i$) for $1 \leq i \leq m$.

⁴ We leave the straightforward proof to the reader.

The proposition below is immediate from Lemma 1.

Proposition 1. *Let the setting be as in Lemma 1, and let $Pr_{\mathcal{A}} : U \rightarrow \mathcal{A}$ be the orthogonal projection. Then (i) There is a unique linear operator $M : \mathcal{A} \rightarrow \mathcal{B}$ such that $L = M \circ Pr_{\mathcal{A}}$; (ii) If $\dim \mathcal{A} = m$ (resp. $\dim \mathcal{B} = m$) then M is surjective (resp. injective); (iii) If $\dim \mathcal{A} = \dim \mathcal{B} = m$ then M is an isomorphism.*

Let $G, H \subset U$ be affine subspaces in a vector space. We say that they are *parallel* if $H = G + u_0$ for some $u_0 \in U$.

Corollary 1. *Let U, V be vector spaces, and let $L : U \rightarrow V$ be as in Eq. (4). Let $\mathcal{A} \subset U, \mathcal{B} \subset V$ be the subspaces spanned by the vectors u_1, \dots, u_m and v_1, \dots, v_m respectively. Let $M : \mathcal{A} \rightarrow \mathcal{B}$ be the operator from Proposition 1.*

Let $G \subset U$ be an affine subspace containing a subspace parallel to \mathcal{A} . Let $\Gamma \subset G$ be an arbitrary set. If M is injective, then $Pr_{\mathcal{A}}(\Gamma)$ and $L(\Gamma)$ are affinely isomorphic.

Proof. Let $u_0 \in U$ be such that $\mathcal{A} = G + u_0$. Let $\mathcal{B}_0 \subset \mathcal{B}$ be the range of M . Thus, $M : \mathcal{A} \rightarrow \mathcal{B}_0$ is a linear isomorphism. By Proposition 1, we have

$$L(\Gamma + u_0) = M Pr_{\mathcal{A}}(\Gamma + u_0) = M(\Gamma + u_0) \subset \mathcal{B}_0.$$

The mapping $u \mapsto L(u + u_0)$ induces an affine isomorphism of G and \mathcal{B}_0 . \square

We will now apply the above material to joint numerical ranges.

Theorem 1. *Let $A_1, \dots, A_m \in \mathcal{M}_N$ be traceless, linearly independent hermitian operators. Let $\mathcal{A} \subset \mathcal{M}_N$ be the subspace spanned by them. Then (i) The joint numerical range of A_1, \dots, A_m is affinely isomorphic to $Pr_{\mathcal{A}}(\Omega_N)$; (ii) The convex hull of the joint numerical range of A_1, \dots, A_m is affinely isomorphic to $Pr_{\mathcal{A}}(\mathcal{Q}_N)$.*

Proof. Let $U = \mathcal{M}_N, V = \mathbb{R}^m$. Let v_1, \dots, v_m be the standard basis in \mathbb{R}^m . Then the map jnr_{A_1, \dots, A_m} has the form Eq. (4) with $u_i = A_i$. By Proposition 1, the map $M : \mathcal{A} \rightarrow \mathbb{R}^m$ is a linear isomorphism. Set $G = \{X \in \mathcal{M}_N : \text{tr}(X) = 1\}$. Then $G \subset \mathcal{M}_N$ is an affine hyperplane containing the affine subspace $\mathcal{A} + \mathbf{1}_N$. Claim (i) now follows from Corollary 1.

The convex hulls of affinely isomorphic sets are affinely isomorphic. Hence, claim (i) implies claim (ii). \square

Corollary 2. *Let the setting be as in Theorem 1, with $m = N^2 - 1$. Then $W(A_1, \dots, A_m)$ is affinely isomorphic to Ω_N and $\text{conv}[W(A_1, \dots, A_m)]$ is affinely isomorphic to \mathcal{Q}_N .*

Proof. This is a special case of Theorem 1. In this case the affine hyperplane G is parallel to \mathcal{A} . Hence $Pr_{\mathcal{A}}$ induces an affine isomorphism of Ω_N and $Pr_{\mathcal{A}}(\Omega_N)$. \square

Example 1. Let $N = 2$. The Pauli matrices

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{7}$$

form an orthogonal basis in the space of traceless Hermitian 2×2 matrices. By Corollary 2, $W(\sigma_1, \sigma_2, \sigma_3) \subset \mathbb{R}^3$ is the unit sphere. Indeed, this also holds by an elementary computation. Let $\psi = [z_1, z_2]$. Denote by $F : \mathbb{C}^2 \rightarrow \mathbb{R}^3$ the joint numerical range map; (see Eq. (3)). Then

$$F(\psi) = [z_1\bar{z}_2 + z_2\bar{z}_1, i(z_1\bar{z}_2 - z_2\bar{z}_1), z_1\bar{z}_1 + z_2\bar{z}_2],$$

and the sum of squared coordinates is $(|z_1|^2 + |z_2|^2)^2$.

Let ρ be a 2×2 density matrix. Its Bloch vector $(\tau_1, \tau_2, \tau_3) = \tau \in \mathbf{R}^3$ is defined by the decomposition

$$\rho = \frac{1}{2} \mathbf{1}_2 + \sum_{j=1}^3 \tau_j \sigma_j \tag{8}$$

where $\tau_j = \text{tr}(\sigma_j \rho) / 2$. Since $\rho = \rho^*$, we have $\tau \in \mathbf{R}^3$, and since $\rho \geq 0$

$$\sqrt{\tau_1^2 + \tau_2^2 + \tau_3^2} = \|\tau\| \leq 1/2.$$

The equality $\|\tau\| = 1/2$ holds if and only if ρ is a pure state. Thus, the map $\text{jnr}_{(\sigma_1, \sigma_2, \sigma_3)}$ yields an isomorphism of \mathcal{Q}_2 (resp. Ω_2) and the Bloch ball (resp. Bloch sphere).

Example 1 generalizes to $N > 2$ as follows. Denote by $d = N^2 - 1$ the dimension of the space of traceless $N \times N$ hermitian matrices with the scalar product (2). Let $\lambda_1, \dots, \lambda_d$ be an orthogonal, but not necessarily an orthonormal, basis. For instance, for $N = 3$ the Gell-Mann matrices $(\lambda_1, \dots, \lambda_8)$ satisfy the orthogonality relations $\text{tr}(\lambda_j \lambda_k) = 2\delta_{jk}$ [19]. Let ρ be a $N \times N$ state. The counterpart of Eq. (8) is

$$\rho = \frac{1}{N} \mathbf{1}_N + \sum_{j=1}^d \tau_j \lambda_j. \tag{9}$$

The generalized Bloch vector $\tau = \tau(\rho)$ has d components $\tau_j = \text{Tr}(\rho \lambda_j) / \text{Tr}(\lambda_j^2)$. If the basis $\lambda_1, \dots, \lambda_d$ is orthonormal and $\rho_\psi = |\psi\rangle\langle\psi|$ is a pure state, then $\tau_\psi = \tau(\rho_\psi)$ satisfies $\tau_j = \langle\psi|\lambda_j|\psi\rangle$. Hence $\tau_\psi \in W(\lambda_1, \dots, \lambda_d)$.

By Corollary 2, the convex hull $\text{conv}[W(\lambda_1, \dots, \lambda_d)]$ is affinely isomorphic to the set \mathcal{Q}_N of quantum states. By Theorem 1, if $\lambda_1, \dots, \lambda_m$ are linearly independent, then the joint numerical range $W(\lambda_1, \dots, \lambda_m)$ is affinely isomorphic to a projection of \mathcal{Q}_N to \mathbf{R}^m . Compare with the results in [17, 12].

3. Quantum maps

We recall that the set \mathcal{L}_N of operators on \mathcal{H}_N is a Hilbert space with the scalar product given by Eq. (2). Let \mathcal{LL}_N be the space of linear operators on \mathcal{L}_N . For $\Phi \in \mathcal{LL}_N$ we denote by $\Phi^* \in \mathcal{LL}_N$ its adjoint. For $X, Y \in \mathcal{L}_N$ we define $\Phi_{X|Y} \in \mathcal{LL}_N$ by

$$\Phi_{X|Y}(A) = XAY^*. \tag{10}$$

Let $X_1, \dots, X_k \in \mathcal{L}_N$ be arbitrary. We set

$$\Phi_{(X_1, \dots, X_k)} = \sum_{i=1}^k \Phi_{X_i|X_i}. \tag{11}$$

We will often use the notation $\Phi = \Phi_{(X_1, \dots, X_k)}$. In the physics literature the transformations of \mathcal{P}_N defined by

$$\rho_1 = \Phi(\rho) = \sum_{i=1}^k X_i \rho X_i^*. \tag{12}$$

are called *quantum maps*. They correspond to generalized quantum measurements with k possible outcomes [18]. Operators $X_i \in \mathcal{L}_N$ are the *measurement operators* or *Kraus operators*. Let $\Phi = \Phi_{(X_1, \dots, X_k)}$. Then $\Phi_{(X_1^*, \dots, X_k^*)} = \Phi^*$, the adjoint operator with respect to the scalar product (2). In the physics

literature the quantum map Φ^* is dual to Φ . The duality of quantum maps corresponds to the duality between the Schrödinger and the Heisenberg representations in quantum mechanics. In the former, the quantum states $\rho \in \mathcal{P}_N$ evolve via $\rho_1 = \Phi(\rho)$, while the observables $A \in \mathcal{M}_N$ do not. In the latter, the quantum states do not change and the observables evolve by $A \mapsto A_1 = \Phi^*(A)$.

We will now establish a few basic properties of quantum maps. The following lemma, whose proof is left to the reader, will be used in Proposition 1.

Lemma 2

1. The operators $\Phi_{X|Y}$ span the vector space $\mathcal{L}\mathcal{L}_N$.
2. Let $X_1, Y_1, X_2, Y_2 \in \mathcal{L}_N$. Then

$$\Phi_{X_2|Y_2} \Phi_{X_1|Y_1} = \Phi_{X_2X_1|Y_2Y_1}.$$

3. We have

$$(\Phi_{X|Y})^* = \Phi_{X^*|Y^*}.$$

Definition 2. We will denote by $\mathbf{1} \in \mathcal{L}_N$ the identity operator. Let $k \geq 1$. We say that the k -tuple of operators $X_1, \dots, X_k \in \mathcal{L}_N$ is an identity resolution if

$$\sum_{i=1}^k X_i X_i^* = \mathbf{1}. \tag{13}$$

The dual property

$$\sum_{i=1}^k X_i^* X_i = \mathbf{1}, \tag{14}$$

holds if and only if X_1^*, \dots, X_k^* is an identity resolution.

Proposition 1. Let $X_1, \dots, X_k \in \mathcal{L}_N$ be arbitrary, and let $\Phi = \Phi_{(X_1, \dots, X_k)} \in \mathcal{L}\mathcal{L}_N$. Then the following holds.

1. The quantum map $\Phi : \mathcal{L}_N \rightarrow \mathcal{L}_N$ preserves \mathcal{M}_N and \mathcal{P}_N .
2. The operators X_1, \dots, X_k satisfy Eq. (13) if and only if $\Phi : \mathcal{L}_N \rightarrow \mathcal{L}_N$ preserves $\mathbf{1}$.
3. The operators X_1, \dots, X_k satisfy Eq. (14) if and only if (i) the map Φ preserves the trace; (ii) the map Φ^* preserves $\mathbf{1}$.
4. If X_1, \dots, X_k satisfy Eq. (14), then Φ preserves \mathcal{Q}_N .

Proof. It suffices to prove (1) for operators $\Phi = \Phi_{X|X}$. The former property is immediate from $(XAX^*)^* = XA^*X^*$, and the latter from $\langle XAX^*v, v \rangle = \langle AX^*v, X^*v \rangle$. Claim (2) is immediate from the definition of Φ . We have

$$\text{Tr}(\Phi(A)) = (\Phi(A), \mathbf{1}) = (A, \Phi^*(\mathbf{1})).$$

Claim (3) now follows from Lemma 2. Claim (4) is immediate from (1) and (3). \square

Set $\Phi = \Phi_{(X_1, \dots, X_k)}$. We say that the quantum map Φ is unital (resp. trace preserving) if $\Phi(\mathbf{1}) = \mathbf{1}$ (resp. $\text{Tr}(\rho(\Phi)) = \text{Tr}\rho$). By Proposition 1, a quantum map is unital (resp. trace preserving) if and only if Eq. (13) (resp. Eq. (14)) is satisfied. Since

$$\text{Tr}(\Phi(\rho)A) = \text{Tr}\left(\sum_i X_i \rho X_i^* A\right) = \text{Tr}\left(\sum_i \rho X_i^* A X_i\right) = \text{Tr}(\rho \Phi^*(A)),$$

Φ is trace preserving (resp. unital) if and only if Φ^* is unital (resp. trace preserving). For $\psi \in \mathcal{H}_N$ we set $\rho_\psi = |\psi\rangle\langle\psi|$. Then $\rho_\psi \in \mathcal{P}_N$, and for any $a \in \mathbb{C}$ we have

$$\rho_{a\psi} = |a|^2 \rho_\psi. \tag{15}$$

In particular, $\rho_\psi = 0$ if and only if $\psi = 0$ and $\rho_\psi \in \mathcal{Q}_N$ if and only if $\|\psi\| = 1$.

Proposition 2. Let $X_1, \dots, X_k \in \mathcal{L}_N$, and let $\Phi = \Phi_{(X_1, \dots, X_k)}$. Let $\psi \in \mathcal{H}_N$ be any vector. For $1 \leq i \leq k$ such that $X_i \psi \neq 0$, set $\psi_i = X_i \psi / \|X_i \psi\|$.

Then

$$\Phi(\rho_\psi) = \sum_i \|X_i \psi\|^2 \rho_{\psi_i}. \tag{16}$$

Proof. Let $X, Y \in \mathcal{L}_N$ and $v \in \mathcal{H}_N$. Then

$$(\Phi_{X|Y}(\rho_\psi))v = X\langle\psi, Y^*v\rangle\psi = \langle Y\psi, v\rangle X\psi.$$

In particular, we have

$$\Phi_{X|X}(\rho_\psi) = \rho_{X\psi}. \tag{17}$$

The claim now follows from Eqs. (11) and (15). \square

Corollary 3. Let X_1, \dots, X_k satisfy Eq. (14) and set $\Phi = \Phi_{(X_1, \dots, X_k)}$. Let $\psi \in \mathcal{H}_N$ be a unit vector. Let $1 \leq m \leq k$ be the number of unit vectors ψ_i from Proposition 2. Then

$$\Phi(\rho_\psi) = \sum_{i=1}^m p_i \rho_{\psi_i},$$

where $p_1, \dots, p_m > 0$ and $\sum p_i = 1$.

Proof. Set $p_i = \|X_i \psi\|^2$. The claim follows from Eq. (16) and the identity

$$\sum_i \|X_i \psi\|^2 = \left\langle \left(\sum_{i=1}^k X_i^* X_i \right) \psi, \psi \right\rangle. \quad \square$$

Corollary 4. Let $X_1, \dots, X_k \in \mathcal{L}_N$, and set $\Phi = \Phi_{(X_1, \dots, X_k)}$. Suppose that X_1, \dots, X_k satisfy Eq. (13). Then

1. For any $A \in \mathcal{L}_N$ we have

$$W(\Phi(A)) \subset W(A); \tag{18}$$

2. For any $A_1, \dots, A_m \in \mathcal{M}_N$ we have

$$W(\Phi(A_1), \dots, \Phi(A_m)) \subset \text{conv}(W(A_1), \dots, W(A_m)); \tag{19}$$

3. If $W(A_1, \dots, A_m)$ is convex, then we have

$$W(\Phi(A_1), \dots, \Phi(A_m)) \subset W(A_1, \dots, A_m). \tag{20}$$

Proof. Let $A \in \mathcal{L}_N$ be arbitrary, and let $\psi \in \mathcal{H}_N$ be a unit vector. Then

$$\text{Tr}(\Phi(A)\rho_\psi) = \text{Tr}\left(\left(\sum_{i=1}^k X_i A X_i^*\right)\rho_\psi\right) = \text{Tr}\left(A\left(\sum_{i=1}^k X_i^* \rho_\psi X_i\right)\right).$$

Note that the operators X_1^*, \dots, X_k^* satisfy Eq. (14). By Proposition 2 and Corollary 4, there are unit vectors $\psi_j \in \mathcal{H}_N$ and probabilities p_j such that

$$\text{Tr}(\Phi(A)\rho_\psi) = \sum_j p_j \text{Tr}(A\rho_{\psi_j}) = \sum_j p_j \langle \psi_j | A | \psi_j \rangle. \tag{21}$$

Claim 2 now follows from Eq. (3). Claim 3 is a special case of Claim 2. Claim 3 and the Toeplitz–Hausdorff theorem yield Claim 1. \square

The examples below illustrate the relationship between quantum maps and numerical ranges, which is the subject of Corollary 4.

Example 2. Let $N = 2$. The *decaying channel* is the discrete dynamics $\Phi = \Phi_{X_1, X_2}$ corresponding to

the Kraus operators $X_1 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{bmatrix}$ and $X_2 = \begin{bmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{bmatrix}$, where $p \in (0, 1)$ is a free parameter.

Set $A^{(j)} = \Phi^{j-1}(A^{(1)})$, where $A^{(1)} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Note that $\text{tr}(A^{(1)}) = 0$. The set $W(A^{(1)})$ is the

disc of radius $1/2$ centered at the origin, i.e., at the *barycenter point* $\text{tr}(A^{(1)})/2$. We have $A^{(j)} = (1-p)^{(j-1)/2} A^{(1)}$, and $W(A^{(j)}) = (1-p)^{(j-1)/2} W(A^{(1)})$ is the disc of radius $\frac{1}{2}(1-p)^{(j-1)/2}$ with

the center at 0. For instance, $A^{(2)} = \Phi(A^{(1)}) = \begin{bmatrix} 0 & \sqrt{1-p} \\ 0 & 0 \end{bmatrix}$, and $A^{(3)} = \Phi^2(A^{(1)}) = \begin{bmatrix} 0 & 1-p \\ 0 & 0 \end{bmatrix}$.

The limit of $W(A^{(j)})$, as $j \rightarrow \infty$, is $\{0\}$. Fig. 1a shows $W(A^{(1)})$, $W(A^{(2)})$, $W(A^{(3)})$ for $p = 0.5$.

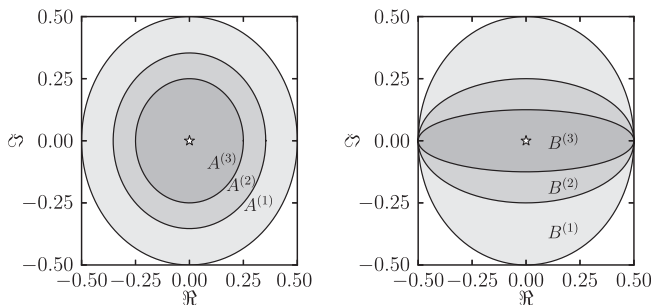


Fig. 1. Numerical ranges $W(A^{(1)})$, $W(A^{(2)})$, $W(A^{(3)})$ from Example 2 for $p = 0.5$ and $W(B^{(1)})$, $W(B^{(2)})$, $W(B^{(3)})$ from Example 3 for $p = 0.25$. The stars mark the barycenter points in these examples. They are preserved by the dynamics.

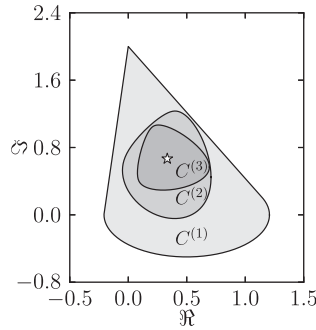


Fig. 2. This figure illustrates Example 4 and the inclusion relation (18). It shows the numerical ranges $W(C^{(1)})$, $W(C^{(2)})$, $W(C^{(3)})$ for $p = 0.5$ and $q = 0.4$. The limit of $W(C^{(j)})$, as $j \rightarrow \infty$, is the barycenter point $(1 + 2i)/3$ marked by the star.

Example 3. Let again $N = 2$. The *phase-flip channel* is the discrete dynamics $\Psi = \Phi_{X_1, X_2}$ corresponding to the Kraus operators $X_1 = \sqrt{1-p} \mathbf{1}_2$ and $X_2 = \sqrt{p}\sigma_1$. Here again, $p \in (0, 1)$ is a free parameter. We set $B^{(1)} = A^{(1)}$ and $B^{(j+1)} = \Psi^j(B^{(1)})$. Then $B^{(2)} = \begin{bmatrix} 0 & 1-p \\ p & 0 \end{bmatrix}$ and $B^{(3)} = \begin{bmatrix} 0 & 1-2p(1-p) \\ 2p(1-p) & 0 \end{bmatrix}$. The numerical ranges of all $B^{(j)}$ are ellipses. Fig. 1b shows $W(B^{(1)})$, $W(B^{(2)})$, $W(B^{(3)})$ for $p = 0.25$.

Example 4. This example is a generalization of Example 3 to $N = 3$. The *double flip channel acting on a qutrit* is the discrete dynamics $\Xi = \Phi_{X_1, X_2, X_3}$; the Kraus operators are $X_1 = \sqrt{1-p-q}\mathbf{1}_3$,

$$X_2 = \sqrt{p} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } X_3 = \sqrt{q} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}. \text{ The parameters } p, q \text{ satisfy } 0 \leq p, q, p + q \leq 1.$$

The operators X_2 and X_3 correspond to bit flips with probabilities p and q . The operator Ξ is trace

preserving. The operator $C^{(1)} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2i \end{bmatrix}$ was studied in [16]; the numerical range $W(C^{(1)})$ is an

ellipse. The *barycenter* of $W(C^{(1)})$ is $\text{tr}(C^{(1)})/3 = (1 + 2i)/3$. Set $C^{(j)} = \Xi^{j-1}(C^{(1)})$. For instance,

$$C^{(2)} = \begin{bmatrix} p & 1-p-q & q \\ p & 1-p-q(1-2i) & 0 \\ 0 & 0 & 2i+q(1-2i) \end{bmatrix}. \text{ In Fig. 2 } p = 0.5 \text{ and } q = 0.4, \text{ hence } 1-p-q = 0.1.$$

4. Joint numerical shadows

We will first recall the notion of the *numerical shadow* of an operator on \mathcal{H}_N [14,13]. Let μ be the normalized Haar measure on $S(\mathcal{H}_N) = \{\psi \in \mathcal{H}_N : \|\psi\| = 1\}$, i.e., the unit sphere S^{2N-1} . We also denote by μ its push-forward to $\Omega_N \simeq \mathbb{C}P^{N-1}$.⁵ Let $A \in \mathcal{L}_N$. The numerical shadow ν_A is the

⁵ It is known as the Fubini–Study measure in the physics literature.

push-forward of μ to \mathbb{C} under the numerical range map. If dz denotes the Lebesgue measure on \mathbb{C} , then $d\nu_A(z) = P_A(z)dz$ where

$$P_A(z) = \int_{\Omega_N} d\mu(\psi) \delta(z - \langle \psi | A | \psi \rangle) \tag{22}$$

is a probability distribution.

Let now $A_1, \dots, A_m \in \mathcal{M}_N$. Their *joint numerical shadow* ν_{A_1, \dots, A_m} is the push-forward of μ under the joint numerical range map $\text{jnr}_{(A_1, \dots, A_m)} : \Omega_N \rightarrow \mathbb{R}^m$. Thus, ν_{A_1, \dots, A_m} is a probability measure supported on $W(A_1, \dots, A_m)$. Let dx^m denote the Lebesgue measure on \mathbb{R}^m . Then $d\nu_{A_1, \dots, A_m} = P_{A_1, \dots, A_m}(x_1, \dots, x_m)dx^m$ where

$$P_{A_1, \dots, A_m}(x_1, \dots, x_m) = \int_{\Omega_N} d\mu(\psi) \prod_{j=1}^m \delta(x_j - \langle \psi | A_j | \psi \rangle). \tag{23}$$

is a probability distribution. Numerical shadow is the special case of the joint numerical shadow corresponding to $m = 2$. See the examples below for illustration.

Let $k_1, \dots, k_m \in \mathbb{N}$. The *moments*

$$I_{k_1, \dots, k_m}(A_1, \dots, A_m) = \int_{\mathbb{R}^m} x_1^{k_1} \cdots x_m^{k_m} d\nu_{A_1, \dots, A_m}(x_1, \dots, x_m). \tag{24}$$

are well defined and, by the Stone–Weierstrass theorem, uniquely determine the joint numerical shadow. When $m = 2$, we recover the moments of the numerical shadow ν_A for the matrix $A = A_1 + iA_2 \in \mathcal{L}_N$ introduced in [14]. Some of the results in [14] extend to the moments of joint numerical shadows for arbitrary m . We will report on this in a separate publication.

If ν is a measure on \mathbb{R}^m and $a \in \mathbb{R}$, we denote by ν^a the push-forward of ν under the self-map $\nu \mapsto a\nu$ of \mathbb{R}^m . If ν_1, ν_2 are measures on \mathbb{R}^m , we denote by $\nu_1 * \nu_2$ their *convolution*. The following proposition exposes a few basic properties of joint numerical shadows. We leave the proof to the reader.

Proposition 3.

1. Let $A_1, \dots, A_m \in \mathcal{M}_N$. Let $U \in \mathcal{L}_N$ be a unitary operator. Set $B_i = UA_iU^*$, $1 \leq i \leq m$. Then

$$\nu_{B_1, \dots, B_m} = \nu_{A_1, \dots, A_m}.$$

2. Let $A_1, \dots, A_m \in \mathcal{M}_N$ and $B_1, \dots, B_m \in \mathcal{M}_N$ be arbitrary. Let $a, b \in \mathbb{R}$. Then

$$\nu_{aA_1 + bB_1, \dots, aA_m + bB_m} = \nu_{A_1, \dots, A_m}^a * \nu_{B_1, \dots, B_m}^b.$$

Example 5. We review Example 1. The set $W(\sigma_1, \sigma_2, \sigma_3) \subset \mathbb{R}^3$ is the unit sphere. The joint numerical shadow $\nu_{\sigma_1, \sigma_2, \sigma_3}$ is the normalized Haar measure.

Example 6. We use the isomorphism $\mathcal{H}_4 = \mathcal{H}_2 \otimes \mathcal{H}_2$ to define the extensions of Pauli matrices $A_j = \sigma_j \otimes \mathbf{1}_2$, $j = 1, 2, 3$. We use the fact that the Haar measure on the space of pure states in $\mathcal{H}_A \otimes \mathcal{H}_B$ induces by partial trace, $\omega = \text{Tr}_B |\psi\rangle\langle\psi|$, the Lebesgue measure on the space of mixed states in \mathcal{H}_2 [20]. Using the equality between the expected values of an operator on \mathcal{H}_2 and the extended operator on \mathcal{H}_4 , we obtain that ν_{A_1, A_2, A_3} is the normalized Lebesgue measure on the Bloch ball. Let $B_j = \mathbf{1}_2 \otimes \sigma_j$, $j = 1, 2, 3$. The *swap operator*

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

is a unitary operator on \mathcal{H}_4 satisfying $B_j = SA_jS^*$. By Proposition 3 and the above discussion, ν_{B_1, B_2, B_3} is the normalized Lebesgue measure on the Bloch ball.

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Note added in Proof

We thank D.C. Brody for pointing out to us the papers [21,22]. These works study the probability distribution that we call ‘numerical shadow’ [14,16]. The authors of [21,22] name it the “normalised density of states”. They study this distribution for Hermitian operators from the viewpoint of quantum statistical mechanics.

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