Semiclassical Spectra without Periodic Orbits for a Kicked Top

Peter Gerwinski, Fritz Haake, and Harald Wiedemann
Fachbereich Physik, Universität-Gesamthochschule Essen, 54117 Essen, Germany

Marek Kuś
Centrum Fizyki Teoretycznej, Polska Akademia Nauk, PL-02-668 Warsaw, Poland

Karol Życzkowski
Instytut Fizyki, Uniwersytet Jagielloński, ul. Reymonta 4, PL-30-059 Kraków, Poland
(Received 24 June 1994)

We present an $\hbar \to 0$ approximation for the quasienergy spectrum of a periodically kicked top, valid under conditions of both regular and chaotic classical motion. In contrast to conventional periodic-orbit theory we implement the semiclassical limit for each matrix element of the Floquet operator rather than for the trace of the propagator. Even though a classical looking action is involved, the approximate matrix elements are specified in terms of complex ghost trajectories instead of real classical orbits. Our mean error for the quasienergies is a surprisingly small 3% of the mean spacing.

PACS numbers: 05.45.+b, 03.65.Sq

Semiclassical approximations for quantum systems whose classical limit is fully chaotic are mostly based on Gutzwiller's periodic-orbit theory [1]. Even though existence problems due to the exponential proliferation of periodic orbits with increasing period seem to have been overcome recently, it remains notoriously difficult to extract more than a few energy or quasienergy levels. Apart from exceptional systems like the cat map [2,3] and some billiards on spaces of constant negative curvature [4–6] for which periodic-orbit theory happens to be exact, we know of no case where a large number of levels has been determined semiclassically. The main obstacle thus far lies in the practical impossibility to gather sufficient knowledge about very long periodic orbits.

In applying periodic-orbit theory to the baker's map, Dittes, Doron, and Smilansky [7] have found a quasienergy uncertainty scaling with $\hbar$ as $\Delta\phi \propto \hbar^{1/2}$; the ratio of $\Delta\phi$ and the mean quasienergy spacing thus diverges with $\hbar \to 0$ as $\hbar^{-1/2}$ which means impossibility in calculating the baker-map spectrum semiclassically.

Alternative semiclassical methods deserve exploration. For instance, Heller's Gaussian wave packet dynamics has yielded surprisingly accurate long-time results which might eventually be employed to determine spectra [8].

We propose here a different scheme for implementing the limit $\hbar \to 0$ for periodically driven systems. We represent the Floquet operator (i.e., the unitary time evolution operator over a period of the driving) as a matrix in a basis of coherent states. Each such state may be associated with a classical phase space cell which shrinks to a point in the limit $\hbar \to 0$. After employing a suitable integral representation for each matrix element, for instance *á la* Feynman, the semiclassical limit is implemented through a saddle-point approximation. Upon diagonalizing the resulting matrix we obtain our semiclassical quasienergy spectrum.

Systems with compact phase spaces and thus finite-dimensional Hilbert spaces should be the first to test our strategy. Any error possibly due to cutting infinite spaces to finite size is thus excluded. We have chosen a periodically kicked top.

The effective size of $\hbar$ is here controlled by an angular-momentum quantum number j as $\hbar \propto 1/j$. With j fixed we are facing a Hilbert space of $2j + 1$ dimensions. The spectrum then consists of $2j + 1$ eigenphases alias quasienergies of the Floquet operator. The specific Floquet operator we work with is expressed in terms of the components of an angular-momentum operator $\vec{J}$ obeying $[J_x, J_y] = iJ_z$, etc., as

$$F = e^{-ikJ_z^2/2j} e^{-i\pi J_y/2}. \tag{1}$$

This accounts for a rotation by an angle $\pi/2$ about the y axis followed by a nonlinear torsion about the z axis. We have chosen the torsion strength as $k = 8$ in order to make sure that in the classical limit the phase space is dominated by chaos [9]; moreover, as a nearly regular case, we have treated $k = 1$.

The classical phase space, incidentally, is the sphere $\lim_{j\to\infty} \vec{J}^{\,2}/j^2 = 1$. A point on that sphere may be specified by angles defined by $J_x = j\sin\theta\cos\phi$, $J_y = j\sin\theta\sin\phi$, $J_z = j\cos\theta$. We have found it convenient to work with the complex stereographic projection variable

$$\gamma = e^{i\phi} \tan\tfrac{\theta}{2}\,. \tag{2}$$

In terms of this coordinate the classical limit yields the stroboscopic map of a point γ into its image Γ as

$$\Gamma = \frac{1+\gamma}{1-\gamma} \exp\left(-ik\frac{\gamma+\gamma*}{1+\gamma\gamma*}\right). \tag{3}$$

Our semiclassical goal suggests to represent the Floquet operator in a coherent-state basis. An angular-momentum

 0031-9007/95/74(9)/1562(4)$06.00 © 1995 The American Physical Society

coherent state [10], to be denoted by $|\theta, \phi\rangle$ or $|\gamma\rangle$, assigns a direction to the angular momentum as $\langle\gamma|J_x|\gamma\rangle = j\sin\theta\cos\phi$, etc., with an uncertainty $\sin\theta\Delta\theta\Delta\phi \propto 1/j$, i.e., the best directional precision permitted by the commutation relations. In particular, such a state is the joint eigenstate $|j,j\rangle$ of $\vec{J}^{\,2}$ and J_z with eigenvalues $j(j+1)$ and j, respectively. All the other coherent states can be generated from this particular one by applying the rotation operator which turns the positive z direction into the direction (θ, ϕ).

We shall need the matrix element of the Floquet operator F between two coherent states [11]

$$\langle\Gamma|F|\gamma\rangle = \left[\frac{(1-\gamma)^2 e^{-ik/2}}{2(1+\Gamma\Gamma^*)(1+\gamma\gamma^*)}\right]^j \sqrt{\frac{ij}{2\pi k}} \int_{-\infty}^{+\infty} dt\, e^{jV} \tag{4}$$

with the exponent

$$V = -\frac{i}{2k}t^2 + 2\ln(1+v), \qquad v = \Gamma^* \frac{1+\gamma}{1-\gamma} e^{ik+t}. \tag{5}$$

In the semiclassical limit $j \to \infty$, the t integral arising here lends itself to a saddle-point approximation. Saddle points are determined by

$$\frac{dV}{dt} = -\frac{i}{k}t + \frac{2v}{1+v} = 0. \tag{6}$$

Not too surprisingly, this saddle-point equation is a slightly disguised form of the classical map, and the saddle-point value of $V(t)$ is closely related to the action of that map [11]. However, by specifying both the initial point γ and the final one Γ one overdetermines the classical orbit, unless Γ happens to be the image of γ under the classical map. For generic pairs (γ, Γ) we may think of a solution of the saddle-point equation as of a ghost trajectory which lives in a complexified phase space and has a complex action [12–14].

The saddle-point approximation begins to make sense for j upwards of $j \approx 1$. However, it is only for $j \gg 1$ that it suffices to use the saddle with the largest real part of $V(t)$ from which both $+\infty$ and $-\infty$ can be reached in the complex t plane by going downhill on the surface $|\exp[j\,\mathrm{Re}V(t)]|$ above that plane. For j smaller than about 30 we have found it necessary to establish a path in the t plane connecting $-\infty$ to $+\infty$ through a sequence of saddles and interlacing zeros of the integrand, keeping $\mathrm{Im}V(t)$ constant, and thus collecting the contributions of, in general, several saddles (see Fig. 1). Incidentally, the quality of the semiclassical approximation for each matrix element can be checked quite easily by evaluating the matrix element rigorously according to (4).

In a first semiclassical attempt at determining the Floquet spectrum we have chosen $2j+1$ different coherent states $|\gamma_n\rangle$ as a complete basis. We have calculated the $(2j+1)^2$ matrix elements $\langle\gamma_n|F|\gamma_m\rangle$ both rigorously and semiclassically. Figure 2 displays, as a function of j, the modulus of the semiclassical error for a typical matrix element. As expected for a saddle-point approximation, that error is inversely proportional to j. Finally, the semiclassical eigenvalues $z_n = r_n \exp(i\phi_n)$ of the Floquet operator are determined from the generalized eigenvalue equation

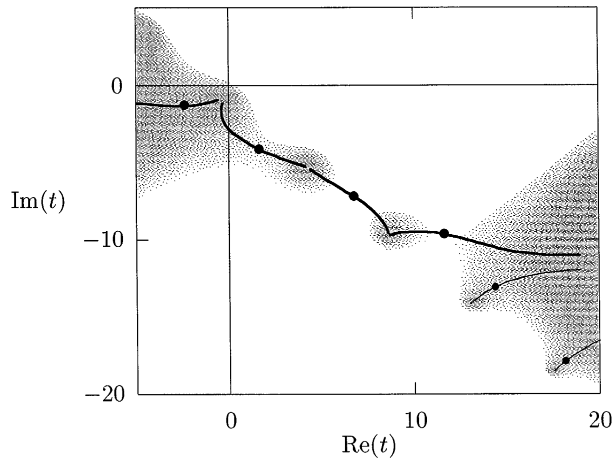

Fig. 1. Gray-scale coded contour plot of $|e^{jV(t)}|$ in the complex t plane. White regions indicate $|e^{jV}| \gg 1$ and dark ones $|e^{jV}| \ll 1$. The thick black line shows the path of constant phase from $-\infty$ to $+\infty$ through saddles (thick dots) and interlacing zeros of e^{jV}. Small dots refer to saddles not on allowed paths of integration.

$$\det(\langle\gamma_m|F|\gamma_n\rangle - z\langle\gamma_m|\gamma_n\rangle) = 0. \tag{7}$$

The nonorthogonality of the coherent states notwithstanding this equation would yield the exact eigenvalues z_n if we were to use the exact matrix elements $\langle\gamma_m|F|\gamma_n\rangle$; in particular, all moduli r_n would come out as unity, due to the unitarity of F.

Our interest is, of course, in the roots arising with the semiclassically approximated matrix. Figure 3 displays the rms deviation Δr of their moduli r_n from unity, again as a function of j; the mean is now taken over all $2j+1$ eigenvalues. Like for the matrix elements, the error scales roughly as $1/j$. The same scaling is found for the rms deviation $\Delta\phi$ of the phases ϕ_n from

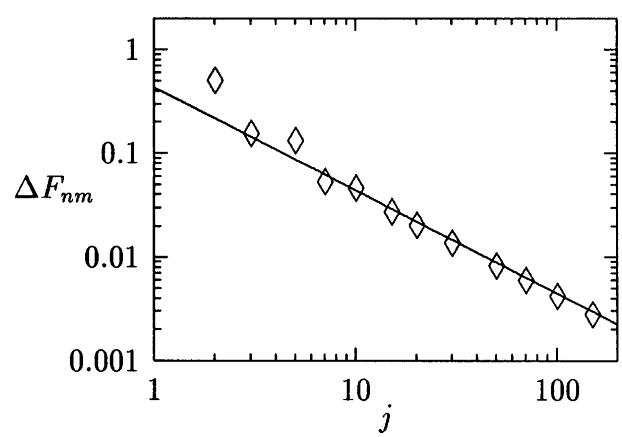

FIG. 2. Modulus of the semiclassical error of a typical matrix element of the Floquet operator versus j. The optimal fit is $0.429j^{-0.993}$, i.e., very close to a proportionality to $1/j$.

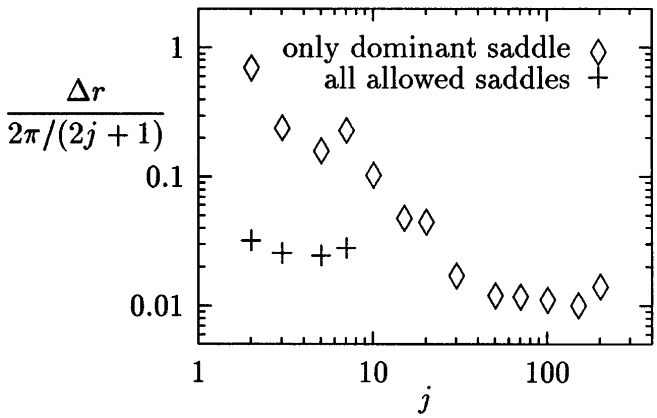

FIG. 3. The radial error of the eigenvalues scaled by the mean spacing versus j.

their exact counterparts, as shown in Fig. 4. While the proportionality of the semiclassical errors to $1/j$ should be expected, it is, in fact, a most welcome surprise that each approximate eigenphase can be uniquely associated with its exact counterpart. This is illustrated in Fig. 5 for $j = 10$. A more quantitative manifestation of the astounding success of our approximation can be read off from Fig. 4: The phase error is as small as 3% of the mean spacing.

The small ratio of the eigenvalue uncertainty to the mean spacing here incurred for the kicked top demands theoretical explanation. To that end we are presently working out the next term in the asymptotic expansion of the matrix elements of which the saddle-point contribution is but the leading term.

In order to check on the reliability of our results, we have explored a second method for establishing the Floquet spectrum from the semiclassically approximated matrix elements $\langle\Gamma|F|\gamma\rangle$. We now work with an overcomplete basis of N coherent states with $N > 2j + 1$ for which it is convenient to adopt a slightly modified normalization,

$$|\tilde{\gamma}_n\rangle = \sqrt{\frac{2j + 1}{N}}|\gamma_n\rangle, \qquad n = 1, 2, \ldots, N. \tag{8}$$

This gives rise to an $N \times N$ "extended" Floquet matrix

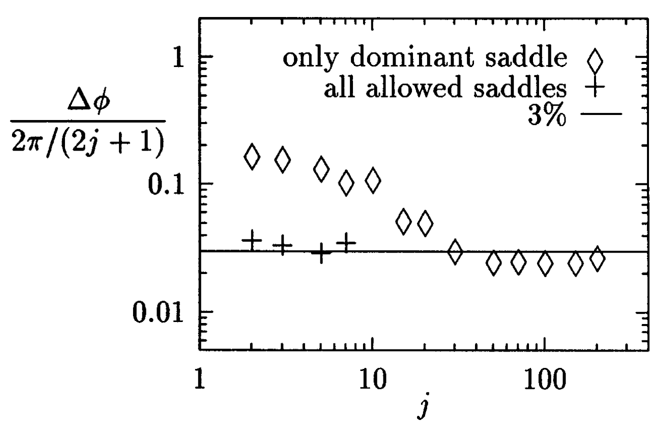

FIG. 4. The phase (quasienergy) error of the eigenvalues is roughly 3% of the mean spacing $2\pi/(2j + 1)$.

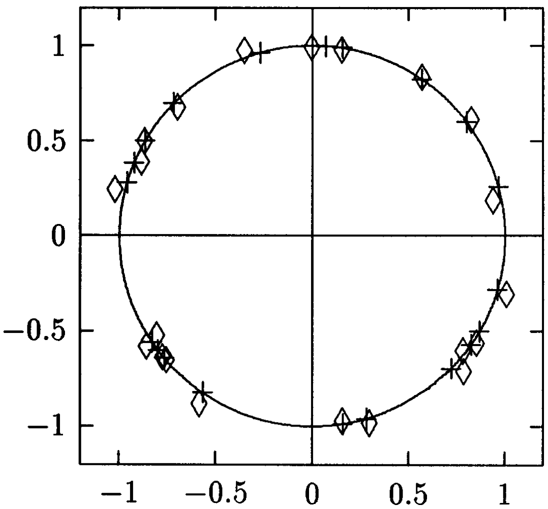

FIG. 5. Exact (+) and semiclassical (◇) Floquet eigenvalues for $j = 10$. In all 21 cases a unique association is possible.

$$\tilde{F}_{mn} = \langle\tilde{\gamma}_m|F|\tilde{\gamma}_n\rangle, \tag{9}$$

which may be related to the unitary $(2j + 1) \times (2j + 1)$ Floquet matrix $F_{\mu\nu}$ with respect to an arbitrary complete orthonormal basis $\{|\mu\rangle, \mu = 1, 2, \ldots, 2j + 1\}$ as

$$\tilde{F}_{mn} = \sum_{\mu,\nu=1}^{2j+1} A^\dagger_{m\mu} F_{\mu\nu} A_{\nu n}, \qquad A_{\nu n} = \langle\nu|\tilde{\gamma}_n\rangle. \tag{10}$$

Here we have introduced a rectangular $(2j + 1) \times N$ matrix A and its adjoint $A^\dagger$.

A piece of linear algebra now comes in handy. As long as no approximation is made in evaluating the matrices involved, among the N eigenvalues of $\tilde{F} = A^\dagger F A$ there must be $N - (2j + 1)$ vanishing ones while the remaining $2j + 1$ eigenvalues must coincide with those of $FAA^\dagger$.

To appreciate this lemma we inspect the matrix

$$(AA^\dagger)_{\mu\nu} = \frac{(2j + 1)\Delta S}{N\Delta S} \sum_{n=0}^{N} \langle\mu|\gamma_n\rangle\langle\gamma_n|\nu\rangle. \tag{11}$$

We have sneaked in a factor $\Delta S/\Delta S$ and choose, in the denominator, $\Delta S = 4\pi/N$, i.e., the mean surface element on the unit sphere around each of the N points (θ_n, ϕ_n). Upon expressing that surface element, in the numerator, in terms of the variable γ_n we obtain

$$AA^\dagger = \frac{2j + 1}{\pi} \sum_{n=0}^{N} |\gamma_n\rangle\langle\gamma_n| \frac{\Delta\mathrm{Re}\gamma_n \Delta\mathrm{Im}\gamma_n}{(1 + \gamma_n\gamma_n^*)^2} \underset{N\to\infty}{\longrightarrow} 1. \tag{12}$$

Indeed, for $N \to \infty$ the foregoing sum can be written as the integral defining the resolution of unity by coherent states [10]. In that limit, therefore, the eigenvalues of the matrix $(A^\dagger F A)_{\mu\nu}$ include the $2j + 1$ eigenvalues of the Floquet matrix $F_{\mu\nu}$ as well as $N - (2j + 1)$ zeros.

We have diagonalized the semiclassical version of the extended matrix $\tilde{F}$, after approximating each of its N^2 elements as described above, for various values of the quantum number j and the extended dimension N. In all cases we have found $N - 2j - 1$ eigenvalues with very small moduli, $r_n \ll 1$, and $2j + 1$ nearly unimodular eigenvalues which, in fact, lie close to the exact ones. Figure 6 displays the dependence of the phase error $\Delta\phi$ of the nearly unimodular eigenvalues, in units of the mean level spacing, on the two dimensions $2j + 1, N$. To our surprise the data suggest that $\Delta\phi[2\pi/(2j+1)]^{-1}$ depends on $2j + 1$ and N only through the single scaling variable $\mu = N/(2j + 1)$. Roughly, $\Delta\phi[2\pi/(2j+1)]^{-1}$ decreases like μ^{-4} until μ has grown to about 2.5 and then becomes independent of μ, staying close to 0.03. This means that all levels can be determined to within 3% of the mean level spacing by increasing the extended dimension to a little less than 3 times the "physical" dimension $2j + 1$. It is most gratifying to see that the limiting accuracy obtained is practically identical to the one incurred in our first semiclassical strategy. This strengthens our expectation that the error can be explained by the next-order term of the asymptotic $1/j$ expansion for the matrix elements of F.

Our strategy is not restricted to conditions of classical chaos. We have carried out calculations like the ones described above for $k = 1$, a case close to integrable. Results of similar, actually even slightly better accuracy were obtained. We should not forget to mention our considerable efforts to check the insensitivity of our semiclassical spectra to the precise locations of the coherent states on the unit sphere, as long as these locations form a more or less regular grid with no pair of points much closer together than a mean nearest-neighbor spacing.

In concluding, it may be well to highlight the difference of our method to the more familiar one of Gutzwiller's in terms of the number of trajectories employed. We work with $(2j + 1)^2$ ghost orbits rather than with exponentially many real periodic orbits (for a Hilbert space of $2j + 1$ dimensions all periodic orbits with periods from 1 to j would be needed). Incidentally, our ghost-orbit approach also differs from Ref. [15] where real orbits were used to approximate matrix elements.

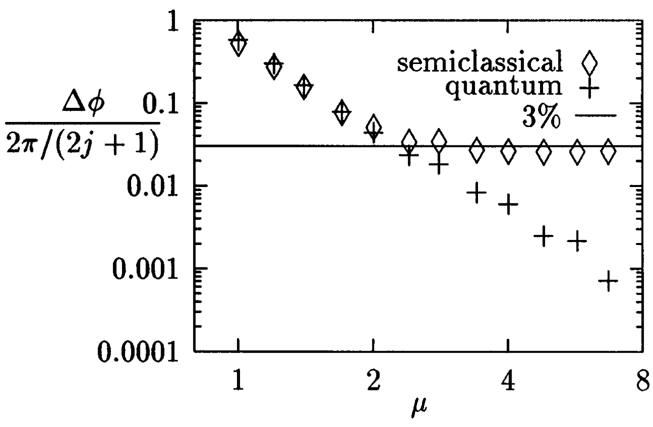

FIG. 6. Phase error determined with an overcomplete basis (dimension $N > 2j + 1$) versus the scaling variable $\mu = N/(2j + 1)$, for $j = 30$. The semiclassical error (◇) saturates at 3% of the mean spacing. With exact matrix elements the phase error (+), due to the nonorthogonality of the coherent states, decays roughly as μ^{-4}. The data for other values of j yield the same two "curves."

We are not spared the efficiency problem which still so notoriously plagues semiclassical approximations of large spectra. In our case that problem is not due to any proliferation of classical orbits. The extra burden we heap on ourselves, compared to the diagonalization of the exact Floquet matrix, is the saddle-point approximation for each matrix element. But efficiency, even though it would be most welcome and should indeed be striven for, was not the primary goal for the moment. We are already glad to see that a full spectrum with generic fluctuations of a dynamical system with global classical chaos is indeed accessible by semiclassical methods, i.e., by approximations exploiting the effective smallness of Planck's constant.

We gratefully acknowledge support by the Sonderforschungsbereich Unordnung und große Fluktuationen of the Deutsche Forschungsgemeinschaft. M. K. and K. Ż. acknowledge partial support by KBN Grant No. P302 102 06. We have enjoyed fruitful discussions with Eugene Bogomolny, Stephen Creagh, Patricio Leboeuf, and Nils Lehmann. We are especially indebted to Jonathan Keating for his patient help with the intricacies of the saddle-point technique.

[1] M. C. Gutzwiller, *Chaos in Classical and Quantum Mechanics*, (Springer, New York, 1990).
[2] J. P. Keating, Nonlinearity **4**, 309 (1991).
[3] S. Creagh (private communication).
[4] C. Matthies and F. Steiner, Phys. Rev. A **44**, R7877 (1991); R. Aurich, C. Matthies, M. Sieber, and F. Steiner, Phys. Rev. Lett. **68**, 1629 (1992).
[5] P. A. Boasman, Nonlinearity **7**, 485 (1994).
[6] J. P. Keating and M. Sieber, Proc. R. Soc. London (to be published)
[7] F.-M. Dittes, E. Doron, and U. Smilansky, Phys. Rev. E **49**, R963 (1994).
[8] M. A. Sepúlveda, S. Tomsovic, and E. J. Heller, Phys. Rev. Lett. **69**, 402 (1992).
[9] F. Haake, *Quantum Signatures of Chaos* (Springer, Berlin, 1991).
[10] F. T. Arecchi, E. Courtens, R. Gilmore, and H. Thomas, Phys. Rev. A **6**, 2211 (1972); R. Glauber and F. Haake, Phys. Rev. A **13**, 357 (1976).
[11] M. Kuś, F. Haake, and B. Eckhardt, Z. Phys. B **92**, 221 (1993).
[12] M. Kuś, F. Haake, and D. Delande, Phys. Rev. Lett. **71**, 2167 (1993).
[13] R. Scharf and B. Sundaram, Phys. Rev. E **49**, R4767 (1994).
[14] R. Balian, G. Parisi, and A. Voros, Phys. Rev. Lett. **41**, 1141 (1978); R. Balian, in *Discourses in Mathematics and Its Applications, No. 1,* edited by S. A. Fulling and F. J. Narcovich (Department of Mathematics, Texas A&M University, College Station, 1991).
[15] M. L. Koszykowski, D. W. Noid, and R. A. Marcus, J. Phys. Chem. **86**, 2113 (1982).