Subnormalized states and trace-nonincreasing maps

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We investigate the set of completely positive, trace-nonincreasing linear maps acting on the set $\mathcal{M}_N$ of mixed quantum states of size N. Extremal point of this set of maps are characterized and its volume with respect to the Hilbert-Schmidt (HS) (Euclidean) measure is computed explicitly for an arbitrary N. The spectra of partially reduced rescaled dynamical matrices associated with trace-nonincreasing completely positive maps belong to the N cube inscribed in the set of subnormalized states of size N. As a by-product we derive the measure in $\mathcal{M}_N$ induced by partial trace of mixed quantum states distributed uniformly with respect to the HS measure in $\mathcal{M}_{N^2}$. © *2007 American Institute of Physics.*
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I. INTRODUCTION

Modern application of quantum mechanics increased the interest in the space of quantum states: positive operators normalized by the assumption that their trace is fixed, Tr $\rho = 1$. For applications in the theory of quantum information processing, it is often sufficient to restrict the attention to the operators acting on a finite dimensional Hilbert space.

The set $\mathcal{M}_N$ of density matrices of size N forms a convex body embedded in $\mathbb{R}^{N^2-1}$. In other words it forms a cross section of the cone of positive operators with a hyperplane corresponding to the normalization condition. In the simplest case of one qubit the set $\mathcal{M}_2$ is equivalent, with respect to the Hilbert-Schmidt (HS) (Euclidean) geometry, to a three-dimensional ball, $\mathbf{B}_3$. For higher dimensions the geometry of $\mathcal{M}_N$ gets more complicated and differs from the ball $\mathbf{B}_{N^2-1}$.[1,2]

Nontrivial properties of the set of mixed quantum states attracted recently a lot of attention. The volume V, the hyperarea A, and the radius R of the maximal ball inscribed into $\mathcal{M}_N$ was computed with respect to the HS measure,[3] which leads to the Euclidean geometry. The volume of the set of quantum states was computed with respect to the Bures measure related to quantum distinguishability[4] and also with respect to a wide class of measures induced by monotone Riemanian metrics.[5,6] The set $\mathcal{M}_N$ is known to be of a constant height,[7] so the ratio A/V coincides with the dimensionality of this set, equal to N^2-1.

If N is a composite number, the density operators from $\mathcal{M}_N$ can represent states of a composed physical system. In this case one defines the set of separable states, which forms a convex

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subset of the set of all states, $\mathcal{M}_N^{\text{sep}} \subset \mathcal{M}_N$. Although a lot of work has been done to estimate the volume of the subset of separable states,[8–13] the problem of finding the exact value of the ratio $\text{Vol}(\mathcal{M}_N^{\text{sep}})/\text{Vol}(\mathcal{M}_N)$ remains open even in the simplest case of two qubits.[14]

In parallel with investigation of the set of quantum states, one studies the properties of the set of completely positive (CP) maps which act on $\mathcal{M}_N$. Such maps are important not only from the theoretical point of view: for instance, linear CP maps acting on a two-level quantum system correspond to linear optical devices used in polarization optics.[15] Due to the Jamiołkowski isomorphism,[16,17] the set of trace preserving (TP) CP maps acting on $\mathcal{M}_N$ forms an $(N^4 - N^2)$-dimensional subset of the $(N^4 - 1)$-dimensional set $\mathcal{M}_{N^2}$ of states acting on an extended Hilbert space, $\mathcal{H}_N \otimes \mathcal{H}_N$. In the simplest case of $N=2$, the structure of this 12-dimensional convex set of maps was studied in Ref. 18.

We start this paper by reviewing the properties of the set of subnormalized states for which $\text{Tr}\,\rho \leq 1$. Such states are obtained by taking the convex hull of the set of normalized states and the particular "zero state." In the classical case, one could argue that such a step is equivalent to increasing the number of distinguishable events by 1, and renaming 0 into $N+1$. This reasoning is based on the fact that the set of subnormalized states of size N as well as the set of normalized states of size $N+1$ form N-dimensional simplices. However, this is not the case in the quantum setup, in which the set of subnormalized states $\mathcal{M}_N^{\text{sub}}$ has N^2 dimensions, in contrast to $(N^2 + 2N)$-dimensional set $\mathcal{M}_{N+1}$. The fact that the dimensionality of the set of subnormalized states grows with the number N of distinguishable states exactly as N^2 plays a key role in an axiomatic approach to the quantum mechanics of Hardy.[19]

The main aim of this work is to describe the set of CP trace-nonincreasing (TNI) maps which act on the set $\mathcal{M}_N$ of mixed states. We compute the exact volume of this N^4-dimensional set with respect to the Euclidean (HS) measure and characterize its extremal points. The TNI maps have a realistic physical motivation, since they describe an experiment for which with a certain probability the apparatus does not work. This could be an interpretation of the "zero map" after action of which no result is recorded. Such maps are sometimes called *trace decreasing*,[20] but to emphasize that the set of these maps contains also all TP maps, we prefer to use a more precise name of TNI maps.

The paper is organized as follows. In Sec. II we analyze the set of subnormalized states and compute its volume. The measures in the set of mixed states induced by partial trace are investigated in Sec. III. In Sec. IV we define the set of TNI maps and provide its characterization, while in Sec. V the volume of this set is calculated with respect to the flat (HS) measure. The set of extremal TNI maps is studied in Sec. VI.

II. SUBNORMALIZED QUANTUM STATES

Let $\mathcal{M}_N$ denote the set of normalized quantum states acting on N-dimensional Hilbert space,

$$\mathcal{M}_N := \{\rho : \rho^\dagger = \rho, \rho \geq 0, \text{Tr}\,\rho = 1\}.$$

$\mathcal{M}_N$ forms a convex set of dimensionality $(N^2 - 1)$. In the simplest case $N=2$, this set is equivalent to the *Bloch ball*, $\mathcal{M}_2 = \mathbf{B}_3 \subset \mathbb{R}^3$.

Definition 1: A Hermitian, positive operator σ is called a *subnormalized state*, if $\text{Tr}\,\sigma \leq 1$.

The set of subnormalized states acting on the N-dimensional Hilbert space $\mathcal{H}_N$ will be denoted by $\mathcal{M}_N^{\text{sub}}$. By construction this set has N^2 dimensions and can be defined as a convex hull of the zero operator and the set of quantum states (see Fig. 1),

$$\mathcal{M}_N^{\text{sub}} := \{\sigma : \sigma^\dagger = \sigma, \sigma \geq 0, \text{Tr}\,\sigma \leq 1\} = \text{conv hull}\{0, \mathcal{M}_N\}. \tag{1}$$

For instance, the set $\mathcal{M}_2^{\text{sub}}$ forms a four-dimensional cone with apex at 0 and base formed by the Bloch ball $\mathcal{M}_2 = \mathbf{B}_3$. Note that the dimension of $\mathcal{M}_N^{\text{sub}}$ grows *exactly* as squared dimension of the Hilbert space. Due to this fact the subnormalized states are a convenient notion to be used in an axiomatic approach to quantum theory.[19]
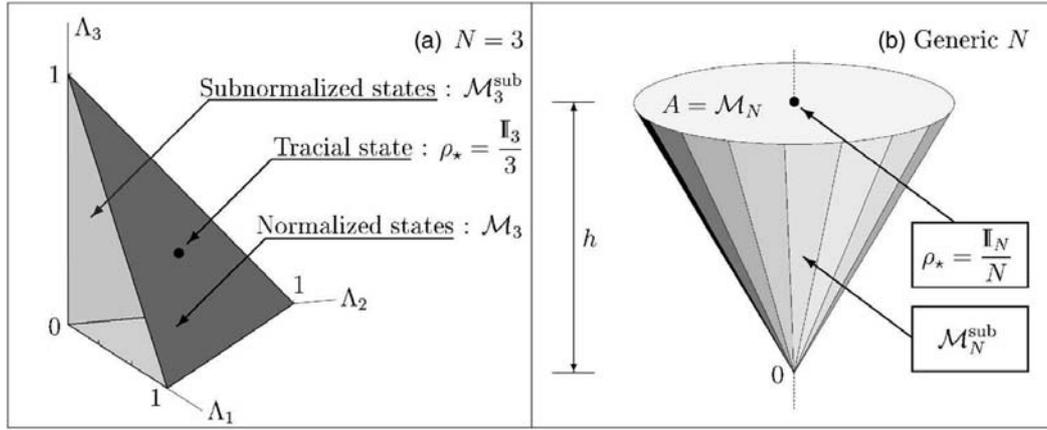
FIG. 1. The set of subnormalized states: (a) the set of eigenvalues for $N=3$, and (b) the convex cone of N^2 dimensions with zero state at the apex and the (N^2-1)-dimensional set $\mathcal{M}_N$ as the base.

In order to characterize the set $\mathcal{M}_N^{\mathrm{sub}}$ of subnormalized states, we compute in this section its volume with respect to the flat HS measure induced by the HS metric.[3] Consider the set $\mathcal{M}_2$ of 2×2 density matrices, parametrized by the real *Bloch coherence vector* $\boldsymbol{\xi}\in\mathbf{B}_3$,

$$\rho = \frac{\mathbb{I}_2}{2} + \boldsymbol{\xi}\cdot\boldsymbol{\Xi}, \tag{2}$$

where $\boldsymbol{\Xi}$ denotes the vector of three rescaled traceless Pauli matrices $(\sigma_x/\sqrt{2}, \sigma_y/\sqrt{2}, \sigma_z/\sqrt{2})$. Note that with such a normalization the radius of the Bloch ball $\mathbf{B}_3$ is given by $R_2 = 1/\sqrt{2}$, as it can be obtained from the relation $\mathrm{Tr}\,\rho^2\leq1$. With this definition, the HS distance between any two density operators, defined as the HS (Frobenius) norm of their difference, proves to be equal to the Euclidean distance $D_\mathrm{E}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2)$ between the labeling Bloch vectors $\boldsymbol{\xi}_1,\boldsymbol{\xi}_2\in\mathbf{B}_3\subset\mathbb{R}^3$,

$$D_\mathrm{HS}(\rho_{\boldsymbol{\xi}_1},\rho_{\boldsymbol{\xi}_2}) = \sqrt{\mathrm{Tr}[(\rho_{\boldsymbol{\xi}_1}-\rho_{\boldsymbol{\xi}_2})^2]} = \|\boldsymbol{\xi}_1-\boldsymbol{\xi}_2\| = D_\mathrm{E}(\boldsymbol{\xi}_1,\boldsymbol{\xi}_2). \tag{3}$$

The above formula holds for an arbitrary N, provided that

$$\rho = \frac{\mathbb{I}_N}{N} + \boldsymbol{\xi}\cdot\boldsymbol{\Xi}. \tag{4}$$

The real *coherence vector* $\boldsymbol{\xi}$ in Eq. (4) is taken as (N^2-1) dimensional and $\boldsymbol{\Xi}$ now represents an operator valued vector which consists of (N^2-1) traceless Hermitian generators of SU(N), fulfilling $\mathrm{Tr}(\Xi_i\Xi_j)=\delta_{ij}$. Note, however, that in this case the geometry of the space of coherence vectors $\boldsymbol{\xi}$ does not coincide with the ball $\mathbf{B}_{N^2-1}$ but constitutes instead a convex subset of it.[2] The condition $\mathrm{Tr}\,\rho^2\leq1$ yields the upper bound for the length of the coherence vector, $|\boldsymbol{\xi}|\leq\sqrt{(N-1)/N}=:R_N$. In the case $N=3$, the vector $\boldsymbol{\Xi}$ consists of the set of eight normalized Gell-Mann matrices $\{\Xi_i\}_{i=1}^8$.

A metric space consisting of a set $\mathcal{M}_N$ and a distance d is automatically endowed with a measure induced by the metric. The measure is defined by the assumption that all balls of a fixed radius defined in $\mathcal{M}_N$ with respect to the distance d have the same volume.

The infinitesimal HS measure around any matrix $\rho\in\mathcal{M}_N$ factorizes as[3]

$$d\mu_\mathrm{HS}(\rho) = \frac{\sqrt{N}}{N!}d\nu^\Delta(\Lambda_1,\ldots,\Lambda_N)\,d\nu^\mathrm{Haar}, \tag{5}$$

where the factor $d\nu^\Delta$ represents the measure in the simplex $\boldsymbol{\Delta}_{N-1}$ of eigenvalues $\Lambda_1,\ldots,\Lambda_N$, while $d\nu^\mathrm{Haar}$ depends on the eigenvectors of ρ. The prefactor $\sqrt{N}$ emerges in Eq. (5) when we force the variables $\Lambda_1,\ldots,\Lambda_N$ to live on the simplex $\boldsymbol{\Delta}_{N-1}$. This reduces the number of independent variables by 1, since $\Sigma_{i=1}^N\Lambda_i=1$, and introduces a factor $\sqrt{\det g}$ in Eq. (5), where g denotes the metric

tensor in the $(N-1)$-dimensional simplex. Such a metric arises due to a change from N linearly dependent variables $\{\Lambda_i\}_{i=1}^{N}$ to the $(N-1)$ linearly independent ones $\{\Lambda_i\}_{i=1}^{N-1}$.

The last factor $d\nu^{\text{Haar}}$ can be integrated on the entire complex *flag manifold*[3,21] defined by the coset space, $\text{Fl}_{\mathbb{C}}^{(N)} := U(N)/[U(1)]^N$, that is, the space of equivalence classes of matrices U diagonalizing the given ρ. The volume of the flag manifold induced by the parametrization used in Eq. (4) is given by[3]

$$\text{Vol}[\text{Fl}_{\mathbb{C}}^{(N)}] = \int_{\text{Fl}_{\mathbb{C}}^{(N)}} d\nu^{\text{Haar}} = \frac{(2\pi)^{N(N-1)/2}}{1!2!\cdots(N-1)!}. \tag{6}$$

Even after splitting off the N phases of $[U(1)]^N$, a residual arbitrariness still remains in the diagonalization of ρ, related to the fact that different permutations of N generically different eigenvalues Λ_i belong to the same unitary orbit. This explains the factor $N!$ in Eq. (5), given by the number of equivalent *Weyl chambers* of the simplex $\boldsymbol{\Delta}_{N-1}$.

The measure $d\nu^{\Delta}(\Lambda_1, \ldots, \Lambda_N)$ reads[3]

$$d\nu^{\Delta}(\Lambda_1, \ldots, \Lambda_N) = \delta\left(\sum_{i=1}^{N} \Lambda_i - 1\right)\prod_{i=1}^{N} \Theta(\Lambda_i)\prod_{i<j}(\Lambda_i - \Lambda_j)^2 d\Lambda_1, \ldots, d\Lambda_N, \tag{7}$$

where the Dirac delta and the product of the Heaviside step functions Θ ensure that the measure is concentrated on the simplex $\boldsymbol{\Delta}_{N-1}$. Up to a normalization constant, the right-hand side (rhs) of the above equation defines a probability distribution on the simplex,

$$d\nu^{\Delta}(\Lambda_1, \ldots, \Lambda_N) \propto P_{\text{HS}}^{(2)}(\Lambda_1, \ldots, \Lambda_N)d\Lambda_1, \ldots, d\Lambda_N, \tag{8}$$

where the upper index 2 refers to the exponent in the last factor of Eq. (7). In random matrix theory it is commonly denoted by β and called *repulsion exponent*, as it determines the repulsion between adjacent eigenvalues. It is equal to 1, 2, and 4 for real, complex, and symplectic ensembles of random matrices, respectively.[22] In this work we are going to restrict our attention to complex density matrices so we fix $\beta=2$, but one may also repeat our analysis for other universality classes.

For later purpose, we now introduce a bigger family of probability distributions, indexed by a real parameter α,

$$P_N^{(\alpha,2)}(\Lambda_1, \ldots, \Lambda_N) := C_N^{(\alpha,2)}\delta\left(1 - \sum_{i=1}^{N} \Lambda_i\right)\prod_{i=1}^{N}\Theta(\Lambda_i)\Lambda_i^{\alpha-1}\prod_{i<j}(\Lambda_i - \Lambda_j)^2, \tag{9a}$$

with normalization constant $C_N^{(\alpha,2)}$ given by

$$\frac{1}{C_N^{(\alpha,2)}} := \int \delta\left(1 - \sum_{i=1}^{N} \Lambda_i\right)\prod_{i=1}^{N}\Theta(\Lambda_i)\Lambda_i^{\alpha-1}\prod_{i<j}(\Lambda_i - \Lambda_j)^2 d\Lambda_1, \ldots, d\Lambda_N = \frac{\displaystyle\prod_{j=1}^{N}\Gamma(1+j)\Gamma[j+(\alpha-1)]}{\Gamma[N^2+(\alpha-1)N]}. \tag{9b}$$

The probability distribution in Eq. (8) represents a special case of $P_N^{(\alpha,2)}$, being

$$P_{\text{HS}}^{(2)}(\Lambda_1, \ldots, \Lambda_N) = P_N^{(\alpha,2)}(\Lambda_1, \ldots, \Lambda_N)\big|_{\alpha=1}. \tag{10}$$

Putting together Eqs. (5)–(7) and (9b), one derives[3] the (HS) volume of the set $\mathcal{M}_N$ of mixed quantum states,

$$\mathrm{Vol}_{\mathrm{HS}}(\mathcal{M}_N) = \int_{\mathcal{M}_N} \mathrm{d}\mu_{\mathrm{HS}}(\rho) = \frac{\sqrt{N}}{N!} \frac{\mathrm{Vol}[\mathrm{Fl}_{\mathbb{C}}^{(N)}]}{C_N^{(1,2)}} = \sqrt{N}(2\pi)^{N(N-1)/2} \frac{\Gamma(1)\Gamma(2)\cdots\Gamma(N)}{\Gamma(N^2)}. \tag{11}$$

As will be shown later in this section, the volume of the set $\mathcal{M}_N^{\mathrm{sub}}$ of subnormalized states can be easily obtained using the (flat) Euclidean geometry as the volume of the cone with base given by $\mathcal{M}_N$. However, for later use, we shall first derive this formula directly by integrating an extension of the HS measure over the entire cone. Let us first formulate the following lemma proved in the appendixes.

Lemma 1: Consider a one parameter family of probability measures $\mathrm{d}\nu_K(\Lambda_1, \ldots, \Lambda_N)$ defined on the set $\mathrm{CH}(N) := \mathrm{conv\ hull}\{0, \mathbf{\Delta}_{N-1}\}$,

$$\mathrm{d}\nu_K(\Lambda_1, \ldots, \Lambda_N) = \prod_{i=1}^{N} \Theta(\Lambda_i)\Lambda_i^{K-N} \prod_{i<j} (\Lambda_i - \Lambda_j)^2 \mathrm{d}\Lambda_1, \ldots, \mathrm{d}\Lambda_N, \tag{12}$$

and labeled by integers $K \geq N$. Then the volume $\nu_K(\mathrm{CH}(N))$ reads

$$\nu_K(\mathrm{CH}(N)) = \int_{\mathrm{CH}(N)} \mathrm{d}\nu_K(\Lambda_1, \ldots, \Lambda_N) = \frac{1}{KNC_N^{(K-N+1,2)}},$$

where $C_N^{(K-N+1,2)}$ is the coefficient defined in Eq. (9b), with $\alpha = K-N+1$.

The measure $\mathrm{d}\mu_{\mathrm{HS}}^{\mathrm{sub}}(\sigma)$ on the set of subnormalized states $\mathcal{M}_N^{\mathrm{sub}}$ is given by

$$\mathrm{d}\mu_{\mathrm{HS}}^{\mathrm{sub}}(\sigma) = \frac{1}{N!} \mathrm{d}\nu^{\mathrm{sub}}(\Lambda_1, \ldots, \Lambda_N) \mathrm{d}\nu^{\mathrm{Haar}} \tag{13}$$

and differs from the one of Eq. (5), relative to quantum normalized states, by the factor $\sqrt{N}$. In Eq. (5), such a factor was due to the change of variables needed to express the volume elements in terms of the set of $(N-1)$ independent variables on the simplex $\mathbf{\Delta}_{N-1}$; in the present case we do not need to change the variables anymore, and the factor $\sqrt{N}$ does not appear.

Moreover, the definition of the measure $\mathrm{d}\nu^{\mathrm{sub}}$ on the entire set $\mathrm{CH}(N)$ of Lemma 1 used in Eq. (13) differs from the one defined on the simplex $\mathbf{\Delta}_{N-1}$ and used in Eq. (7). In particular, for $K = N$ Eq. (12) implies

$$\mathrm{d}\nu^{\mathrm{sub}}(\Lambda_1, \ldots, \Lambda_N) = \mathrm{d}\nu_N(\Lambda_1, \ldots, \Lambda_N) = \Theta\left(\sum_{i=1}^{N} \Lambda_i - 1\right) \prod_{i=1}^{N} \Theta(\Lambda_i) \prod_{i<j} (\Lambda_i - \Lambda_j)^2 \mathrm{d}\Lambda_1, \ldots, \mathrm{d}\Lambda_N. \tag{14}$$

In analogy to the derivation of the volume [Eq. (11)] of the set of normalized states, the HS volume of the set $\mathcal{M}_N^{\mathrm{sub}}$ of subnormalized states can be earned by Eqs. (13), (14), (6), and (9b) together with Lemma 1, in which we set $K = N$,

$$\mathrm{Vol}_{\mathrm{HS}}(\mathcal{M}_N^{\mathrm{sub}}) = \frac{1}{N!} \mathrm{Vol}[\mathrm{Fl}_{\mathbb{C}}^{(N)}]\nu_N(\mathrm{CH}(N)) = (2\pi)^{N(N-1)/2} \frac{\Gamma(1)\Gamma(2)\cdots\Gamma(N)}{N^2\Gamma(N^2)}. \tag{15}$$

The above result can be obtained directly using the Archimedean formula for the Euclidean volume of the D-dimensional cone of Fig. 1 [panel (b)],

$$V = \frac{1}{D} \cdot A \cdot h, \tag{16}$$

where V is the D-dimensional volume of the (hyper)cone representing $\mathcal{M}_N^{\mathrm{sub}}$, A is the area of its $(D-1)$-dimensional base $\mathcal{M}_N$, and h denotes its height, that is, the distance between the base and the apex (the latter corresponding to the state $\sigma_0 = 0$). Making use of the definition of HS distance [Eq. (3)], one gets the results

$$V = \mathrm{Vol}_{\mathrm{HS}}(\mathcal{M}_N^{\mathrm{sub}}),$$

$$D = \mathrm{dimension}(\mathcal{M}_N^{\mathrm{sub}}) = N^2,$$

(17)

$$A = \mathrm{Vol}_{\mathrm{HS}}(\mathcal{M}_N),$$

$$h = D_{\mathrm{HS}}(\mathcal{M}_N, 0) = \inf_{\rho \in \mathcal{M}_N} D_{\mathrm{HS}}(\rho, 0) = \frac{1}{\sqrt{N}}.$$

Note that $h = D_{\mathrm{HS}}(\rho_\star, 0)$, where $\rho_\star = \mathbb{I}_N/N$ denotes the maximally mixed state of Fig. 1, due to the following chain of relations:

$$h^2 = \inf_{\rho \in \mathcal{M}_N} D_{\mathrm{HS}}^2(\rho, 0) = \inf_{\rho \in \mathcal{M}_N} \mathrm{Tr}\, \rho^2 = \inf_{\Lambda \in \Delta_{N-1}} \sum_{i=1}^{N} \Lambda_i^2 = \inf_{\Lambda_1 + \cdots + \Lambda_N = 1} [\Lambda_1^2 + \cdots \Lambda_N^2] = \frac{1}{N}. \quad (18)$$

Substituting results of Eq. (17) into Eq. (16), we arrive at

$$\mathrm{Vol}_{\mathrm{HS}}(\mathcal{M}_N^{\mathrm{sub}}) = \frac{\mathrm{Vol}_{\mathrm{HS}}(\mathcal{M}_N)}{N^{5/2}}, \quad (19)$$

which, due to Eq. (11), is consistent with the volume given by Eq. (15).

Generating random subnormalized mixed states with respect to Hilbert-Schmidt measure

As we have already seen, the HS measure endows $\mathcal{M}_N$ with the flat, Euclidean geometry. Moreover, as emphasized in Fig. 1, the set $\mathcal{M}_N^{\mathrm{sub}}$ of subnormalized states constitutes with respect to this geometry an N^2-dimensional cone, whose $(N^2 - 1)$-dimensional base is $\mathcal{M}_N$. Thus, directly from Eq. (1), every state $\sigma \in \mathcal{M}_N^{\mathrm{sub}}$ can be decomposed as

$$\sigma = a\rho + (1 - a)0, \quad (20)$$

where $\rho \in \mathcal{M}_N$, 0 is the null state, and a is a positive number less than or equal to 1. Therefore generating states $\{\sigma_i\}$ uniformly in the cone means to distribute homogeneously ρ_i in $\mathcal{M}_N$ and to use them in Eq. (20). The weights $a_i \in [0,1]$ must be distributed accordingly to a density function $f(a)$ scaling as a^{N^2-1}. These random numbers may be obtained by inverting the cumulative distribution function $F(x) := \int_0^x f(x)\mathrm{d}x$ over uniformly distributed random numbers $\xi_i \in [0,1]$, that is, generating homogeneously $\xi_i \in [0,1]$ and taking $a_i := F^{-1}(\xi_i) = \xi_i^{1/N^2}$. As a result, given a sequence of HS-distributed mixed states $\{\rho_i\} \in \mathcal{M}_N$, as the one studied in Sec. III A, and a sequence $\{\xi_i\}$ of random numbers uniformly distributed in [0,1], one obtains an algorithmic prescription to generate a sequence of HS-distributed random subnormalized states $\{\sigma_i = \xi_i^{1/N^2} \rho_i\}$.

III. MEASURE INDUCED BY PARTIAL TRACE OF MIXED STATES

The aim of this section is to determine the measure induced on a bipartite $K \times N$ quantum system AB, represented by means of an extended Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B = \mathbb{C}^K \otimes \mathbb{C}^N$, by partial tracing over one of the two subsystems. Such measures will be essential in determining the volumes of various sets of maps that is done in the subsequent sections of this work. Without loss of generality, we consider ancillary systems A whose dimension K is greater or equal to the dimension N of the system B. The induced measure depends on the choice of the systems to trace over and on the initial distribution of the density operators ρ_{AB} on the composite system $\mathcal{H}_{AB}$. We start by analyzing a propaedeutic example.

A. Partial tracing over pure states

Consider random pure states $\rho_{AB}=|\psi_{AB}\rangle\langle\psi_{AB}|$ distributed according to the natural Fubini-Study (FS) measure. Such a measure is the only unitarily invariant one on the space of pure states, and for $N=2$ it gives the measure uniformly covering the entire boundary of the Bloch ball.[2] A general pure state can be represented in a product basis as

$$|\psi_{AB}\rangle = \sum_{i=1}^{K}\sum_{j=1}^{N} M_{ji}|i\rangle_A \otimes |j\rangle_B.$$

The positive matrix $\rho_B=MM^\dagger$ is equal to the density matrix obtained by a partial trace on the K-dimensional space A, $\rho_B=\mathrm{Tr}_A(\rho_{AB})$. The spectrum of ρ_B coincides with the set of Schmidt coefficients Λ_i of the pure state $|\psi_{AB}\rangle$. The matrix M need not be Hermitian; the only constraint is the trace condition, $\mathrm{Tr}\, MM^\dagger=1$, that makes ρ_B a density matrix. Furthermore, the natural measure on the space of pure states corresponds to taking M from the Ginibre ensemble[22] and then renormalize them to ensure $\mathrm{Tr}\, MM^\dagger=1$.[23] The probability distribution of the Schmidt coefficients implied by the FS measure on $\mathcal{H}_A \otimes \mathcal{H}_B$ is given by[24]

$$P_{N,K}^{(2)}(\Lambda_1,\ldots,\Lambda_N) = B_{N,K}^{(2)}\delta\left(1 - \sum_i \Lambda_i\right)\prod_i \Theta(\Lambda_i)\Lambda_i^{K-N}\prod_{i<j}(\Lambda_i - \Lambda_j)^2, \tag{21a}$$

in which the upper index 2 is the repulsion exponent, as we deal with complex density matrices. The normalization constant $B_{N,K}^{(2)}$ reads[23]

$$B_{N,K}^{(2)} := \frac{\Gamma(KN)}{\prod_{j=0}^{N-1}\Gamma(K-j)\Gamma(N-j+1)}. \tag{21b}$$

Observe that the entire distribution [Eq. (9)], including the normalization constant, can be tuned into Eq. (21), provided that we choose $\alpha-1=K-N$. In particular, for $\alpha=1$, that is, $K=N$, Eq. (10) shows that the induced distribution for the eigenvalues of ρ_B coincides with the HS distribution. Thus, generating normalized Wishart matrices $MM^\dagger$, with M belonging to the Ginibre ensemble of $N\times N$ matrices, becomes a useful procedure for producing $N\times N$ density matrices HS distributed, and the algorithm described in Sec. II becomes effective.

B. Partial tracing over random Hilbert-Schmidt-distributed mixed states

Consider now a related problem of determining the probability distribution of states $\rho_B=\mathrm{Tr}_A(\rho_{AB})$, as it is assumed that the mixed states ρ_{AB} are distributed according to the HS measure on $\mathcal{M}_{KN}$.

From Sec. III A we know that ρ_{AB} itself can be generated by taking FS-distributed pure state $\rho_{A'B'AB}=|\psi_{A'B'AB}\rangle\langle\psi_{A'B'AB}|$ in an extended Hilbert space $\mathcal{H}_{A'B'AB}=\mathcal{H}_{A'B'}\otimes\mathcal{H}_{AB}=\mathbb{C}^{KN}\otimes\mathbb{C}^{KN}$ and by partial tracing over the ancillary subsystem $A'B'$. Putting all together we obtain

$$\rho_B = \mathrm{Tr}_A[\rho_{AB}] = \mathrm{Tr}_A[\mathrm{Tr}_{A'B'}(\rho_{A'B'AB})] = \mathrm{Tr}_{AA'B'}[|\psi_{A'B'AB}\rangle\langle\psi_{A'B'AB}|]. \tag{22}$$

The latter equation implies that the desired distribution is obtained by coupling the N-dimensional system B to an environment of size NK^2, generating FS-distributed pure states in this overall N^2K^2 system, and finally tracing out the environment. Hence the distribution of the spectrum of ρ_B is given by $P_{N,NK^2}^{(2)}(\Lambda_1,\ldots,\Lambda_N)$ of Eq. 21.

In the rest of this section we will consider the special case $K=N$. This assumption corresponds to generating mixed states distributed according to the HS measure on the space $\mathcal{M}_{N^2}$ of bipartite systems, and tracing out one of them. In particular, we will focus on the simplest cases of $N=2$ (qubits) and $N=3$ (qutrits).
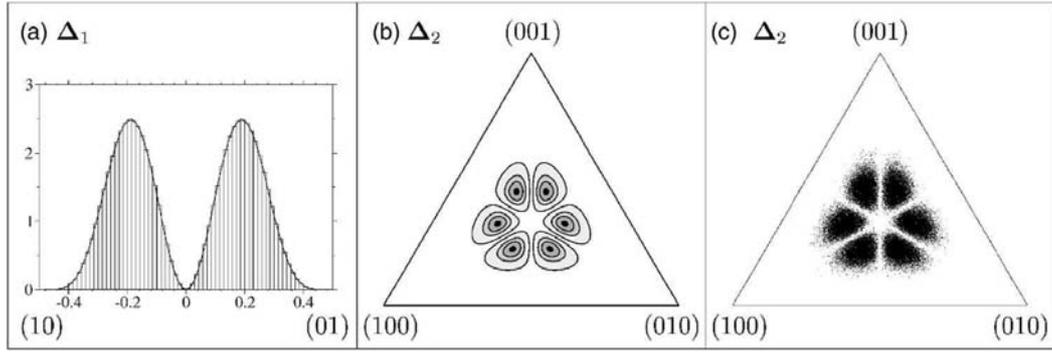
FIG. 2. Induced distributions in the simplices of eigenvalues obtained by partial trace of mixed states of $N \times N$ systems distributed according to the HS measure. (a) $N=2$, $\hat{\Delta}_1 = [-1/2, 1/2]$. Comparison of a histogram with the theoretical prediction [Eq. (25)] represented by solid line. (b) $N=3$. Contour lines of the distribution [Eq. (28)] in the simplex $\hat{\Delta}_2$. (c) $N=3$. Plot obtained numerically for 10^5 random states of size $N^2=9$.

C. Partial trace of two-qubit mixed quantum states

Let us start with the simplest case $N=2$, for which $B_{N,N^3}^{(2)} = B_{2,8}^{(2)} = 180\,180$ and the simplex Δ_1, corresponding to the positive Λ_1, Λ_2 such that $\Lambda_1 + \Lambda_2 = 1$, is nothing but an interval $[0,1]$. The probability distribution [Eq. (21)] reads in this case as

$$P_{2,8}^{(2)}(\Lambda_1, \Lambda_2) = 180\,180\, \delta(1 - \Lambda_1 - \Lambda_2) \Theta(\Lambda_1) \Theta(\Lambda_2) \Lambda_1^6 \Lambda_2^6 (\Lambda_1 - \Lambda_2)^2. \tag{23}$$

We express the eigenvalues Λ_1 and Λ_2 in terms of a real parameter $r \in \left[-\frac{1}{2}, \frac{1}{2}\right] =: \Delta_1$ as follows:

$$\Lambda_1 = \tfrac{1}{2} + r,$$
$$\Lambda_2 = \tfrac{1}{2} - r, \tag{24}$$

and we earn from Eq. (23) the radial distribution inside the Bloch ball,

$$\widetilde{P}(r) = 720\,720\, \chi_{\hat{\Delta}_1}(r) \left(\frac{1}{4} - r^2 \right)^6 r^2, \tag{25}$$

where $\chi_{\hat{\Delta}_1}(r)$ denotes the indicator function of the simplex $\hat{\Delta}_1$ [see Fig. 2(a)].

D. Partial trace of two-qutrit mixed quantum states

For $N=3$, the eigenvalues Λ_1, Λ_2, and Λ_3 can be expressed in polar coordinates (r, ϕ) in a form similar to Eq. (24),

$$\Lambda_1 = \tfrac{1}{3} + r \cos\left(\phi + \tfrac{2}{3}\pi \right),$$

$$\Lambda_2 = \tfrac{1}{3} + r \cos(\phi), \tag{26}$$

$$\Lambda_3 = \tfrac{1}{3} + r \cos\left(\phi - \tfrac{2}{3}\pi \right).$$

Similarly as before, we indicate with $\hat{\Delta}_2$ the counterimage of the simplex Δ_2, that is, the set in the (r, ϕ) plane such that $(\Lambda_1(r, \phi), \Lambda_2(r, \phi), \Lambda_3(r, \phi)) \in \Delta_2$.

Computing the Jacobian of the transformation [Eq. (26)] (with $\Lambda_3 = 1 - \Lambda_1 - \Lambda_2$), we see that the volume element transforms as

$$dV = d\Lambda_1 d\Lambda_2 = \frac{\sqrt{3}}{2} \times \chi_{\hat{\Delta}_2}(r,\phi) \times r\,dr\,d\phi.$$

The value of the radial variable r is related to the purity of the mixed state, $3/2 r^2 = \mathrm{Tr}\,\rho^2 - 1/3$, where $\mathrm{Tr}\,\rho^2 = \Lambda_1^2 + \Lambda_2^2 + \Lambda_3^2$. The constant $B_{N,N^3}^{(2)}$ follows from Eq. (21b),

$$B_{3,27}^{(2)} = \frac{80!}{12 \cdot 24! \cdot 25! \cdot 26!}.$$

We are now in the position to compute for this case the explicit probability distribution [Eq. (21)],

$$P_{3,27}^{(2)}(\Lambda_1,\Lambda_2,\Lambda_3) = B_{3,27}^{(2)} \chi_{\Delta_2}(\Lambda_1,\Lambda_2,\Lambda_3)\Lambda_1^{24}\Lambda_2^{24}\Lambda_3^{24}(\Lambda_1-\Lambda_2)^2(\Lambda_1-\Lambda_3)^2(\Lambda_2-\Lambda_3)^2, \quad (27)$$

that in the polar plane reads

$$\widetilde{P}(r,\phi) = \frac{9}{64} \frac{80!}{24! \cdot 25! \cdot 26!} \chi_{\hat{\Delta}_2}(r,\phi)r^6 \sin^2(3\phi)\left(\frac{1}{27} - \frac{1}{4}r^2 + \frac{1}{4}r^3 \cos(3\phi)\right)^{24}. \quad (28)$$

The latter distribution is invariant under the transformations $\phi \to -\phi$ and $\phi \to \phi + 2k\pi/3$, $k \in \mathbb{Z}$, as shown in Fig. 2.

IV. TRACE-NONINCREASING MAPS

A generic state of an N-dimensional quantum system is completely described once a positive, Hermitian, and normalized density matrix $\rho \in \mathcal{M}_N$ is given. In order to analyze and classify the set of all possible physical operations on a quantum system, we need to describe the set of *superoperators* Φ mapping $\mathcal{M}_N$ onto itself. The linearity of physical operations can be expressed by forcing the superoperator $\Phi : \mathcal{M}_N \mapsto \mathcal{M}_N$ to act as a matrix action on the "vector" ρ, that is,

$$\rho' = \Phi\rho \quad \text{or} \quad \rho'_{m\mu} = \Phi_{\substack{m\mu \\ n\nu}}\rho_{n\nu}.$$

We use Einstein summation convention for indices appearing twice. In order to map the domain $\mathcal{M}_N$ onto itself, the linear superoperator Φ must fulfill these additional properties.

Properties 1:

$$\rho' = (\rho')^{\dagger} \Leftrightarrow \Phi_{\substack{m\mu \\ n\nu}} = \Phi_{\substack{\mu m \\ \nu n}}^{*}, \quad (29a)$$

$$\mathrm{Tr}\,\rho' = \mathrm{Tr}\,\rho = 1 \Leftrightarrow \Phi_{\substack{mm \\ n\nu}} = \delta_{n\nu}, \quad (29b)$$

$$\rho' \geqslant 0 \Leftrightarrow \Phi_{\substack{m\mu \\ n\nu}}\rho_{n\nu} \geqslant 0 \quad \text{when } \rho \geqslant 0. \quad (29c)$$

In particular, all the superoperators Φ fulfilling property (29b) are called TP. For any given Φ, it proves convenient to introduce the *dynamical matrix* D_Φ, uniquely and linearly obtained from the superoperator by reshuffling the indices,[25,26]

$$D_{\substack{mn \\ \mu\nu}} = \Phi_{\substack{m\mu \\ n\nu}}. \quad (30)$$

The dynamical matrix can be thought as a matrix on a bipartite $N \times N$ quantum system AB, represented by means of an extended Hilbert space $\mathcal{H}_{AB}$ of the same kind of the one in Sec. III, so that $D_{\substack{mn \\ \mu\nu}} = {}_A\langle m| \otimes {}_B\langle n|D_\Phi|\mu\rangle_A \otimes |\nu\rangle_B$.

In terms of D_Φ, Properties 1 can be reexpressed as follows.

Properties 2:

$$\rho' = (\rho')^\dagger \Leftrightarrow D_{mn \atop \mu\nu} = D^*_{\mu\nu \atop mn} \quad \text{so } D_\Phi = D_\Phi^\dagger, \tag{31a}$$

$$\text{Tr}\,\rho' = \text{Tr}\,\rho = 1 \Leftrightarrow D_{mn \atop m\nu} = \delta_{n\nu} \quad \text{so } \text{Tr}_A\,D_\Phi = \mathbb{I}_N, \tag{31b}$$

$$\rho' \geqslant 0 \Leftrightarrow D_{mn \atop \mu\nu}\,\rho_{n\nu} \geqslant 0 \quad \text{when } \rho \geqslant 0. \tag{31c}$$

In the following we will focus on the set $\mathcal{CP}_N$ of maps $\Phi \colon \mathcal{B}(\mathbb{C}^N) \mapsto \mathcal{B}(\mathbb{C}^N)$ that are CP,[27] that is, on maps Φ such that for every identity map $\mathbb{I}_K \colon \mathbb{C}^K \mapsto \mathbb{C}^K$ the extended maps $\Phi \otimes \mathbb{I}_K \colon \mathcal{B}(\mathbb{C}^N) \otimes \mathbb{C}^K \mapsto \mathcal{B}(\mathbb{C}^N) \otimes \mathbb{C}^K$ are positive [here $\mathcal{B}(\mathbb{C}^N)$ is the Banach space of bounded linear operators on $\mathbb{C}^N$]. A very important property characterizes the dynamical matrix D_Φ of a CP map Φ.

Theorem (Choi[28]): *A linear map Φ is CP if and only if the corresponding dynamical matrix D_Φ is positive.*

Note that property (31c) holds true in general once the positivity of the matrix D_Φ is proved for product states of $\mathbb{C}^N \otimes \mathbb{C}^N$, as stated in the Jamiołkowski theorem.[16] This property is implied by complete positivity, that is, a stronger condition. If we combine Eqs. (31a) and (31b) with the complete positivity, we observe that for any given $\Phi \in \mathcal{CP}_N^{\text{TP}}$ (i.e., the set of CP TP maps), its *rescaled dynamical matrix* $\rho_\Phi := D_\Phi/N$ possesses the three properties of Hermiticity, positivity, and normalization. Thus ρ_Φ belongs to the set $\mathcal{M}_{N^2}$ of density matrices of size N^2. This leads to the celebrated Jamiołkowski isomorphism,[16] that maps the entire set $\mathcal{CP}_N^{\text{TP}}$ of quantum maps onto a proper $(N^4 - N^2)$-dimensional subset of the set $\mathcal{M}_{N^2}$ of quantum states on extended system. This subset, denoted by $\mathcal{M}_{N^2}^{\mathrm{I}}$, contains all states ρ such that $\text{Tr}_A\,\rho = \mathbb{I}_N$. The N^2 constraints reducing the dimension of $\mathcal{M}_{N^2}$ come from Eq. (31b). With the aim of removing these N^2 constraints, we introduce a family of linear maps.

Definition 2: A linear positive map Φ is called TNI, if $\text{Tr}\,\Phi(\rho) \leqslant \text{Tr}\,\rho = 1$ for any $\rho \in \mathcal{M}_N$.

We state now a lemma that makes a link between CP-TNI maps and their images given by the Jamiołkowski isomorphism, that is, the set of their (rescaled) dynamical matrices.

Lemma 2: For any given $\Phi \in \mathcal{CP}_N^{\text{TNI}}$, its rescaled dynamical matrix $\sigma_\Phi := D_\Phi/N$ is Hermitian [according to Eq. (31a)], positive (as in the statement of the theorem of Choi), and fulfills the following constraint:

$$\text{Tr}\,\Phi(\rho) = \text{Tr}\,\rho' \leqslant \text{Tr}\,\rho = 1 \Leftrightarrow \text{Tr}_A\,\sigma_\Phi \leqslant \frac{\mathbb{I}_N}{N}. \tag{32}$$

For a proof of this lemma, see Appendix B. Note that the above constraint is a kind of a relaxation of Eq. (31b). The *reduced and rescaled dynamical matrix* $\text{Tr}_A\,\sigma_\Phi$ belongs then to a subset of the set $\mathcal{M}_N^{\text{sub}}$ defined in Eq. (1). To specify this subset consider the following set of *subtracial states:*

$$\mathcal{M}_N^{\square} := \left\{ \sigma \in \mathcal{M}_N^{\text{sub}} : \sigma \leqslant \frac{\mathbb{I}_N}{N} \right\} = \left\{ \sigma \in \mathcal{M}_N^{\text{sub}} : \max[\text{EV}(\sigma)] \leqslant \frac{1}{N} \right\}, \tag{33}$$

where $\text{EV}(\sigma)$ denotes the set of eigenvalues of σ. Note that the set $\mathcal{M}_N^{\square}$ coincides, up to rescaling by N, with the set of positive Hermitian operators E_i fulfilling $E_i \leqslant \mathbb{I}_N$. A collection of such operators, which satisfies the relation $\Sigma_i E_i = \mathbb{I}_N$, is called positive operator valued measure (POVM) and plays a crucial role in the theory of quantum measurement.[2]

Definition (33) allows us to rewrite condition (32) as

$$\Phi \in \mathcal{CP}_N^{\text{TNI}} \Leftrightarrow \text{Tr}_A\,\sigma_\Phi \in \mathcal{M}_N^{\square}. \tag{34}$$

Finally, we can summarize the result of Eq. (34) in the following.

Proposition: Every TNI map $\Phi \in \mathcal{CP}_N^{\text{TNI}}$ can be represented by a subtracial state $\check{\sigma}_\Phi \in \mathcal{M}_N^{\square} \subset \mathcal{M}_N^{\text{sub}}$, whose explicit expression is given in terms of the rescaled dynamical matrix σ_Φ of Lemma 2 by

TABLE I. The two kinds of CP-TP and CP-TNI maps analyzed in this section are mapped in the set of their correspondent superoperators Φ, the set of their rescaled dynamical matrices σ_Φ, and the set of their representative subtracial states $\check{\sigma}_\Phi$. Inclusion relations between the sets under consideration are explicitly shown in the table.

	Map Φ	Rescaled dynamical matrix σ_Φ	Reduced and rescaled dynamical matrix $\check{\sigma}_\Phi$
CP Trace Preserving maps	$\mathcal{CP}_N^{\text{TP}}$	$\mathcal{M}_{N^2}^{\mathbb{I}} = (\mathcal{M}_{N^2} \cap \mathcal{M}_{N^2}^{\boxplus})$	$\mathbb{I}_N/N \in (\mathcal{M}_N \cap \mathcal{M}_N^{\square})$
CP Trace Non Increasing maps	$\mathcal{CP}_N^{\text{TNI}}$	$\mathcal{M}_{N^2}^{\boxplus} \subset \mathcal{M}_{N^2}^{\text{sub}}$	$\mathcal{M}_N^{\square} \subset \mathcal{M}_N^{\text{sub}}$

$$\check{\sigma}_\Phi := \text{Tr}_A \, \sigma_\Phi = \frac{1}{N} \text{Tr}_A \, D_\Phi.$$

Moreover, the rescaled dynamical matrix σ_Φ offers itself another representation of Φ into an N^4-dimensional proper subset of $\mathcal{M}_{N^2}^{\text{sub}}$, as one can derive from the rhs of Eq. (32) that

$$\text{Tr} \, \sigma_\Phi = \text{Tr}_{AB} \, \sigma_\Phi = \text{Tr}_B(\text{Tr}_A \, \sigma_\Phi) \leqslant \text{Tr}_B(\mathbb{I}_N/N) = 1.$$

In Definition 3, we will denote this set with $\mathcal{M}_{N^2}^{\boxplus}$.

Definition 3:

$$\mathcal{M}_{N^2}^{\boxplus} := \{\sigma \in \mathcal{M}_{N^2}^{\text{sub}} : \text{Tr}_A \, \sigma \in \mathcal{M}_N^{\square}\}.$$

To clarify the notation used we collect the sets of states and maps considered in Table I.

In the space of the N eigenvalues of $N \times N$ positive Hermitian matrices, the set $\mathcal{M}_N^{\square}$ of subtracial states defined in Eq. (33) consists of a cube inscribed into the set $\mathcal{M}_N^{\text{sub}}$ of subnormalized states of size N. Such a multidimensional cube has a vertex in the origin, is oriented along axes, and touches the simplex of quantum states $\mathcal{M}_N$ in a single point $\mathbb{I}_N/N$, as shown in Fig. 3 for $N=2$ and $N=3$. Observe that every $\check{\sigma}_\Phi$ of $\mathcal{M}_N$ can be rescaled in order to let it belong to the cube of $\mathcal{M}_N^{\square}$. In terms of maps it means that every $\Phi \in \mathcal{CP}_N$ can be mixed with the 0 map in order to become TNI.

The set $\mathcal{M}_{N^2}^{\boxplus}$ of CP-TNI maps has N^4 dimensions. It contains the $(N^4 - N^2)$-dimensional set $\mathcal{M}_{N^2}^{\mathbb{I}}$ of CP-TP maps, which satisfy $\text{Tr} \, \Phi(\rho) = \text{Tr}(\rho)$, as a subset. The set $\mathcal{CP}_N^{\text{TP}}$ forms extremal points in $\mathcal{CP}_N^{\text{TNI}}$.
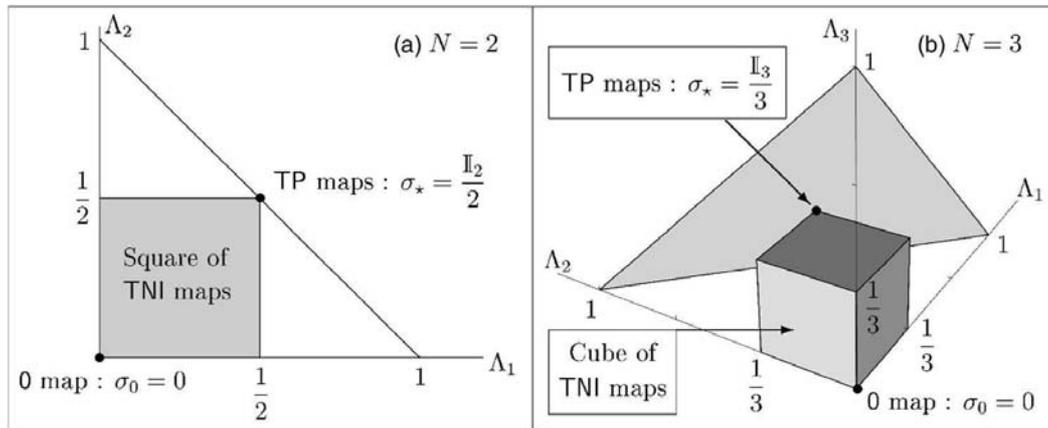

FIG. 3. The set of CP-TNI maps $\Phi \in \mathcal{CP}_N^{\text{TNI}}$ represented, by means of the Jamiołkowski isomorphism, as the subset $\mathcal{M}_N^{\square}$ of the set of subnormalized states $\mathcal{M}_N^{\text{sub}}$, in terms of the reduced and rescaled dynamical matrices $\check{\sigma}_\Phi = \text{Tr}_A \, \sigma_\Phi$. The eigenvalues of these matrices $\check{\sigma}_\Phi$ span an N-dimensional cube inscribed into the set of subnormalized states of size N, here plotted for (a) $N=2$ and (b) $N=3$.

V. VOLUME OF THE SET OF CP-TNI MAPS

In Sec. IV, the Jamiołkowski isomorphisms allowed us to establish links between superoperators $\Phi \in \mathcal{CP}_N^{\mathrm{TNI}}$, rescaled dynamical matrices $\sigma_\Phi = \mathrm{Tr}_A\, \sigma_\Phi \in \mathcal{M}_{N^2}^{\boxplus}$, and subtracial states $\check{\sigma}_\Phi \in \mathcal{M}_N^{\square}$. In this section we will make use of this representation aiming to define a measure in $\mathcal{CP}_N^{\mathrm{TNI}}$ and compute its volume.

We start by defining the H S measure $d\mu_{\mathrm{HS}}^{\boxplus}$ on the space of TNI maps $\Phi \in \mathcal{CP}_N^{\mathrm{TNI}}$ as the HS measure of their representative reduced dynamical matrices $\sigma_\Phi \in \mathcal{M}_{N^2}^{\boxplus} \subset \mathcal{M}_{N^2}^{\mathrm{sub}}$. Such a measure is analogous to the one introduced in Eqs. (13) and (14), restricted from $\mathcal{M}_{N^2}^{\mathrm{sub}}$ to $\mathcal{M}_{N^2}^{\boxplus}$, and reads

$$d\mu_{\mathrm{HS}}^{\boxplus}(\Phi) = d\mu_{\mathrm{HS}}^{\mathrm{sub}}(\sigma_\Phi)|_{\mathcal{M}_{N^2}^{\boxplus}\subset\mathcal{M}_{N^2}^{\mathrm{sub}}} = \frac{1}{N^2!} \, d\nu^{\mathrm{sub}}(\Lambda_1, \ldots, \Lambda_{N^2})|_{\mathcal{M}_{N^2}^{\boxplus}\subset\mathcal{M}_{N^2}^{\mathrm{sub}}} d\nu^{\mathrm{Haar}}. \tag{35}$$

Here $\Lambda_1, \ldots, \Lambda_{N^2}$ denote the eigenvalues of the matrix σ_Φ of size N^2.

The restriction of the space from $\mathcal{M}_{N^2}^{\mathrm{sub}}$ to $\mathcal{M}_{N^2}^{\boxplus}$ does not affect the Haar measure on the entire complex flag manifold $\mathrm{Fl}_{\mathbb{C}}^{(N^2)}$, given by the coset space $\mathrm{U}(N^2)/[\mathrm{U}(1)]^{N^2}$ of equivalence classes of unitary matrices of eigenstates of σ_Φ. The volume of this flag manifold follows from Eq. (6),

$$\mathrm{Vol}[\mathrm{Fl}_{\mathbb{C}}^{(N^2)}] = \int_{\mathrm{Fl}_{\mathbb{C}}^{(N^2)}} d\nu^{\mathrm{Haar}} = \frac{(2\pi)^{N^2(N^2-1)/2}}{1!2!\cdots(N^2-1)!}. \tag{36}$$

In order to parallel Eqs. (11) and (15), we need to compute the volume spanned by the eigenvalues of matrices contained in $\mathcal{M}_{N^2}^{\boxplus}$ according to the measure $d\nu^{\mathrm{sub}}(\Lambda_1, \ldots, \Lambda_{N^2})$, multiply it by the volume [Eq. (36)] of the flag manifold $\mathrm{Fl}_{\mathbb{C}}^{(N^2)}$, and divide the result by $N^2!$ as in Eq. (35). Performing this task one obtains

$$\mathrm{Vol}_{\mathrm{HS}}(\mathcal{CP}_N^{\mathrm{TNI}}) = \frac{1}{N^2!}\mathrm{Vol}[\mathrm{EV}(\mathcal{M}_{N^2}^{\boxplus})]\mathrm{Vol}[\mathrm{Fl}_{\mathbb{C}}^{(N^2)}] = \frac{1}{N^2!}\int_{\mathrm{EV}(\mathcal{M}_{N^2}^{\boxplus})} d\nu^{\mathrm{sub}}(\Lambda_1, \ldots, \Lambda_{N^2})\mathrm{Vol}[\mathrm{Fl}_{\mathbb{C}}^{(N^2)}].$$

$$\tag{37}$$

Note that the operation of partial trace maps the set $\mathcal{M}_{N^2}^{\boxplus}$ into $\mathcal{M}_N^{\square}$. Assuming that mixed states from $\mathcal{M}_{N^2}^{\boxplus} \subset \mathcal{M}_{N^2}^{\mathrm{sub}}$ are distributed according to the HS measure, we need to analyze the measure induced in $\mathcal{M}_N^{\square} \subset \mathcal{M}_N^{\mathrm{sub}}$ by partial trace.

In Sec. III B we described how the normalized HS distribution on $\mathcal{M}_{NK}$, whose expression is equivalent to the induced distribution $P_{NK,NK}^{(2)}(\Lambda_1, \ldots, \Lambda_{NK})$ of Eq. (21), is mapped by partial tracing into the distribution $P_{N,NK^2}^{(2)}(\Lambda_1, \ldots, \Lambda_N)$. In particular, for $K=N$, the HS distribution on $\mathcal{M}_{N^2}$ is mapped into $P_{N,N^3}^{(2)}(\Lambda_1, \ldots, \Lambda_N)$.

In a completely equivalent way, the measure $d\nu^{\mathrm{sub}}(\Lambda_1, \ldots, \Lambda_{N^2}) = d\nu_{N^2}(\Lambda_1, \ldots, \Lambda_{N^2})$ of Eq. (35) is transformed by partial tracing into the measure $d\nu_{N^3}(\Lambda_1, \ldots, \Lambda_N)$, and Eq. (37) yields

$$\mathrm{Vol}_{\mathrm{HS}}(\mathcal{CP}_N^{\mathrm{TNI}}) = \frac{1}{N^2!}\int_{\mathrm{EV}(\mathcal{M}_{N^2}^{\boxplus})} d\nu_{N^2}(\Lambda_{1^2}, \ldots, \Lambda_{N^2})\mathrm{Vol}[\mathrm{Fl}_{\mathbb{C}}^{(N^2)}]$$

$$= \frac{\mathrm{Vol}[\mathrm{Fl}_{\mathbb{C}}^{(N^2)}]}{N^2!}\int_{\mathrm{EV}(\mathcal{M}_N^{\square})} d\nu_{N^3}(\Lambda_1, \ldots, \Lambda_N). \tag{38}$$

Making use of Eqs. (12) and (36), we arrive at

$$\text{Vol}_{\text{HS}}(\mathcal{CP}_N^{\text{TNI}}) = \frac{(2\pi)^{N^2(N^2-1)/2}}{1!2!\cdots N^2!} \int \prod_{i=1}^N \Theta\left(\frac{1}{N} - \Lambda_i\right) \prod_{i=1}^N \Theta(\Lambda_i)\Lambda_i^{N^3-N} \prod_{i<j}(\Lambda_i - \Lambda_j)^2 d\Lambda_1, \ldots, d\Lambda_N,$$

(39)

where we have made explicit the domain of integration using the Heaviside step function Θ. The last integral can be computed using Lemma 3, whose proof is in Appendix C.

Lemma 3: On the N-dimensional cube $\square_N := [0, 1/N]^N$, consider the one parameter family of measures $d\nu_K^{\square}(\Lambda_1, \ldots, \Lambda_N)$,

$$d\nu_K^{\square}(\Lambda_1, \ldots, \Lambda_N) = \prod_{i=1}^N \Theta\left(\frac{1}{N} - \Lambda_i\right)\Theta(\Lambda_i)\Lambda_i^{K-N}\prod_{i<j}(\Lambda_i - \Lambda_j)^2 d\Lambda_1, \ldots, d\Lambda_N,$$

(40)

labeled by any integer $K \geq N$. Then the volume $\nu_K^{\square}(\square_N)$ reads

$$\int_{\square_N} d\nu_K^{\square}(\Lambda_1, \ldots, \Lambda_N) = \frac{I(K - N + 1, 1, 1, N)}{N^{KN}},$$

(41)

where $I(K-N+1, 1, 1, N)$ is the Selberg's integral[22]

$$I(K - N + 1, 1, 1, N) = \prod_{j=1}^N \frac{\Gamma(1+j)\Gamma(K-N+j)\Gamma(j)}{\Gamma(2)\Gamma(K+j)}.$$

(42)

Finally, using Lemma 2, Eq. (39) yields the final formula for the HS volume of the set of TNI maps,

$$\text{Vol}_{\text{HS}}(\mathcal{CP}_N^{\text{TNI}}) = \frac{(2\pi)^{N^2(N^2-1)/2}}{1!2!\cdots N^2!}\int_{\square_N} d\nu_{N^3}^{\square}(\Lambda_1, \ldots, \Lambda_N) = \frac{(2\pi)^{N^2(N^2-1)/2}}{1!2!\cdots N^2!}\prod_{j=1}^N \frac{\Gamma(1+j)\Gamma(N^3-N+j)\Gamma(j)}{N^{N^3}\Gamma(N^3+j)}.$$

(43)

Equation (38) implies that the HS volume of the set of CP-TNI maps is proportional to the volume of the set of subtracial states $\mathcal{M}_N^{\square}$ of Eq. (33) computed according to the induced measure $d\nu_{N^3}(\Lambda_1, \ldots, \Lambda_N)$. In particular, Lemma 3 allows us to compute such a volume, which is in turn proportional to the volume spanned by all possible elements of a POVM, according to any given induced measure $d\nu_K(\Lambda_1, \ldots, \Lambda_N)$. Moreover, as we explained in the Remark, also the volume of all subnormalized states $\mathcal{M}_N^{\text{sub}}$ can be computed according to any induced measure. For comparison, we present below their ratio for any value of parameter K, reminding the reader that in the former calculations we set $K = N^3$.

$$\frac{\text{Vol}_{N,K}(\mathcal{M}_N^{\square})}{\text{Vol}_{N,K}(\mathcal{M}_N^{\text{sub}})} = \frac{NK}{N^{NK}}C_N^{(K-N+1,2)}I(K-N+1, 1, 1, N) = \frac{(NK)!}{N^{NK}}\prod_{j=1}^N \frac{\Gamma(j)}{\Gamma(K+j)}.$$

VI. EXTREMAL CP-TNI MAPS

We start this section by introducing another representation for CP-TNI maps $\Phi: \rho \mapsto \rho'$, alternative to the one described in Sec. IV and summarized in Lemma 4 (for a proof see Appendix D).

Lemma 4: For any given CP-TNI map Φ, its action on density matrices $\rho \in \mathcal{M}_N$ can be described by using a discrete family of $N \times N$ operators A_μ, whose number S does not exceed N^2. The explicit action of the map Φ reads

$$\Phi(\rho) = \sum_{\mu=1}^{S} A_\mu \rho A_\mu^\dagger, \tag{44a}$$

and the matrices A_μ fulfill

$$\sum_{\mu=1}^{S} A_\mu^\dagger A_\mu = N\check{\sigma}_\Phi^{\mathrm{T}}, \tag{44b}$$

where the subtracial state $\check{\sigma}_\Phi \in \mathcal{M}_N^\square$ is given by the reduced and rescaled dynamical matrix correspondent to the map Φ, as in the Proposition and σ^{T} means *transpose of* σ. Conversely, if $S \leq N^2$ operators A_μ of size $N \times N$ fulfill

$$\frac{1}{N}\sum_{\mu=1}^{S} A_\mu^\dagger A_\mu = \check{\sigma} \in \mathcal{M}_N^\square, \tag{45a}$$

then

$$\Phi(\rho) := \sum_{\mu=1}^{S} A_\mu \rho A_\mu^\dagger \tag{45b}$$

defines a CP-TNI map.

When Φ is a CP-TP map, that is, when $\check{\sigma}_\Phi = \check{\sigma}_\Phi^{\mathrm{T}} = \mathbb{I}_N/N$, then Eqs. (44) define the usual Kraus-Stinespring (KS) representation and the operators A_μ are said *Kraus operators*. In this sense Lemma 4 can be seen as an extension of the KS representation.

We aim to use the previous lemma on a particular subclass of CP-TNI maps, represented by very trivial subtracial states, as in the following.

Definition 4: We say that a map $\Phi \in \mathcal{CP}_N^{\mathrm{TNI}}$ is *k extremal* if its representative subtracial state $\check{\sigma}_\Phi \in \mathcal{M}_N^\square$ is diagonal and possesses exactly k nonvanishing entries, all equal to $1/N$. Those subtracial states represent maximally mixed states in k-dimensional subspace. In the following such maps will be denoted by Φ_k.

Let Ω_N denote the ordered set of positive integer less than or equal to N, and let $\mathcal{P}(\Omega_N)$ be the set " sorted parts of Ω_N," that is, the set of all possible ordered collections of elements of Ω_N. Consider the partition

$$\mathcal{P}(\Omega_N) = \bigcup_{k=0}^{N} \mathcal{P}_k(\Omega_N), \tag{46}$$

where each $\mathcal{P}_k(\Omega_N)$ contains exactly k elements, $\mathcal{P}_0(\Omega_N) = \{\varnothing\}$ and $\mathcal{P}_N(\Omega_N) = \{\Omega_N\}$. Thus it proves convenient to use the natural isomorphism

$$\zeta \in \mathcal{P}_k(\Omega_N) \sim \check{\sigma}_\zeta = \frac{1}{N}\sum_{\ell \in \zeta} |\ell\rangle\langle\ell|, \tag{47}$$

for labeling the set of k-extremal CP-TNI maps: See Fig. 4 for an example with $N=3$, on which

$$\mathcal{P}_3(\Omega_3) = \{\Omega_3\} = \{\{1,2,3\}\}, \quad \mathcal{P}_2(\Omega_3) = \{\{1,2\},\{1,3\},\{2,3\}\},$$

$$\mathcal{P}_1(\Omega_3) = \{\{1\},\{2\},\{3\}\}, \quad \text{and } \mathcal{P}_0(\Omega_3) = \{\varnothing\}.$$

The vectors $|\ell\rangle$ are meant to be orthonormal. Figure 4 illustrates another geometrical meaning of the integer number k labeling a k-extremal map. It is proportional to the *taxi distance* in $\mathbb{R}^N$ between $\mathbf{0}$ and the vector of diagonal entries of $\check{\sigma}_\zeta$, that is, the minimal number of sides connecting 0 to $\check{\sigma}_\zeta$ in the plot of eigenvalues.

From Eq. (47) we learn that $\check{\sigma}_\zeta$ is essentially a projector operator rescaled by N, $\check{\sigma}_\zeta = P_\zeta/N$. We denote as $\mathcal{M}_\zeta$ the image of $\mathcal{M}_N$ obtained by projection,
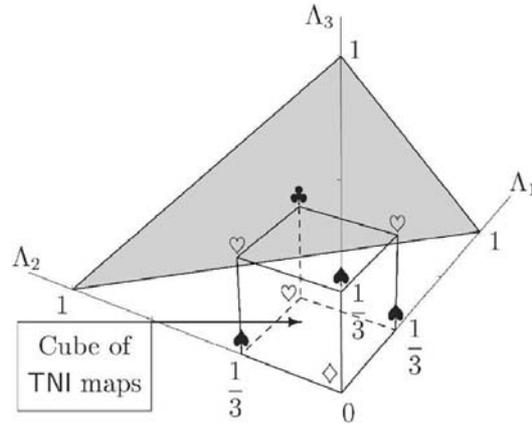
FIG. 4. Set of CP-TNI maps with extremal points marked. In particular, (♣) marks the CP-TP map $\Phi_3 = \mathbb{I}_3/3$, (♢) marks the 0 map $\Phi_0 = 0$, and (♡) and (♠) mark maps Φ_2, and Φ_1, respectively.

$$\mathcal{M}_\zeta := \bigcup_{\rho \in \mathcal{M}_N} P_\zeta \rho P_\zeta \subset \mathcal{M}_N^{\text{sub}}. \tag{48}$$

Using the isomorphism [Eq. (47)], Lemma 4 can be rephrased as follows.

Lemma 5: For any given k-extremal CP-TNI map Φ_k, that is, for any given $\zeta \in \mathcal{P}_k(\Omega_N)$, its action on density matrices $\rho \in \mathcal{M}_N$ can be described by using a discrete family of $N \times N$ operators A_μ, whose number S does not exceed N^2. The explicit action of the map Φ_k reads

$$\Phi(\rho) = \sum_{\mu=1}^{S} A_\mu \rho A_\mu^\dagger, \tag{49a}$$

and the matrices A_μ fulfill

$$\sum_{\mu=1}^{S} A_\mu^\dagger A_\mu = N \check{\sigma}_\zeta. \tag{49b}$$

Simply using Lemma 5 we deduce the following.

Properties 3: Let Φ_k be a k-extremal CP-TNI map, represented by some $\check{\sigma}_\zeta$, with $\zeta \in \mathcal{P}_k(\Omega_N)$. Then it holds true with the following conditions.

(1) When $k=N$, Φ_N belongs to $\mathcal{CP}_N^{\text{TP}}$. When $k<N$ the TP condition becomes ρ dependent: nevertheless $\Phi_k \in \mathcal{CP}_N^{\text{TNI}}$ acts as a TP map on $\rho \in \mathcal{M}_\zeta$.
(2) Acting on the maximally mixed state $\rho_\star := \mathbb{I}_N/N$ reveals the parameter k, being $\text{Tr}[\Phi_k(\rho_\star)] = k/N$.
(3) $\mathcal{M}_N^{\text{sub}} \supseteq \Phi_k(\rho_\star) = (k/N)\omega$, for some $\omega \in \mathcal{M}_N$. When $k=N$ and Φ is *unital*, then $\omega = \rho_\star$.

Proof of Properties 3:

(1) Tracing both sides of Eq. (49a), using the cyclicity property of the trace, and inserting Eq. (49b), one obtains $\text{Tr}[\Phi_k(\rho)] = \text{Tr}(N\check{\sigma}_\zeta \rho) = \text{Tr}(P_\zeta \rho)$, where P_ζ is the projector of Eq. (48). Then using the property of projector and cyclicity, together with definition (48), one gets

$$\text{Tr}[\Phi_k(\rho)] = \text{Tr}(P_\zeta^2 \rho) = \text{Tr}(P_\zeta^2 \rho P_\zeta^2). \tag{50}$$

Assuming that $\rho \in \mathcal{M}_\zeta$ means, according to definition (48), that it exists a quantum state $x \in \mathcal{M}_N$ such that $\rho = P_\zeta x P_\zeta$, so that

$$\mathrm{Tr}[\Phi_k(\rho)] = \mathrm{Tr}(P_\zeta^2 x P_\zeta^2) = \mathrm{Tr}(P_\zeta^2 x P_\zeta^2) = \mathrm{Tr}(\rho), \tag{51}$$

and Φ_k shows trace preservation on $\mathcal{M}_\zeta$. In particular, for $k=N$, $\mathcal{M}_\zeta = \mathcal{M}_N$ and Eq. (51) holds in full generality.

(2)   From the same lines of previous point (1) one gets $\mathrm{Tr}[\Phi(\rho_\star)] = \mathrm{Tr}(N\check{\sigma}_\zeta \mathbb{I}_N/N) = \mathrm{Tr}(P_\zeta)/N = k/N$.

(3)   We deduce from Lemma 5 that

$$\Phi(\rho_\star) = \frac{1}{N}\sum_{\mu=1}^{S} A_\mu A_\mu^\dagger, \tag{52}$$

that is, positive and Hermitian, too. Its normalization, computed in previous point (2), makes this state a subnormalized state. The second statement in (3) is actually the definition of CP unital maps.   ∎

As a particular example, we will consider now the case of k-extremal CP-TNI maps Φ_k associated with rescaled dynamical matrices that are *product states*, $\sigma_\zeta = \omega \otimes \check{\sigma}_\zeta$ for some $\omega \in \mathcal{M}_N$ and $\zeta \in \mathcal{P}_k(\Omega_N)$. For this very special case, the action of Φ_k on a generic $\rho \in \mathcal{M}_N$ reads

$$\Phi(\rho) = \omega\,\mathrm{Tr}(P_\zeta \rho). \tag{53}$$

The map Φ_k sends the entire k-dimensional subspace spanned by $\mathcal{M}_\zeta$ into ω, whereas the complementary subspace is annihilated.

VII. CONCLUDING REMARKS

In this work we investigated the set of subnormalized quantum states and the set of CP-TNI maps. In particular, we computed the volume of the set of subnormalized states and provided an algorithm to generate such states randomly with respect to the (Euclidean) measure.

We described the structure of the set of CP-TNI maps and computed its volume. It is worth it to emphasize that untill now no exact result for the volume of the set of CP-TP maps is known (although some estimates can be done[29]). On the other hand, in this work we obtained an exact result for the HS volume of the set of TNI maps, which includes the set of TP maps.

Our paper contains several side results worth mentioning. In particular, we found the following.

(a)   The probability distribution of states obtained by partial trace of mixed states of the bipartite system distributed according to the measure.

(b)   The volume of the sets of normalized, subnormalized, and subtracial states with respect to the measures induced by partial trace.

(c)   An interpretation of the extremal TNI maps, which act as TP on certain subspaces.

APPENDIX A: PROOF OF LEMMA 1

For any given $t \in \mathbb{R}^+$, consider the quantity

$$G(t,N,K) := \int \Theta\left(t - \sum_{i=1}^{N} \Lambda_i\right) \prod_{i=1}^{N} \Theta(\Lambda_i) \Lambda_i^{K-N} \prod_{i<j} (\Lambda_i - \Lambda_j)^2 d\Lambda_1, \ldots, d\Lambda_N, \quad (A1)$$

where Θ is the Heaviside step function, fulfilling $\Theta(tx) = \Theta(x)$ for every positive t. By rescaling the variables $\Lambda_i \rightarrow t\lambda_i$, we get

$$G(t,N,K) = t^{NK} \int \Theta\left(1 - \sum_{i=1}^{N} \lambda_i\right) \prod_{i=1}^{N} \Theta(\lambda_i) \lambda_i^{K-N} \prod_{i<j} (\lambda_i - \lambda_j)^2 d\lambda_1, \ldots, d\lambda_N.$$

Comparing with Eq. (A1) gives us the scaling relation $G(t,N,K) = t^{NK} G(1,N,K)$. By taking the derivative at $t=1$, we obtain

$$\left[\frac{d}{dt} G(t,N,K)\right]\bigg|_{t=1} = NK \cdot G(1,N,K), \quad (A2)$$

and Eq. (A1), together with Eq. (9b), yield

$$\left[\frac{d}{dt} G(t,N,K)\right]\bigg|_{t=1} = \int \delta\left(1 - \sum_{i=1}^{N} \Lambda_i\right) \prod_{i=1}^{N} \Theta(\Lambda_i) \Lambda_i^{K-N} \prod_{i<j} (\Lambda_i - \Lambda_j)^2 d\Lambda_1, \ldots, d\Lambda_N = \frac{1}{C_N^{(K-N+1,2)}}.$$

Observe that $G(1,N,K)$ on the rhs of Eq. (A2) is nothing but the integral in the statement of Lemma 1, so the result follows. ∎

Remark [Volume of $\mathcal{M}_N$ and $\mathcal{M}_N^{sub}$ with respect to induced measures: The one parameter measure $d\nu_K(\Lambda_1, \ldots, \Lambda_N)$ introduced in Lemma 1 is called induced measure,[23] and appears as a natural measure for reduced density matrices of pure states equidistributed on a bipartite system of dimension $N \times K$. As it is evident by comparing Eqs. (7) and (12), the HS distribution coincides with $d\nu_N(\Lambda_1, \ldots, \Lambda_N)$, and with respect to this specific measure we compute the volumes of $\mathcal{M}_N$ and of $\mathcal{M}_N^{sub}$, given by Eqs. (11) and (15). As a by-product of Lemma 1, we are able to compute the volumes of quantum states and that of subnormalized states also in the case of induced measure for arbitrary K,

$$\mathrm{Vol}_{N,K}(\mathcal{M}_N) = \frac{\sqrt{N}}{N!} \frac{\mathrm{Vol}[\mathrm{Fl}_{\mathbb{C}}^{(N)}]}{C_N^{(K-N+1,2)}} = \sqrt{N}(2\pi)^{N(N-1)/2} \frac{\Gamma(K-N+1)\Gamma(K-N+2)\cdots\Gamma(K-1)\Gamma(K)}{\Gamma(N^2)}$$

and

$$\mathrm{Vol}_{N,K}(\mathcal{M}_N^{sub}) = \frac{1}{N!} \frac{\mathrm{Vol}[\mathrm{Fl}_{\mathbb{C}}^{(N)}]}{NK C_N^{(K-N+1,2)}} = (2\pi)^{N(N-1)/2} \frac{\Gamma(K-N+1)\Gamma(K-N+2)\cdots\Gamma(K-1)\Gamma(K)}{NK\Gamma(N^2)}.$$

APPENDIX B: PROOF OF LEMMA 2

Let us define $\varepsilon_{n\nu}^{(\Phi)} := \Phi_{mm} = D_{mn}$. From Eq. (31a) it follows that $\varepsilon^{(\Phi)}$ is Hermitian. For any given $\rho \in \mathcal{M}_N$, the hypothesis $\Phi \in \mathcal{CP}_N^{TNI}$ implies $\rho'_{mm} = \Phi_{mm} \rho_{n\nu} = \varepsilon_{n\nu}^{(\Phi)} \rho_{n\nu} = \mathrm{Tr}(\varepsilon^{(\Phi)}\rho) \leq 1 = \mathrm{Tr}(\mathbb{I}_N\rho)$.

In other words,

$$\mathrm{Tr}[(\mathbb{I}_N - \varepsilon^{(\Phi)})\rho] \geq 0, \quad \forall \rho \in \mathcal{M}_N. \quad (B1)$$

The matrix $(\mathbb{I}_N - \varepsilon^{(\Phi)})$ is Hermitian; thus there exists a unitary $U_{(\Phi)}$ such that $U_{(\Phi)}(\mathbb{I}_N - \varepsilon^{(\Phi)})U_{(\Phi)}^\dagger = \Xi^{(\Phi)}$, with $\Xi^{(\Phi)}$ diagonal. Equation (B1) must hold for all $\rho \in \mathcal{M}_N$, so that it will hold, in particular, on the sequence of density matrices

$$\rho^{(\Phi,i)} = U_{(\Phi)}^\dagger |i\rangle\langle i| U_{(\Phi)}. \tag{B2}$$

Inserting each of the N matrices of Eq. (B2) into Eq. (B1), and using the cyclic property of the trace, one obtains

$$\Xi_{ii}^{(\Phi)} \geq 0, \quad \forall\, 1 \leq i \leq N \tag{B3}$$

or equivalently

$$\varepsilon_{ii}^{(\Phi)} \leq 1, \quad \forall\, 1 \leq i \leq N. \tag{B4}$$

Now Eq. (32) follows by observing that $\varepsilon^{(\Phi)} = N \operatorname{Tr}_A \sigma_\Phi$.

APPENDIX C: PROOF OF LEMMA 3

Directly from Eqs. (40) and (41) we write

$$\int_{\Box_N} \mathrm{d}\nu_K(\Lambda_1, \ldots, \Lambda_N) = \int \prod_{i=1}^N \Theta\!\left(\frac{1}{N} - \Lambda_i\right)\Theta(\Lambda_i)\Lambda_i^{K-N}\prod_{i<j}(\Lambda_i - \Lambda_j)^2 \mathrm{d}\Lambda_1, \ldots, \mathrm{d}\Lambda_N.$$

By rescaling variables $\Lambda_i \to \lambda_i/N$, one obtains the result

$$\int \mathrm{d}\nu_K(\Lambda_1, \ldots, \Lambda_N) = N^{-NK} \int \prod_{i=1}^N \Theta(1 - \lambda_i)\Theta(\lambda_i)\lambda_i^{K-N}\prod_{i<j}(\lambda_i - \lambda_j)^2 \mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_N$$

$$= \frac{1}{N^{KN}} \int_0^1 \cdots \int_0^1 \prod_{i=1}^N \lambda_i^{K-N}\prod_{i<j}(\lambda_i - \lambda_j)^2 \mathrm{d}\lambda_1, \ldots, \mathrm{d}\lambda_N$$

equivalent to Eq. (41) (see Ref. 22 for the definition of the Selberg's integral)   ∎.

APPENDIX D: PROOF OF LEMMA 4

In Sec. IV we decided to express every CP-TNI map $\Phi\colon \rho \mapsto \rho'$ in terms of a linear superoperator Φ and a related dynamical matrix D_Φ as follows:

$$\rho'_{ik} = \Phi_{\substack{ik \\ j\ell}}\, \rho_{j\ell} = D_{\substack{ij \\ k\ell}}\, \rho_{j\ell}. \tag{D1}$$

According to Lemma 2, the $N^2 \times N^2$ dynamical matrix D_Φ is Hermitian and positive; thus one can benefit of its spectral decomposition. Defining as $|m^{(\mu)}\rangle := \Sigma_{ij} m_{ij}^{(\mu)}|i;j\rangle$ the μth (bi-indexed) eigenvector of D_Φ, corresponding to the eigenvalue $m^{(\mu)} \in \mathbb{R}^+$, the square root decomposition reads

$$D_\Phi = \sum_{\mu=1}^S m^{(\mu)}|m^{(\mu)}\rangle\langle m^{(\mu)}| = \sum_{\mu=1}^S \left(\sqrt{m^{(\mu)}}|m^{(\mu)}\rangle\right)\left(\sqrt{m^{(\mu)}}\langle m^{(\mu)}|\right) = \sum_{\mu=1}^S |A_\mu\rangle\langle A_\mu|, \tag{D2}$$

with $[A_\mu]_{ij} := \sqrt{m^{(\mu)}}m_{ij}^{(\mu)}$ and S is given by the number of eigenvalues of D_Φ different from zero. Equation (D2) can be reexpressed in matrix form as

$$D_{\substack{ij \\ k\ell}} = \sum_{\mu=1}^S [A_\mu]_{ij}\overline{[A_\mu]}_{k\ell}, \tag{D3}$$

where the symbol $\bar{x}$ denotes the complex conjugate of x. Now we observe that the each bi-indexed vector $[A_\mu]_{ij}$, of size N^2, can be seen as a square matrix of size N, and reshuffling its indexes one gets $\overline{[A_\mu]}_{k\ell} = [A_\mu^\dagger]_{\ell k}$. Equation (D1) becomes

$$\rho'_{ik} = D_{\substack{ij \\ k\ell}} \, \rho_{j\ell} = \sum_{\mu=1}^{S} [A_\mu]_{ij}\rho_{j\ell}[A_\mu^\dagger]_{\ell k}, \tag{D4}$$

that is, the matrix form of Eq. (44a).

The explicit matrix form of the left-hand side of Eq. (44b) reads

$$\sum_{\mu=1}^{S} [A_\mu^\dagger A_\mu]_{ij} = \sum_{\mu=1}^{S}\sum_{\ell=1}^{N} [A_\mu^\dagger]_{i\ell}[A_\mu^\dagger]_{\ell j} = \sum_{\ell=1}^{N}\sum_{\mu=1}^{S} [A_\mu]_{\ell j}\overline{[A_\mu]}_{\ell i} = \sum_{\ell=1}^{N} D_{\substack{\ell j \\ \ell i}} = [\mathrm{Tr}_A \, D_\Phi]_{ji}, \tag{D5}$$

where we made use of Eq. (D3), so that Eq. (44b) follows from the Proposition.

To prove the second part of the lemma, one simply notes that the rhs of Eq. (D2) always defines a positive and Hermitian $N^2 \times N^2$ matrix D, no matter of which S complex $N \times N$ matrices have been used. The additional constraint that makes D becoming a dynamical matrix, representative of some CP-TNI map Φ, is given by Eq. (34). Imposing Eq. (34), in terms of the family of matrices A_μ, is equivalent of imposing Eq. (45a), as it can be earned by reversing the chain [Eq. (D5)] and by noting that $\check\sigma \in \mathcal{M}_N^\square \Rightarrow \check\sigma^{\mathrm{T}} \in \mathcal{M}_N^\square$. Finally, the claim can be obtained through the same steps [Eqs. (D2)–(D4)] as before and this ends the proof. ∎

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