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# Higher-rank numerical ranges and compression problems

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**Abstract**

We consider higher-rank versions of the standard numerical range for matrices. A central motivation for this investigation comes from quantum error correction. We develop the basic structure theory for the higher-rank numerical ranges, and give a complete description in the Hermitian case. We also consider associated projection compression problems.


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## 1. Introduction

In this paper, we initiate the study of higher-rank versions of the standard numerical range for matrices. A primary motivation for us arises through the basic problem of error correction in quantum computing. Specifically, the development of theoretical and ultimately experimental

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techniques to overcome the errors associated with quantum operations is central to continued advances in quantum computing. As it turns out, the numerical ranges $\Lambda_k(T)$, for $k > 1$, defined below are intimately related to this problem of "quantum error correction". In the paper [1] we give applications of the results from the present paper to this problem.

Let $T$ be an $N \times N$ matrix with complex entries. For $k \geqslant 1$, define the rank-$k$ *numerical range of* $T$ as the subset $\Lambda_k(T)$ of the complex plane given by

$$\Lambda_k(T) = \{\lambda \in \mathbb{C} : PTP = \lambda P \text{ for some rank-}k \text{ projection } P\}, \tag{1}$$

where we use the term "projection" to mean "orthogonal projection". Observe that the numerical range of $T$ is obtained as

$$\Lambda_1(T) = W(T) = \{\langle T\psi|\psi\rangle : |\psi\rangle \in \mathbb{C}^N, \||\psi\rangle\| = 1\}. \tag{2}$$

In our analysis, it is desirable to explicitly find the scalars $\lambda$ and the associated projections $P$ in Eq. (1). Thus, this "compression problem" will be the focus of this paper. A search of the substantive literature on numerical ranges reveals connections with two lines of investigation. The "$k$-numerical range" introduced by Halmos in [2] is the set of all $\lambda$ that satisfy $\lambda = \mathrm{Tr}(PTP)$ for some rank-$k$ projection $P$. Evidently, this set includes the set $k\Lambda_k(T)$, but the reverse inclusion does not hold in general. The "$k$th matrix numerical range" studied by several authors consists of the set $W(k : T)$ of all matrices $X^*TX$, where $X$ is an $N \times k$ matrix such that $X^*X = I$. The higher-rank numerical ranges $\Lambda_k(T)$ can alternatively be formulated as $\Lambda_k(T) = W(k : T) \cap \{\lambda I_k : \lambda \in \mathbb{C}\}$. See [3–6] as examples of other entrance points into the literature on generalized notions of the numerical range.

The rest of the paper is organized as follows: In Section 2, we discuss the basic structure theory for the sets $\Lambda_k(T)$. In particular, we derive an explicit characterization of these sets for all Hermitian matrices. We state a conjecture and an open problem in the case of normal matrices. We discuss some lower dimensional cases in Section 3, and in the penultimate section (Section 4), we present a method for constructing the associated compression projections that captures all possible projections in the Hermitian case. In the context of quantum error correction, projections that correspond to elements of $\Lambda_k(T)$, for $k > 1$, must be explicitly identified. For instance, in the rank-two case, such projections correspond to quantum bits of information, or "qubits", that can be corrected after particular quantum operations act (see Section 5).

## 2. Compression-values

In this section, we investigate the basic structure theory of the sets $\Lambda_k(T)$. We shall refer to elements of $\Lambda_k(T)$ as "compression-values" for $T$, since $\lambda \in \Lambda_k(T)$ if and only if the $k \times k$ scalar matrix $\lambda I_k$ is the compression of $T$ to a $k$-dimensional subspace. This means that $T$ is unitarily equivalent to a $2 \times 2$ block matrix of the form

$$T = \begin{pmatrix} \lambda I_k & A \\ B & C \end{pmatrix}, \tag{3}$$

where $A$ is a $k \times (N-k)$ matrix, $B$ is an $(N-k) \times k$ matrix, and $C$ is an $(N-k) \times (N-k)$ matrix. Equivalently, $T$ is a "dilation" of the scalar matrix $\lambda I_k$, or, $T - \lambda I$ maps a $k$-dimensional subspace into its orthogonal complement.

The following set inclusions may be readily verified:

$$W(T) = \Lambda_1(T) \supseteq \Lambda_2(T) \supseteq \cdots \supseteq \Lambda_N(T). \tag{4}$$

The following properties are also easily checked:

(i) $\Lambda_k(\alpha T + \beta I) = \alpha\Lambda_k(T) + \beta \ \forall \alpha, \beta \in \mathbb{C}$.
(ii) $\Lambda_k(T^*) = \overline{\Lambda_k(T)}$.
(iii) $\Lambda_k(T) \subseteq \Lambda_k(\mathrm{Re}T) + \mathrm{i}\Lambda_k(\mathrm{Im}T)$.
(iv) $\Lambda_k(T \oplus S) \supseteq \Lambda_k(T) \cup \Lambda_k(S)$.
(v) $\Lambda_{k_1+k_2}(T \oplus S) \supseteq \Lambda_{k_1}(T) \cap \Lambda_{k_2}(S)$.

The numerical range $W(T) = \Lambda_1(T)$ is a non-empty, compact and convex subset of the plane that includes the spectrum of $T$. If $T$ is normal, then $W(T)$ is the convex hull of the eigenvalues for $T$. In particular, if $T$ is Hermitian, then $W(T)$ is the closed interval of the real line determined by the minimal and maximal eigenvalues of $T$. The higher-rank numerical ranges can, of course, be empty. But compactness still holds in general. The proof of the following result is elementary, hence we leave it to the interested reader.

**Proposition 2.1.** *Let $T$ be an $N \times N$ matrix and let $k \geqslant 1$. Then the* rank-$k$ *numerical range $\Lambda_k(T)$ forms a compact set.*

Now we give a description of the higher-rank numerical range for large values of $k$ relative to $N$.

**Proposition 2.2.** *Let $T$ be an $N \times N$ matrix and suppose that $2k > N$. Then the* rank-$k$ *numerical range $\Lambda_k(T)$ is an empty set or a singleton set. If $\Lambda_k(T) = \{\lambda_0\}$ is a singleton set with $2k > N$, then $\lambda_0$ is an eigenvalue of geometric multiplicity at least $2k - N$. In particular, $\Lambda_N(T)$ is non-empty if and only if $T$ is a scalar matrix.*

**Proof.** Given $2k > N$, assume that $\Lambda_k(T)$ is non-empty, and contains $\lambda_0 \neq \lambda_1$. Let $P_0, P_1$ be the corresponding rank-$k$ projections. Then the projection $P = P_0 \wedge P_1$ onto the intersection of the ranges of these two projections is non-zero and satisfies $\lambda_0 P = PTP = \lambda_1 P$. This contradiction shows that $\Lambda_k(T)$ is a singleton set when it is non-empty.

For the second claim, the equality $P(T - \lambda_0 I)P = 0$ implies

$$T - \lambda_0 I = (I - P)(T - \lambda_0 I) + P(T - \lambda_0 I)(I - P). \tag{5}$$

Hence, $\mathrm{rank}(T - \lambda_0 I) \leqslant 2\,\mathrm{rank}(I - P) = 2N - 2k$, and so,

$$\ker(T - \lambda_0 I) \geqslant N - (2N - 2k) = 2k - N. \qquad \square \tag{6}$$

In the normal case the previous result yields more detailed information for large values of $k$.

**Corollary 2.3.** *Let $T$ be an $N \times N$ normal matrix and suppose that $2k > N$. Then the* rank-$k$ *numerical range $\Lambda_k(T)$ is an empty set or a singleton set. In fact, the case $\Lambda_k(T) = \{\lambda_0\}$ occurs if and only if there is a $(2N - 2k) \times (2N - 2k)$ matrix $T_0$ such that $T$ is unitarily equivalent to $\lambda_0 I_{2k-N} \oplus T_0$, and $\lambda_0$ belongs to $\Lambda_{N-k}(T_0)$.*

We now derive a general description of the rank-$k$ numerical range in the Hermitian case for arbitrary $k$.

**Theorem 2.4.** *Let $A$ be an $N \times N$ Hermitian matrix with eigenvalues (counting multiplicities) given by $a_1 \leqslant a_2 \cdots \leqslant a_N$ and let $k \geqslant 1$ be a fixed integer with $1 \leqslant k \leqslant N$. Then the* rank-$k$ *numerical range $\Lambda_k(A)$ coincides with $[a_k, a_{N-k+1}]$ which is*

(i) *a non-degenerate closed interval if* $a_k < a_{N-k+1}$,
(ii) *a singleton set if* $a_k = a_{N-k+1}$,
(iii) *an empty set if* $a_k > a_{N-k+1}$.

*Moreover,* $\Lambda_k(A)$ *coincides with the intersection of the numerical ranges* $W(V^*AV)$, *where* $V$ *runs through all isometries* $V : \mathbb{C}^{N-k+1} \to \mathbb{C}^N$.

**Proof.** Let $\lambda \in \Lambda_k(A)$ and let $P_k$ be a rank-$k$ projection with $P_kAP_k = \lambda P_k$. If $V : \mathbb{C}^{N-k+1} \to \mathbb{C}^N$ is an isometry, then the subspace $P_k\mathbb{C}^N$ and the range space $VV^*(\mathbb{C}^N)$ have non-zero intersection. Thus, there exists a unit vector $|\psi\rangle \in \mathbb{C}^N$ such that $|\psi\rangle = P_k|\psi\rangle = VV^*|\psi\rangle$. Let $|\psi'\rangle$ be the unit vector in $\mathbb{C}^{N-k+1}$ given by $|\psi'\rangle = V^*|\psi\rangle$. Then we have

$$\langle V^*AV\psi'|\psi'\rangle = \langle A\psi|\psi\rangle \tag{7}$$
$$= \langle P_kAP_k\psi|\psi\rangle = \lambda\langle P_k\psi|\psi\rangle = \lambda. \tag{8}$$

Hence, we have shown that $\lambda$ belongs to $W(V^*AV)$. As $V : \mathbb{C}^{N-k+1} \to \mathbb{C}^N$ was an arbitrary isometry, it follows that $\Lambda_k(A)$ is contained in the intersection of all such numerical ranges $W(V^*AV)$.

Next, let $\{|i\rangle : 1 \leqslant i \leqslant N-k+1\}$ be a fixed orthonormal basis for $\mathbb{C}^{N-k+1}$ and let $\{|\psi_i\rangle\}$ be an orthonormal basis for $\mathbb{C}^N$ of eigenvectors for $A$ corresponding to the eigenvalues $a_1, \ldots, a_N$. Consider two linear isometries $V_1, V_2 : \mathbb{C}^{N-k+1} \to \mathbb{C}^N$ defined by $V_1(|i\rangle) = |\psi_i\rangle$, $V_2(|i\rangle) = |\psi_{N-i+1}\rangle$.

Then $V_1^*AV_1$ and $V_2^*AV_2$ are operators on $\mathbb{C}^{N-k+1}$ that are diagonal with respect to the basis $\{|i\rangle\}$, and we have $W(V_1^*AV_1) = [a_1, a_{N-k+1}]$ and $W(V_2^*AV_2) = [a_k, a_N]$. It follows that

$$\Lambda_k(A) \subseteq \bigcap_V W(V^*AV) \subseteq W(V_1^*AV_1) \bigcap W(V_2^*AV_2) \tag{9}$$
$$= [a_k, a_{N-k+1}]. \tag{10}$$

We complete the proof by showing $\Lambda_k(A)$ contains the set $[a_k, a_{N-k+1}]$ when $a_k \leqslant a_{N-k+1}$. Suppose first that $a_{N+1-k} > a_k$ (and so $2k \leqslant N$). Fix $\lambda$ in the interval $[a_k, a_{N+1-k}]$. We shall directly construct a rank-$k$ projection $P_k$ such that $P_kAP_k = \lambda P_k$. Consider the set of $k$ pairs $\{a_{k+1-j}, a_{N-k+j}\}$, $1 \leqslant j \leqslant k$. As a notational convenience we shall write $\{b_j, b'_j\}$ for the ordered pair $\{a_{k+1-j}, a_{N-k+j}\}$, and so $b_j > b'_j$. (The following construction may be easily modified for any joint partition of the sets $\{a_N, \ldots, a_{N-k+1}\}$ and $\{a_k, \ldots, a_1\}$ into ordered pairs.)

We may write $A$, up to unitary equivalence, as a direct sum

$$A = \left(\oplus_j A_j\right) \oplus B, \tag{11}$$

where each $A_j$ is a diagonal $2 \times 2$ matrix with spectrum $\{b_j, b'_j\}$, and $B$ is either vacuous, or is the diagonal matrix with diagonal entries $\{a_{k+1}, \ldots, a_{N-k}\}$. As $\lambda$ satisfies

$$\lambda \in [a_k, a_{N-k+1}] \subseteq [b_j, b'_j] = W(A_j) \quad \forall 1 \leqslant j \leqslant k, \tag{12}$$

we may find angles $\theta_j$ such that

$$\lambda = b_j \cos^2\theta_j + b'_j \sin^2\theta_j \quad \forall 1 \leqslant j \leqslant k. \tag{13}$$

Now define an orthonormal set of $k$ vectors by

$$|\phi_j\rangle = \cos\theta_j|\psi_{N-k+j}\rangle + \sin\theta_j|\psi_{k-j+1}\rangle \quad \forall 1 \leqslant j \leqslant k, \tag{14}$$

and the rank-$k$ projection $P_k$ onto the subspace spanned by these vectors;

$$P = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2| + \cdots + |\phi_k\rangle\langle\phi_k|.$$

It follows that $P_k A P_k = \lambda P_k$. Indeed, observe that for $1 \leqslant j \leqslant k$ we have

$$\langle A\phi_1|\phi_j\rangle = \cos\theta_1\langle A\psi_N|\phi_j\rangle + \sin\theta_1\langle A\psi_1|\phi_j\rangle \tag{15}$$

$$= a_N\cos\theta_1\langle\psi_N|\phi_j\rangle + a_1\sin\theta_1\langle\psi_1|\phi_j\rangle \tag{16}$$

$$= b_1\cos\theta_1\cos\theta_j\delta_{N,N-1+j} + b_1'\sin\theta_1\sin\theta_j\delta_{j,1} \tag{17}$$

$$= \lambda\delta_{j,1}. \tag{18}$$

Similarly, $\langle A\phi_i|\phi_j\rangle = \lambda\delta_{ij}$ for $1 \leqslant i, j \leqslant k$.

The remaining case is characterized by the constraint $\lambda := a_k = a_{N-k+1}$. If, in addition, $a_{N-k+2} > a_{k-1}$, then we may split the sets $\{a_N, \ldots, a_{N-k+2}\}$ and $\{a_{k-2}, \ldots, a_1\}$ into pairs as above, and similarly define $k-1$ vectors $|\phi_1\rangle, \ldots, |\phi_{k-1}\rangle$. As the final vector we can take $|\phi_k\rangle := |\psi_k\rangle$, and define $P_k = \sum_{j=1}^{k} |\phi_j\rangle\langle\phi_j|$. If $a_{N-k+2} = a_{k-1}$, but $a_{N-k+3} > a_{k-2}$, then we will use $|\psi_k\rangle$ and $|\psi_{k-1}\rangle$ as two of the vectors. This process may be continued, if required, to account for degeneracies in the spectrum of $A$ around the eigenvalue $a_k$, and construct a rank-$k$ projection which yields $\lambda \in \Lambda_k(A)$. The result now follows. □

For each real number $r \in \mathbb{R}$, we write $\lceil r \rceil$ for the smallest integer $n$ satisfying $n \geqslant r$. From Theorem 2.4, we see that if $k \leqslant \lceil N/2 \rceil$ (equivalently $2k-1 \leqslant N$), then $\Lambda_k(A)$ is non-empty for each $N \times N$ Hermitian matrix $A$. The following is an analogous result for a general non-Hermitian matrix. (This result can also be derived from Theorem 3.3 of [5].)

**Corollary 2.5.** *Let $T$ be an $N \times N$ complex matrix. Let $k$ be a positive integer satisfying $k \leqslant \lceil N/4 \rceil$ (equivalently $4k-3 \leqslant N$). Then $\Lambda_k(T)$ is non-empty.*

**Proof.** Write $T = A + \mathrm{i}B$ with $A = A^*$ and $B = B^*$. Let $b = b_{2k-1}$ be the $(2k-1)$th smallest eigenvalue of $B$. By Theorem 2.4, $b \in \Lambda_{2k-1}(B)$; and so there is a projection $P$ of rank $2k-1$ such that $P(B-bI)P = 0$. Consider the $(2k-1) \times (2k-1)$ Hermitian matrix $A_0$ given by the restriction of the compression $PAP$ to the range of $P$. It follows from another application of Theorem 2.4 that $\Lambda_k(A_0)$ is a singleton set $\{a\}$, where $a$ is the $k$th smallest eigenvalue of $A_0$. Hence, there exists a projection $Q \leqslant P$ such that $\operatorname{rank} Q = k$ and $QAQ = aQ$. Thus, $QTQ = QAQ + \mathrm{i}QBQ = (a + \mathrm{i}b)Q$ and $\Lambda_k(T)$ is non-empty. □

The construction of projections that is described in the proof of the previous theorem will be further fleshed out in subsequent sections. It is perhaps appropriate to emphasize the most important non-trivial case of this result. Specifically, when $2k \leqslant N$ and the spectrum of $A$ is non-degenerate, Theorem 2.4 shows that the rank-$k$ numerical range is the interval $\Lambda_k(A) = [a_k, a_{N-k+1}]$ – see Fig. 1, which shows generalized numerical ranges for $N = 4$ and $N = 6$. Also note that as an immediate consequence of Theorem 2.4, it follows that the sets $\Lambda_k(A)$ are convex for all $k \geqslant 1$ and Hermitian $A$.

We finish this section by discussing the case of normal matrices. First note that property (iii) above and Theorem 2.4 give a crude containment result for $\Lambda_k(T)$ for arbitrary $T$. Indeed, $\Lambda_k(T)$ is a subset of the rectangular region in the complex plane $\{\alpha + \mathrm{i}\beta : \alpha \in \Lambda_k(\mathrm{Re}(T)), \beta \in \Lambda_k(\mathrm{Im}(T))\}$. In general we can obtain a more refined containment in the normal case.

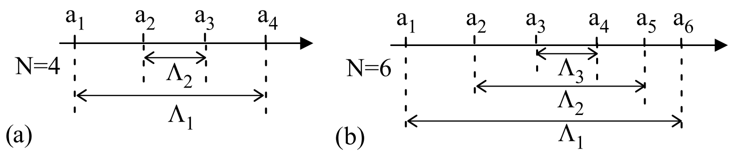


Fig. 1. The numerical range $\Lambda_k(A)$ for a non-degenerate Hermitian operator $A$ of size (a) $N = 4$ and (b) $N = 6$, with spectrum $\{a_i\}$.

**Theorem 2.6.** *Let $T$ be an $N \times N$ normal matrix and let $k \geqslant 1$ be a fixed positive integer. Then*

$$\Lambda_k(T) \subseteq \cap_\Gamma(\operatorname{co}\Gamma), \tag{19}$$

*where $\Gamma$ runs through all $(N + 1 - k)$-point subsets (counting multiplicities) of the spectrum of $T$.*

**Proof.** The relevant parts of the proof of Theorem 2.4 can be easily extended to the normal case to verify the inclusion of Eq. (19). □

**Remark 2.7.** Observe that Theorem 2.4 shows the converse inclusion of Eq. (19) holds in the Hermitian case. We believe this inclusion holds more generally, at least in the normal case, and we plan to undertake this investigation elsewhere.

**Conjecture 2.8.** *If $T$ is an $N \times N$ normal matrix, then $\Lambda_k(T)$ coincides with the intersection of the convex hulls* co$\Gamma$*, where $\Gamma$ is an $(N + 1 - k)$-point subset (counting multiplicities) of the spectrum of $T$.*

Verification of this conjecture would, of course, automatically imply that $\Lambda_k(T)$ is convex, whenever this set is non-empty and $T$ is normal. We state the general case as an open problem.

**Problem 2.9.** Is $\Lambda_k(T)$ a convex set whenever it is non-empty?

As a consequence of Theorem 2.6 and the proof of Theorem 2.4, the conjecture can be seen to hold for $N = 1, 2, 3, 4$ and all values of $k$ in each of these cases. Indeed, Theorem 2.6 shows that $\Lambda_k(T)$ is contained in the desired set, and the construction of projections in the proof of Theorem 2.4 may be adapted for $N \leqslant 4$. In each of these cases, $\Lambda_k(T)$ is either the empty set, a singleton set, or an interval, and hence can never have interior. See Fig. 2 for an illustration of some of the non-interval cases for $N = 4$. (We note that the $N = 4$ unitary case is explicitly worked out in [1].)

The first open case is that of $N = 5$ and $k = 2$. The cyclic 5-shift is a good test example. This is the unitary $U : \mathbb{C}^5 \to \mathbb{C}^5$ defined on an orthonormal basis $\{|\xi_1\rangle, \ldots, |\xi_5\rangle\}$ by $U|\xi_j\rangle = |\xi_{j+1(\mathrm{mod}\,5)}\rangle$. The spectrum of $U$ is given by $z_n = \exp\{\mathrm{i}\frac{2\pi n}{5}\}$, for $n = 0, 1, 2, 3, 4$. Thus, $\Lambda_2(U)$ is a subset of the pentagon shaped region depicted in Fig. 3. The arguments of $(b) \Rightarrow (a)$ in Theorem 2.4 may be used to show that $\Lambda_2(U)$ contains the border points of this region, and also contains the centre $\lambda = 0$. The problem is to determine if the rest of the interior points are included.

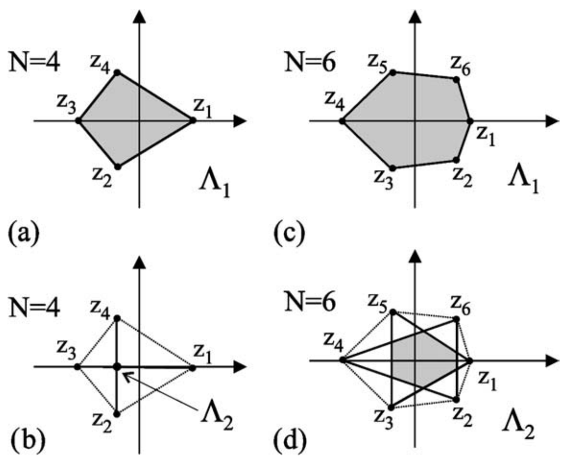


Fig. 2. Examples of the numerical range $\Lambda_k(T)$ for a normal operator $T$ of size (a,b) $N = 4$ and (c,d) $N = 6$, with non-degenerate complex eigenvalues $\{z_i\}$. The set $\Lambda_2$ is contained in the subset depicted (and equal at least in the case $N = 4$).

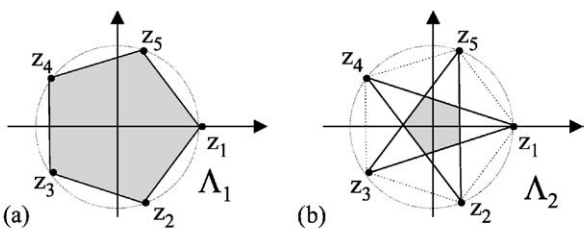


Fig. 3. Numerical ranges $\Lambda_k(U)$ for the cyclic 5-shift, with spectrum consisting of the fifth roots of unity, $z_n$: (a) $k = 1$ and (b) $k = 2$, $\Lambda_2(U)$ is contained in this set.

## 3. Eigenvalue-pairing construction

The method presented in the next section shows how all the higher-rank projections may be obtained through a generalization of the "eigenvalue pairing" approach used in the proof of Theorem 2.4. For illustration purposes, in this section we further discuss the pairing approach in some lower dimensional cases.

First let us recall the $k = 1$ case as motivation for what follows below. If $a_N \geqslant \cdots \geqslant a_1$ are the eigenvalues of $A = A^*$ as above, then the numerical range of $A$ is given by $\Lambda_1(A) = [a_1, a_N]$. (Assume $A$ is non-scalar, so this is truly an interval.) Let $|\psi_j\rangle$ be a choice of eigenvector for each $a_j$. We may write a given $\lambda \in \Lambda_1(A)$ as a linear combination $\lambda = a_1 c_1^2 + a_2 c_2^2 + \cdots + a_N c_N^2$, where $\vec{c} = (c_i)$ are real scalars belonging to the $(N-1)$-dimensional simplex $\left(\sum_{i=1}^N c_i^2 = 1\right)$. Then the typical unit vector $|\phi\rangle$ that satisfies $\lambda = \langle A\phi|\phi\rangle$, and the corresponding rank one projection, are given by

$$|\phi\rangle = \sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\theta_j} c_j |\psi_j\rangle \quad \text{and} \quad P = |\phi\rangle\langle\phi|. \tag{20}$$

Observe that there are $N$ variables $c_j$, with two constraints, and this gives $N-2$ free parameters. There is also an additional $N$ free phases from the choices $\theta_j$, for $1 \leqslant j \leqslant N$. (Although it is $N-1$ free phases up to a global phase allowed in the definition of $P$.)

For instance, if $(N, k) = (2, 1)$, then $\Lambda_1(A) = [a_1, a_2]$ where $a_j$ is an eigenvalue for the eigenstate $|\psi_j\rangle$. In this case, $\lambda \in [a_1, a_2]$ may be written as $\lambda = a_1 \cos^2\beta_1 + a_2 \sin^2\beta_1$, and the angle $\beta_1$ may be computed via the equation

$$\cos^2\beta_1 = \frac{\lambda - a_2}{a_1 - a_2}. \tag{21}$$

The corresponding projection $P_1$ is obtained as a "coherent mixture" of both eigenstates:

$$|\phi_1\rangle = \mathrm{e}^{\mathrm{i}\theta_1}\cos\beta_1|\psi_1\rangle + \mathrm{e}^{\mathrm{i}\theta_2}\sin\beta_1|\psi_2\rangle, \quad P_1 = |\phi_1\rangle\langle\phi_1|. \tag{22}$$

Next consider the case $(N, k) = (3, 1)$. Let $\lambda$ belong to $\Lambda_1(A) = [a_1, a_3]$. In this case, $\lambda$ may be obtained via the equation

$$\lambda = a_1\cos^2\beta_1 + a_2\sin^2\beta_1\cos^2\beta_2 + a_3\sin^2\beta_1\sin^2\beta_2. \tag{23}$$

In this case, $\beta_2 = \beta_2(\lambda, \beta_1)$ depends on both $\lambda$ and $\beta_1$, and hence there is a one parameter family of solutions determined by $\beta_1$. The projection is given by $P_1 = |\phi_1\rangle\langle\phi_1|$ where

$$|\phi_1\rangle = \mathrm{e}^{\mathrm{i}\theta_1}\cos\beta_1|\psi_1\rangle + \mathrm{e}^{\mathrm{i}\theta_2}\sin\beta_1\cos\beta_2|\psi_2\rangle + \mathrm{e}^{\mathrm{i}\theta_3}\sin\beta_1\sin\beta_2|\psi_3\rangle$$

and we have three free phases $\{\theta_1, \theta_2, \theta_3\}$ (two phases up to a global phase). In the case that $\lambda = a_2$, we may also use the solution Eq. (22) to find a vector $|\phi_{13}\rangle$ as a mixture of $|\psi_1\rangle$ and $|\psi_3\rangle$, and then mix it with $|\psi_2\rangle$ to obtain

$$|\phi_1\rangle = \cos\beta_2|\psi_2\rangle + \sin\beta_2|\phi_{13}\rangle.$$

Let us turn now to higher-rank projections obtained from the eigenvalue pairing approach in the case $N = 4$. The case of interest when $N = 4$ is $(N, k) = (4, 2)$. The challenge occurs when $\Lambda_2(A) = [a_2, a_3]$ is a true interval. If we are given $\lambda \in \Lambda_2(A)$, we can consider all pairs $\{a_i, a_{i'}\}$ that contain $\lambda$. Here, there are two possibilities:

(i) $\{a_4, a_2\}, \{a_3, a_1\}$;
(ii) $\{a_4, a_1\}, \{a_3, a_2\}$.

Of course, in the case of arbitrary $N$, there will be many more possible pairings. Now we solve the (2, 1) problem for each of the pairs separately. For instance, in the case of (i), we solve for $\beta_1$ and $\beta_2$ in the equations,

$$\begin{cases} \lambda = a_1\cos^2\beta_1 + a_3\sin^2\beta_1, \\ \lambda = a_2\cos^2\beta_2 + a_4\sin^2\beta_2, \end{cases} \tag{24}$$

and so

$$\cos^2\beta_1 = \frac{\lambda - a_3}{a_1 - a_3} \quad \text{and} \quad \cos^2\beta_1 = \frac{\lambda - a_4}{a_2 - a_4}. \tag{25}$$

We then define coherent combinations of eigenstates grouped in pairs:

$$\begin{cases} |\phi_1\rangle = \mathrm{e}^{\mathrm{i}\theta_1}\cos\beta_1|\psi_1\rangle + \mathrm{e}^{\mathrm{i}\theta_3}\sin\beta_1|\psi_3\rangle, \\ |\phi_2\rangle = \mathrm{e}^{\mathrm{i}\theta_2}\cos\beta_2|\psi_2\rangle + \mathrm{e}^{\mathrm{i}\theta_4}\sin\beta_2|\psi_4\rangle. \end{cases} \tag{26}$$

Then write $P_2 = |\phi_1\rangle\langle\phi_1| + |\phi_2\rangle\langle\phi_2|$, and it follows that $PAP = \lambda P$.

As noted above, this problem is equivalent to finding a unitary matrix $U$ such that the matrix $A' = UAU^*$ includes a $2 \times 2$ block given by the scalar matrix $\lambda \mathbb{1}_2$. In the case $A = \operatorname{diag}(a_1, a_2, a_3, a_4)$, observe that one choice for such a unitary is given by $U = OD$, where $D = \operatorname{diag}(e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4})$ and $O$ is the orthogonal matrix given by

$$O = \begin{pmatrix} \cos\beta_1 & 0 & \sin\beta_1 & 0 \\ 0 & \cos\beta_2 & 0 & \sin\beta_2 \\ -\sin\beta_1 & 0 & \cos\beta_1 & 0 \\ 0 & -\sin\beta_2 & 0 & \cos\beta_2 \end{pmatrix}. \tag{27}$$

## 4. Higher-rank projections

In this section we consider the problem of finding all possible rank-$k$ projections $P$ associated with a compression-value $\lambda \in \Lambda_k(T)$; i.e., to solve for the rank-$k$ projections $P$ such that $PTP = \lambda P$. We shall focus on the Hermitian case $A = A^*$. Recall that in the proof of Theorem 2.4 and the discussion of the previous section we explicitly constructed certain families of projections to show that particular values of $\lambda$ belonged to $\Lambda_k(A)$. However, what we would like is a method for constructing such projections that captures all possibilities. Unlike the standard eigenvalue and eigenspace problem, in the generic case of this compression problem there will be infinitely many projections. Indeed, even in the typical case for the numerical range $W(A) = \Lambda_1(A)$ this is the case. But it is possible, and in fact easy, to write down such a method for the $k = 1$ case. There are of course more complications for $k \geqslant 2$.

First let us note that, while the eigenvalue-pairing approach constructs a diverse set of projections, it is not sufficient to capture all projections associated with values of $\Lambda_k(A)$. Indeed, even consider the $k = 1$ case of a Hermitian matrix $A$ with spectrum $\{0, 1, 2\}$. Here, $\Lambda_1(A) = [0, 2]$. Let $|\psi_i\rangle$, $i = 0, 1, 2$, be unit eigenvectors for the corresponding eigenvalues. The eigenvalue-pairing approach for $\lambda = 1$ in this case yields the family of projections $P = |\psi\rangle\langle\psi|$, where

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(e^{i\theta_1}|\psi_0\rangle + e^{i\theta_2}|\psi_2\rangle\big). \tag{28}$$

But the set of all projections $P = |\psi\rangle\langle\psi|$ such that $\langle A\psi|\psi\rangle = 1$ is the larger set given by unit vectors of the form

$$|\psi\rangle = e^{i\theta_1} c_0|\psi_0\rangle + c_1|\psi_1\rangle + e^{i\theta_2} c_0|\psi_2\rangle. \tag{29}$$

For an arbitrary Hermitian matrix $A$, the rank one projections $P = |\psi\rangle\langle\psi|$ associated with values $\lambda \in \Lambda_1(A)$ may be computed in the following manner. Let $a_1 \leqslant \cdots \leqslant a_N$ be the eigenvalues for $A$, and let $|\psi_i\rangle$, $1 \leqslant i \leqslant N$, be a choice of corresponding eigenvectors. Suppose we have a unit vector $|\psi\rangle = \sum_{i=1}^N c_i|\psi_i\rangle$. Then a simple computation shows that

$$\lambda = \langle A\psi|\psi\rangle \quad \text{if and only if} \quad \lambda = \sum_{i=1}^N a_i|c_i|^2. \tag{30}$$

This constructive condition characterizes the rank one projections associated with elements of the numerical range. Notice that there are infinitely many possibilities for such projections whenever $\lambda$ is not an eigenvalue for $A$. There is a corresponding characterization for arbitrary $k$, though it is not constructive for $k \geqslant 2$. Instead, in what follows we present a constructive, algorithmic approach to find all higher-rank projections associated with compression-values of $\Lambda_k(A)$ for $k \geqslant 2$.

Let $\lambda \in \Lambda_k(A)$. By using a translation, we may assume that $\lambda = 0$. Let $P_+$, $P_0$, and $P_-$ be the projections onto the eigenspaces of $A$ for respectively, the positive eigenvalues, the eigenvalue zero, and the negative eigenvalues. First, we consider the case when there is no degeneracy in the spectrum of $A$; that is, $\lambda = 0$ is not an eigenvalue of $A$. Next choose a $k$-dimensional subspace $\mathscr{V}_+$ of $P_+\mathbb{C}^N$. Note that this is possible by Theorem 2.4, and our assumptions $\lambda = 0 \in \Lambda_k(A) = [a_k, a_{N-k+1}]$, and the non-degeneracy of the spectrum. By the same reasoning, $P_-\mathbb{C}^N$ is at least $k$-dimensional, and hence we may choose an isometry $U : \mathscr{V}_+ \to P_-\mathbb{C}^N$. Now we define a $k$-dimensional subspace of $\mathbb{C}^N$:

$$\mathscr{V} = \left\{v_+ + Uv_+ : v_+ \in \mathscr{V}_+\right\}. \tag{31}$$

Next define a $k$-dimensional subspace $\mathscr{W} = f(A)\mathscr{V}$ where

$$f(x) = \begin{cases} |x|^{-1/2} & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases} \tag{32}$$

Observe that $f(A)Af(A) = P_+ - P_-$, and hence $\forall v_1, v_2 \in \mathscr{V}$ we have

$$\langle Af(A)v_1|f(A)v_2\rangle = \langle f(A)Af(A)v_1|v_2\rangle \tag{33}$$

$$= \langle (P_+ - P_-)v_1|v_2\rangle \tag{34}$$

$$= \langle P_+v_1|P_+v_2\rangle - \langle P_-v_1|P_-v_2\rangle = 0. \tag{35}$$

It follows that $P_{\mathscr{W}}$ is a rank-$k$ projection such that $P_{\mathscr{W}}AP_{\mathscr{W}} = 0$.

Now we show that every rank-$k$ projection $P$ such that $PAP = 0$, can be written in the form $P = P_{\mathscr{W}}$ as above. Let $P$ be such a projection, and let $\mathscr{V}$ be the $k$-dimensional subspace $\mathscr{V} = f(A)^{-1}\mathscr{W}$. Then for all $v \in \mathscr{V}$ we have

$$0 = \langle Af(A)v|f(A)v\rangle = \langle f(A)Af(A)v|v\rangle \tag{36}$$

$$= \langle (P_+ - P_-)v|v\rangle = \|P_+v\|^2 - \|P_-v\|^2. \tag{37}$$

In particular, this implies that the map $U(P_+v) \equiv P_-v$ determines a well defined isometry $U : \mathscr{V}_+ \to \mathscr{V}_-$, where $\mathscr{V}_+ = P_+\mathscr{V}$ and $\mathscr{V}_- = P_-\mathscr{V}$. Thus, $\mathscr{V}_+$ and $\mathscr{V}_-$ are both $k$-dimensional and $\mathscr{V}$ is of the form given in Eq. (31), and hence $P = P_{\mathscr{W}}$ as claimed.

We have presented a constructive method to obtain projections associated with the compression-values of $\Lambda_k(A)$, in the case that there are no degeneracies in the spectrum of $A$. We have also shown that every such projection arises in this manner. Let us summarize the method:

(i) Choose a $k$-dimensional subspace $\mathscr{V}_+$ of $P_+\mathbb{C}^N$.
(ii) Choose a linear isometry $U : \mathscr{V}_+ \to P_-\mathbb{C}^N$.
(iii) Define the $k$-dimensional subspace $\mathscr{V} = \{v_+ + Uv_+ : v_+ \in \mathscr{V}_+\}$.
(iv) Let $\mathscr{W} = f(A)\mathscr{V}$. Then $\dim \mathscr{W} = k$ and $P_{\mathscr{W}}AP_{\mathscr{W}} = 0$.

If there are degeneracies in the spectrum of $A$, the above method may be adjusted by including part of the subspace $P_0\mathbb{C}^N$ in the subspaces $\mathscr{V}_+$ and $\mathscr{V}_-$ as follows: As above, we want to construct all $k$-dimensional subspaces $\mathscr{V}$ of $\mathbb{C}^N$ such that

$$\langle (P_+ - P_-)v|v\rangle = 0 \quad \forall v \in \mathscr{V}. \tag{38}$$

This can be accomplished since $0 \in \Lambda_k(A) = [a_k, a_{N-k+1}]$, and so $k \leqslant \dim P_0\mathbb{C}^N + \dim P_\pm\mathbb{C}^N$. Consider all possible pairs of non-zero integers $(k_1, k_2)$ with $k_1 + k_2 = k$, $k_1 \leqslant \dim P_0\mathbb{C}^N$, and

$$k_2 \leqslant \min\{\dim P_-\mathbb{C}^N, \dim P_+\mathbb{C}^N\}. \tag{39}$$

Choose $\mathscr{V}_0$ as any $k_1$-dimensional subspace of $P_0\mathbb{C}^N$ and choose $\mathscr{V}_+$ as any $k_2$-dimensional subspace of $P_+\mathbb{C}^N$. Let $X : \mathscr{V}_+ \to (\mathscr{V}_0)^\perp \cap P_0\mathbb{C}^N$ be any operator and let $U : \mathscr{V}_+ \to P_-(\mathbb{C}^N)$ be any isometry and define

$$\mathscr{V} = \mathscr{V}_0 + \{v + Xv + Uv : v \in \mathscr{V}_+\}. \tag{40}$$

Then $\mathscr{W} = f(A)\mathscr{V}$ is a $k$-dimensional subspace of $\mathbb{C}^N$ with the desired properties.

## 5. Concluding remark

We conclude by briefly discussing the mathematical context of the work [1], which includes applications of the present work to quantum computing. Every quantum operation $\mathscr{E}$ on a given quantum system is determined operationally by a set of operators $\{A_i\}$ that act on the Hilbert space for the system via the so-called operator-sum representation $\mathscr{E}(\cdot) = \sum_i A_i(\cdot)A_i^*$. (See [7] for a brief introduction to some of the mathematical aspects of quantum computing.) In the context of quantum error correction, the $A_i$ are often called “error operators”. It is the effects of such operators that must be mitigated for whenever there is a transfer of quantum information determined by $\mathscr{E}$. There are numerous strategies that have been, and are being, developed for this type of error correction. We go into detail on this subject in [1], but here we indicate how the mathematical conditions that characterize correction in the fundamental protocol for quantum error correction connects with the higher-rank numerical ranges. In the “standard model” for quantum error correction [8,9], codes are identified with subspaces of the system Hilbert space, and “correctability” of a given code subspace $\mathscr{C}$ in terms of an error model $\mathscr{E}$ is shown to be equivalent to the existence of scalars $\Lambda = (\lambda_{ij})$ such that

$$P_{\mathscr{C}} A_i^* A_j P_{\mathscr{C}} = \lambda_{ij} P_{\mathscr{C}} \quad \forall i, j. \tag{41}$$

Here, $P_{\mathscr{C}}$ denotes the projection of the system space onto $\mathscr{C}$. Thus, the problem of finding correctable codes for a given error model $\mathscr{E} = \{A_i\}$ is equivalent to finding the compression-values inside the higher-rank numerical ranges $\Lambda_k(A_i^* A_j)\ \forall i, j$ and $\forall k > 1$, along with the corresponding projections. As indicated in [1], this problem may be reduced to a system of such problems for Hermitian or normal operators.

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