Journal of Convex Analysis
Volume 24 (2017), No. 3, 903–916

The Accessibility of Convex Bodies and Derandomization of the Hit and Run Algorithm

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Received: August 13, 2015
Accepted: April 19, 2016

We introduce the concept of accessibility and prove that any convex body X in $\mathbb{R}^d$ is accessible with relevant constants depending on d only. This property leads to a new algorithm which may be considered as a natural derandomization of the *hit and run* algorithm applied to generate a sequence of random points covering X uniformly. We prove stability of the Markov chain generated by the proposed algorithm and provide its rate of convergence.

*BC, TK and RK were supported by NSERC discovery grants and Ontario's ERA grants.
†The research of TS was supported by Polish Ministry of Science and Higher Education Grant no. IdP2011 000361 and EC grant RAQUEL. He was also supported by the National Science Centre of Poland, grant number DEC-2012/07/B/ST1/03320.
‡KŻ acknowledges a support by the NSC grant DEC-2011/02/A/ST1/00119.

ISSN 0944-6532 / $ 2.50 ©

1. Introduction

We are concerned with the accessibility of convex bodies in $\mathbb{R}^d$. Generally speaking, a convex set X is accessible if there exist two integers k and l and a set of vectors $\{e_1, \dots, e_l\}$ such that we may get from an arbitrary point $x \in X$ to a fixed point $y \in X$ in k steps walking in the directions given by the vectors and not leaving the set X. Our main result – Theorem 2.4 presented in Section 2 – says that any convex body in $\mathbb{R}^d$ is accessible. We also provide universal constants k, l, which depend only on the dimension d.

In Sections 3 and 4 we propose to consider an algorithm which may be treated as a natural derandomization of the *hit and run* algorithm (see [6, 10, 11, 12, 17]). Recall that the latter algorithm picks a random point along a random direction through the current point. Random directions are chosen uniformly on the sphere $S^{d-1} \subset \mathbb{R}^d$. In the algorithm proposed in this work we also select the line through the current point randomly but their directions are restricted to the set of given vectors $\{e_j\}_{j=1}^l$. If the vectors are such that the set is accessible the algorithm is exponentially convergent to the Lebesgue distribution. We provide its rate of convergence. There is no surprise that the algorithm is generically slower than the hit and run algorithm but the fact that it is also convergent at an exponential rate seems to be interesting per se.

Finally, in Section 5 we apply the algorithm to concrete convex bodies appearing in quantum information theory and statistical physics (quantum states, stochastic matrices and bistochastic matrices).

2. Accessibility of convex bodies

To characterize the set X and the set of vectors $\mathbf{e} = \{e_1, \dots e_l\}$, $l \geq d$, we will need the notion of accessibility in k steps illustrated in Fig. 2.1.

Definition. A compact set $X \subset \mathbb{R}^d$ is called **accessible** in k steps with respect to l vectors of $\mathbf{e}$, if there exists $x_* \in \operatorname{int} X$ such that from any point $x \in X$ one can reach x_* in not more than k moves along the basis vectors. Thus there exist some sets $\{i_1, \dots, i_k\} \subset \{1, \dots, l\}$ and $\{\lambda_1, \dots, \lambda_k\}$, $\lambda_1, \dots, \lambda_k \in \mathbb{R}$ such that

$$x + \sum_{j=1}^{m} \lambda_j \mathbf{e}_{i_j} \in \operatorname{int} X \quad \text{for any } m \leq k \quad \text{and} \quad x + \sum_{j=1}^{k} \lambda_j \mathbf{e}_{i_j} = x_*.$$

If a set X is accessible in k steps with respect to a fixed set of vectors of $\mathbf{e}$ we will briefly call it *k–accessible*. We are aimed at proving that any convex body in $\mathbb{R}^d$ is accessible with proper k and l depending on d only.

At the very beginning we prove the following lemma.

Lemma 2.1. *Assume that a convex body $X \subset \mathbb{R}^d$ is such that* $\operatorname{diam} X \leq R$ *and there is a r–ball B such that $B \subset X$. Then X is accessible in $d+1$ steps with respect to some $\{e_1, \dots, e_l\}$, where $l \leq (1 + 2R/r)^d + d$.*

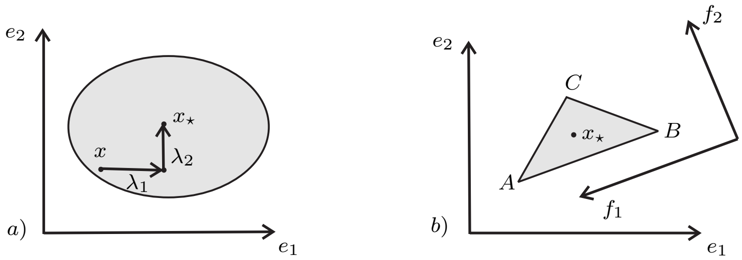

Figure 2.1: Examples of accessibility in $2D$: a) an ellipse is 2–accessible with respect to any orthogonal basis: any point x can be transformed into a selected point x_* in two moves along the basis vectors; b) the triangle ABC is not accessible with respect to basis (e_1, e_2), as starting from the corner A one cannot move along the basis vectors, but this triangle is 2–accessible with respect to the basis (f_1, f_2).

Proof. Let x_* denote the center of the ball B. Let $B(R)$ denote the ball centred at 0 with diameter R and let S be its sphere, i.e.

$$S = \{x \in \mathbb{R}^d : \|x\| = R\}.$$

Choose an open cover $\mathcal{U} = \{U_1, \ldots, U_q\}$ of the sphere S such that $\operatorname{diam} U_i \leq r/2$. We may assume that $q \leq (1 + 2R/r)^d$, by Lemma 4.10 in [14]. Choose $x_i \in U_i$ and let $e_i = [0, x_i]$ for $i = 1, \ldots, q$. Finally set $e_i = \mathfrak{e}_{i-q}$ for $i \in \{q+1, \ldots, q+d\}$, where $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_d\}$ is the standard basis in $\mathbb{R}^d$.

Consider now the sets of vectors $\vec{U_i} = \{[0, x] : x \in U_i\}$ for $i = 1, \ldots, q$. Choose $x \in X$. The set

$$\{x + \|x_* - x\| R^{-1} \cdot \vec{U_i}\}_{i=1}^q$$

is a cover of the sphere $\tilde{S} = \{y \in \mathbb{R}^d : \|y - x\| = \|x_* - x\|\}$. Obviously $x_* \in \tilde{S}$. Since $\|x_* - x\| \leq R$, the diameter of all sets of the cover is less than $r/2$. Thus there is $i \in \{1, \ldots, q\}$ such that

$$x + \|x_* - x\| R^{-1} \cdot \vec{U_i} \subset B(x_*, r) \cap \tilde{S}$$

and consequently

$$x + \|x_* - x\| R^{-1} \cdot e_i \in B(x_*, r).$$

Now after at most d–steps along the natural basis $\{\mathfrak{e}_1, \ldots, \mathfrak{e}_d\}$ we reach the point x_*. The proof is complete. □

A map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ is called an affine map if it is of the form

$$\varphi(x) = Ax + b \quad \text{for } x \in \mathbb{R}^d,$$

where $A : \mathbb{R}^d \to \mathbb{R}^d$ is linear and $b \in \mathbb{R}^d$ is a constant vector. The affine map φ is nonsingular iff $\det A \neq 0$. An **ellipsoid** in $\mathbb{R}^d$ is the image of a unit ball $B \subset \mathbb{R}^d$ under some nonsingular affine map.

To proceed we shall make use of the following famous result of F. John (see [7]).

Theorem 2.2. *Let $X \subset \mathbb{R}^d$ be a convex body. Then there is an ellipsoid E (called the John ellipsoid which turns out to be the ellipsoid of maximal volume contained in X) so that if c is the center of E then the inclusions*

$$E \subset X \subset c + d(E - c)$$

hold.

The following lemma is obvious.

Lemma 2.3. *If a convex body $X \subset \mathbb{R}^d$ is accessible in k steps with respect to $\{e_1, \ldots, e_l\}$, then for any nonsingular affine map $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ the set $\varphi(X)$ is also k–accessible with respect to $\{\varphi(e_1), \ldots, \varphi(e_l)\}$.*

Proof. Let $x_* \in \operatorname{int} X$ be such that for any $x \in X$ we have

$$x + \sum_{j=1}^{m} \lambda_j e_{i_j} \in \operatorname{int} X \quad \text{for any } m \leq k \quad \text{and} \quad x + \sum_{j=1}^{k} \lambda_j e_{i_j} = x_*$$

for some $\{i_1, \ldots, i_k\} \subset \{1, \ldots, l\}$ and $\{\lambda_1, \ldots, \lambda_k\}$, $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$.

Let $\tilde{x}_* = \varphi(x_*)$. We easily check that for any $y \in \varphi(X)$, $y = \varphi(x)$, we have

$$y + \sum_{j=1}^{m} \lambda_j \varphi(e_{i_j}) \in \varphi(\operatorname{int} X) \quad \text{for any } m \leq k \quad \text{and} \quad y + \sum_{j=1}^{k} \lambda_j \varphi(e_{i_j}) = \tilde{x}_*. \qquad \square$$

The main result of this section is the following theorem

Theorem 2.4. *Assume that $X \subset \mathbb{R}^d$ is a convex body. Then X is $d + 1$ accessible with respect to some $\{e_1, \ldots, e_l\}$, where $l \leq (2d + 1)^d + d$.*

Proof. From Theorem 2.2 it follows that

$$\varphi(B) \subset X \subset c + d(\varphi(B) - c)$$

for some nonsingular affine map $\varphi(\cdot) = A \cdot + b$ and a unit ball B. Thus

$$B \subset \varphi^{-1}(X) \subset \varphi^{-1}(c + d(\varphi(B) - c)) = dB + (d - 1)A^{-1}(b - c).$$

Hence $\operatorname{diam} \varphi^{-1}(X) \leq 2d$ and a ball B with radius 1 is contained in $\varphi^{-1}(X)$. From Lemma 2.3 it follows that $\varphi^{-1}(X)$ is accessible in $d + 1$ steps with some $\{\hat{e}_1, \ldots, \hat{e}_l\}$, where $l \leq (2d + 1)^d + d$. Since φ^{-1} is a nonsingular affine map, our hypothesis follows from Lemma 2.3. $\square$

3. Convergence theorem, general setup

Let $(X,\mathcal{A})$ and $(I,\mathcal{B})$ be two measurable spaces and let μ, ν be two probability measures on X and I, respectively.

We shall assume that for any $i\in I$ we have a transition kernel $T_i: X\times\mathcal{A}\to[0,1]$, i.e. $T_i(x,\cdot)$ is a probability measure for any $x\in X$ and for any $A\in\mathcal{A}$ the function $T_i(\cdot,A): X\to[0,1]$ is measurable. Additionally, we assume that μ is invariant with respect to T_i for any $i\in I$:

$$\mu(A)=\int_X T_i(x,A)\mu(dx)\quad\text{for all } A\in\mathcal{A}.$$

Now, if we assume that for any $A\in\mathcal{A}$ the function $T_\cdot(\cdot,A): I\times X\to[0,1]$ is $\mathcal{B}\otimes_\sigma\mathcal{A}$–measurable, then it follows from the Fubini theorem that the measure μ is invariant with respect to the operator Q of the form

$$Q\hat{\mu}(\cdot)=\int_X\int_I T_i(x,\cdot)\,d\hat{\mu}(x)d\nu(i).$$

By $\mathcal{M}$ and $\mathcal{M}_1$ we shall denote the set of all Borel measures and all probability Borel measures on X, respectively.

By $\|\cdot\|_{TV}$ we denote the **total variation norm**, i.e., if $\hat{\mu}\in\mathcal{M}-\mathcal{M}$, then $\|\mu\|_{TV}:=\hat{\mu}^+(X)+\hat{\mu}^-(X)$, where $\hat{\mu}=\hat{\mu}^+-\hat{\mu}^-$ is the Jordan decomposition of the signed measure $\hat{\mu}$.

We start with the following version of Doeblin's theorem [5], which provides sufficient conditions for exponential convergence rates of the transition operator Q.

Proposition 3.1. *Assume that there exist $\theta\in(0,1)$, $M\in\mathbb{N}$ and a measure $\nu\in\mathcal{M}_1$ such that for any measurable set A*

$$Q^M(x,A)\geq\theta\nu(A)\quad\textit{for any } x\in X.$$

Then there exists a unique invariant measure $\mu_*\in\mathcal{M}_1$ such that

$$\|Q^n\mu-\mu_*\|_{TV}\ \leq\ C\alpha^n\quad\textit{for all } \mu\in\mathcal{M}_1 \textit{ and } n\geq 1,\tag{1}$$

with the convergence rate $\alpha=(1-\theta)^{1/M}$ and prefactor $C=2(1-\theta)^{-1}$.

Assume now that X is a bounded metric space and let $\varphi: X\to\mathbb{R}$ be a Lipschitz function such that $\int_X\varphi(x)\mu(dx)=0$. Then we have (see Theorem 17.5.4 in [13]):

Proposition 3.2. *Let (Φ_n) be the Markov chain corresponding to the transition operator Q. Under the hypothesis of Proposition* 3.1*, we have the Central Limit Theorem (CLT),*

$$\frac{\sum_{i=1}^n\varphi(\Phi_i)}{\sqrt{n}}\Longrightarrow W,\quad\textit{as } n\to+\infty,$$

where W is a random variable with normal distribution $\mathcal{N}(0, D)$ for some $D \geq 0$ and the convergence is understood in law. Moreover, we have the Law of the Iterated Logarithm (LIL),

$$\limsup_{n \to +\infty} \frac{\sum_{i=1}^{n} \varphi(\Phi_i)}{\sqrt{2n \log \log n}} = D$$

with probability 1*. Of course the above implies that also*

$$\liminf_{t \to +\infty} \frac{\sum_{i=1}^{n} \varphi(\Phi_i)}{\sqrt{2n \log \log n}} = -D$$

with probability 1*.*

4. The algorithm and its convergence rate

We first describe the algorithm which is a natural derandomization of the hit and run algorithm.

4.1. Description of the algorithm:

We consider a compact set $X \subset \mathbb{R}^d$. In a preliminary step let us choose a set of normed vectors $e = \{e_1, \ldots e_l\}$ in $\mathbb{R}^d$ for $l \geq d$ such that $\mathrm{Lin}\{e_1, \ldots e_l\} = \mathbb{R}^d$. To generate a sequence of random points in X repeat the following steps of the algorithm, illustrated in Fig. 4.1.

(i) Choose an arbitrary starting point $x_0 \in X$,
(ii) Draw randomly a vector e_i, where the direction i is chosen with a uniform distribution among $(1, \ldots, l)$.
(iii) Find boundary points $x_1^{\min}, x_1^{\max} \in \partial X$ along the direction e_i: there exist positive numbers a, b such that $x_1^{\min} = x_0 - ae_i$ and $x_1^{\max} = x_0 + be_i$.
(iv) Select a point x_1 randomly with respect to the uniform measure in the interval $[x_1^{\min}, x_1^{\max}]$.
(v) Repeat the steps (ii)–(iv) to find subsequent random points $x_2, x_3, \ldots$.

This algorithm is very close to slice sampling. The main difference is that slice sampling proceeds recursively. It assumes one is able to simulate a uniform measure on an slice of codimension 1 (obtained by intersecting with an $n-1$ dimensional affine hyper plane).

4.2. Convergence rate with respect to the Lebesgue measure

Let X be a compact subset of $\mathbb{R}^d$ with a nonempty interior. Let a point x_* in the interior of X be given. We introduce two positive constants, r and R, such that $B(x_*, r) \subset X \subset B(x_*, R)$, where $B(x_*, r)$ denotes a closed ball in $\mathbb{R}^d$. Let $\mathbf{e} = \{e_1, \ldots, e_l\}$ such that $\mathrm{Lin}\{e_1, \ldots, e_l\} = \mathbb{R}^d$ be given. We consider the

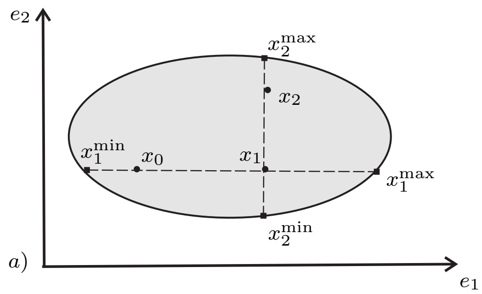

Figure 4.1: Algorithm to generate random points uniformly in a compact set X: one starts with any interior point $x_0 \in X$, picks randomly a direction (in this case e_1) finds two boundary points and draws x_1 randomly in the interval $[x_1^{\min}, x_1^{\max}]$. In the next step one chooses a next direction (here e_2), finds both boundary points and draws x_2 randomly in the interval $[x_2^{\min}, x_2^{\max}]$.

Markov chain $\mathbf{\Phi} = (\Phi_n)_{n\geq 1}$ corresponding to the algorithm illustrated in Fig. 4.1, which is described by the following transition function

$$T(x, A) \;=\; \frac{1}{l} \sum_{i=1}^{l} \nu_{x,i}(A), \tag{2}$$

where $\nu_{x,i}$ is the measure uniformly distributed over the set $\{y \in X : y - x = te_i \text{ for some } t \in \mathbb{R}\}$.

We are in a position to formulate the following theorem.

Theorem 4.1. *If $X \subset \mathbb{R}^d$ is k-accessible with respect to the basis $\mathbf{e} = \{e_1, \ldots, e_l\}$, $x_* \in X$ and $B(x_*, r) \subset X \subset B(x_*, R)$ for some $r, R > 0$, then the chain $\mathbf{\Phi}$ corresponding to the transition function T given by* (2) *satisfies the hypothesis of Proposition* 3.1 *with*

$$M = k + d \quad \text{and} \quad \theta = b_d\, l^{-k-d} (r/R)^{k+d}, \tag{3}$$

where $b_d = \pi^{d/2}/\Gamma(d/2+1)$ denotes the volume of a unit ball in $\mathbb{R}^d$.

Proof. First, there is no restriction in assuming that $r \leq 1$. Fix $x \in X$ and let $\{i_1, \ldots, i_k\}$ and $\{\lambda_1, \ldots, \lambda_k\}$ be given according to k–accessibility of X. We are going to derive lower bounds for subdensities f_m of $T^m(x, A)$ for $m = 1, \ldots, k$ defined on suitable spaces. Firstly, let $\tilde{X}_1 := \{x + t_1 e_{i_1} : t_1 \in \mathbb{R}\} \cap X$. By the fact that $\tilde{X}_1$ contains a non-degenerate interval its one dimensional measure $\mathcal{L}_1$ is positive. Set $f_1(y) = 1/R$ for $y \in \tilde{X}_1$. Finally, we easily check that for any measurable set A the following inequality holds

$$T(x, A) \geq (1/l) \int_{A \cap \tilde{X}_1} f_1 \mathrm{d}\mathcal{L}_1.$$

Define $\tilde{X}_2 = \bigcup_{t_2 \in \mathbb{R}} (\tilde{X}_1 + t_2 e_{i_2}) \cap X$. The set $\tilde{X}_2 \neq 0$ and its t_2–dimensional Lebesgue measure $\mathcal{L}_{t_2}$ is positive, where $t_2 = \dim \mathrm{Lin}\{e_{i_1}, e_{i_2}\}$, by k–accessibility of X. Set $f_2(y) := (1/R)^2$ for $y \in \tilde{X}_2$ and observe that

$$T^2(x, A) \geq (1/l)^2 \int_{A \cap \tilde{X}_2} f_2 \mathrm{d}\mathcal{L}_{t_2}.$$

By induction we define the sets $\tilde{X}_3, \ldots, \tilde{X}_m$. If we have done it for $m < k$, we may do it for $m+1$ as well. Namely, we set $\tilde{X}_m := \bigcup_{t_m \in \mathbb{R}} (\tilde{X}_{m-1} + t_m e_{i_m}) \cap X$. Then the set $\tilde{X}_m$ has positive t_m–dimensional Lebesgue measure, where $t_m = \dim \mathrm{Lin}\{e_{i_1}, \ldots, e_{i_m}\}$. Moreover,

$$T^m(x, A) \geq (1/l)^m \int_{A \cap \tilde{X}_m} f_m \mathrm{d}\mathcal{L}_{t_m},$$

where $f_m(y) := (1/R)^m$ for $y \in \tilde{X}_m$. In this way we obtain the constant $t_k = \dim \mathrm{Lin}\{e_{i_1}, \ldots, e_{i_k}\}$ and the set $\tilde{X}_k \ni x_0$. Set

$$e_{i_{k+1}} = e_{q_1}, \ldots, e_{i_{k+d}} = e_{q_d},$$

where $q_1, \ldots, q_d \in \{1, \ldots, l\}$ are such that $\mathrm{Lin}\{e_{q_1}, \ldots, e_{q_d}\} = \mathbb{R}^d$.

Now we repeat the procedure d more times. Then $\tilde{X}_{k+d} \supset B(x_0, r)$ and $t_{k+d} = d$ and we finally obtain

$$T^{k+d}(x, A) \geq (1/Rl)^{k+d} \mathcal{L}_d(A \cap B(x_0, r)).$$

Since, $B(x_0, r) \subset \tilde{X}_{k+d}$ we obtain

$$T^{k+d}(x, A) \geq (1/Rl)^{k+d} b_d r^d \nu(A) \ \geq \ b_d l^{-k-d} (r/R)^{k+d} \nu(A),$$

where $\nu(A) = \mathcal{L}_d(A \cap B(x_0, r))(\mathcal{L}_d(B(x_0, r)))^{-1}$.

This completes the proof. □

From Proposition 3.2 it follows that

Corollary 4.2. *Let $(\Phi_n)_{n \geq 1}$ be the Markov chain corresponding to the transition function T and let φ be an arbitrary Lipschitz function on X such that $\int_X \varphi d\mathcal{L}_d = 0$. Then $(\varphi(\Phi_n))_{n \geq 1}$ satisfies the CLT and LIL.*

4.3. Case where the density is not uniform

In this section we consider the case where the measure on a compact subset K of $\mathbb{R}^n$ is not just the restriction of the Lebesgue measure, but a more general measure. We assume that the measure has a continuous strictly positive density with respect to the Lebesgue measure, and denote this density by f.

In this case, we consider the subset $\tilde{K}$ of $\mathbb{R}^n \times \mathbb{R}^+$ given by

$$\tilde{K} = \{(x, y) : x \in K, 0 < y \leq f(x)\}.$$

This is clearly a compact set, and we can prove readily the following Proposition, from which we derive our estimates.

Proposition 4.3. *Assume that we are under the hypotheses of Theorem* 4.1 *for the set* $\tilde{K}$*. Then the law of the first component* x *of* (x, y) *converges towards the probability measure that we want to simulate* $f(x)dx$ *at the speed given by Theorem* 4.1.

Proof. This is a direct application of Fubini's theorem. □

5. Applications to quantum information theory and statistical physics

In several problems of statistical and quantum physics one works with states defined on an N dimensional space. The corresponding sets of states form compact convex sets in $\mathbb{R}^d$, were $d = d(N)$. The same is true for the set of linear discrete transformations acting on them. In the classical case one uses *stochastic* and *bistochastic matrices*, which send the set of probability simplex into itself, while in the quantum case one deals with *quantum operations*, formally defined as completely positive, trace preserving maps.

In several concrete applications, related e.g. to the theory of quantum information, one considers often various convex subsets of the above sets, and is interested to analyze properties of their typical elements. To this end it is important to develop an efficient algorithm to generate a sample of random points according to the flat measure in a given convex set X. In some cases there exist such algorithms dedicated to a given set: For instance, procedures to generate random quantum states were studied in [2, 8, 18], while other contributions deal with random subnormalized states [3], random bistochastic matrices [4] and random quantum operations [1].

However, the procedures mentioned above are dedicated to a particular problem and cannot be easily adopted to other convex sets. On the other hand, the sampling algorithm developed in this work is universal, as it allows one to generate sequences of random points distributed uniformly in an arbitrary convex body $X \subset \mathbb{R}^d$. We are going to characterize X by the radius R of the minimal outscribed sphere, the radius r of the maximal inscribed sphere and by the barycenter x_*.

The rest of the paper is devoted to providing explicit estimates of the convergence rate of the algorithm to generate random points in X, which depends on the dimensionality d, the ratio $\mu = r/R$ and on the accessibility parameter k.

5.1. Balls, cubes and simplexes in $\mathbb{R}^d$.

For balls and cubes in $\mathbb{R}^d$ it is not difficult to generate random points according to the uniform measure, so we will not advocate to use the above algorithm for this purpose. However it is illuminating to compare estimations for the parameters determining the convergence rate according to Eq. (3).

5.1.1. The Euclidean ball

For a unit **ball** B^d both radii coincide, $R = r$, so their ratio $\mu = r/R$ is equal to unity. Since for any choice of the basis $\mathbf{e}$ the ball is d accessible, estimation (3) gives $M = 2d$ and $\theta = b_d\, d^{-2d}$ where b_d denotes the volume of a unit d–ball. This implies the convergence rate of our algorithm applied to a d–ball,

$$\alpha = (1-\theta)^{1/M} = (1 - b_d d^{-2d})^{1/2d}.$$

5.1.2. The unit cube

For an unit **cube** C^d the inscribed radius $r = 1/2$, and outscribed radius $R = \frac{1}{2}\sqrt{d}$ so the ratio reads $\mu = r/R = 1/\sqrt{d}$. If the basis $\mathbf{e}$ is determined by the sides of the cube then C^d is d–accessible. This implies $M = 2d$ and $\theta = b_d d^{-3d}$ and yields the convergence rate

$$\alpha = (1 - b_d d^{-3d})^{1/2d}.$$

5.1.3. The simplex

For an N–**simplex** Δ_N embedded in $\mathbb{R}^d$ with $d = N-1$ we have $R = \sqrt{(N-1)/N}$ and $r = 1/\sqrt{N(N-1)}$ so that $\mu = 1/(N-1) = d^{-1}$. Note that the simplex Δ_N describes the set of classical states – N-point probability distributions.

For a d–simplex we can find a basis $\mathbf{e}$ such that the set Δ_{d+1} is d–accessible. This is the case if the first vector e_1 is parallel to a side of the simplex, e_2 and e_1 span the plane parallel to a face of Δ_{d+1}, while adding an additional vector e_n spans a hyperplane containing an n–face of the simplex. For this choice of the basis one obtains therefore $M = 2d$ and $\theta = b_d\, d^{-4d}$, which implies

$$\alpha = (1 - b_d d^{-4d})^{1/2d}$$

in our algorithm.

5.2. Quantum states

The set Ω_N of density matrices (Hermitian and positive operators, $\rho^\dagger = \rho \geq 0$, normalized by the trace condition $\mathrm{Tr}\rho = 1$) of size N has the dimension $d = N^2 - 1$. The radius of the out-sphere, equal to the Hilbert–Schmidt distance between a pure state $\mathrm{diag}(1, 0, \ldots, 0)$ and the maximally mixed state

$\rho_* = \mathbb{I}/N$, reads $R = \sqrt{(N-1)/N}$. The radius of the inscribed sphere given by the distance between ρ_* and the center of a face, $\text{diag}(0, 1, \ldots, 1)/(N-1)$ is equal to $r = 1/\sqrt{N(N-1)}$ hence $\mu = 1/(N-1) = 1/(\sqrt{d+1} - 1) \sim d^{-1/2}$.

Any quantum state $\rho \in \Omega_N$ can be expressed in terms of the generalized *Bloch vector* τ,

$$\rho = \frac{1}{N}\mathbb{I} + \sum_{i=1}^{d} \tau_i \lambda_i. \tag{4}$$

Here $\{\lambda_i\}$ is a set of $d = N^2 - 1$ traceless generators of the group $SU(N)$, which form an orthonormal basis in the Hilbert–Schmidt space of operators of order N. For $N = 2$ one usually takes three Pauli matrices σ_i while for $N = 3$ it is convenient to use eight Gell-Mann matrices [15]. Since the state ρ is hermitian, the coordinates of the corresponding Bloch vector, $\tau_i = \text{Tr}\lambda_i \rho$, are real. Thus the Bloch vector $\tau = (\tau_1, \ldots, \tau_d)$ belongs to $\mathbb{R}^d$ and the conditions for τ to guarantee positivity of ρ are known [9]. Setting some coefficients of τ to zero corresponds to a projection onto a subspace and does not spoil positivity of ρ.

Thus the d–dimensional convex set Ω_N of quantum states is d–accessible with respect to the Bloch basis $(\lambda_1, \ldots, \lambda_N)$. For this choice of the basis one obtains therefore $M = 2d = 2(N^2 - 1)$ and $\theta \sim b_d\ d^{-3d}$, which implies $\alpha \sim (1 - b_d d^{-3d})^{1/2d}$. Interestingly, from the point of view of the estimation for the convergence rate the set Ω_N of mixed quantum states, behaves analogously as a d–cube C^d of dimension $d = N^2 - 1$.

5.3. Stochastic matrices.

Stochastic matrices of order N form a convex body of dimensionality $d = N(N-1)$ and play a role of classical maps, which send the simplex of N–point probability vectors into itself. Each column of a stochastic matrix T consists of non-negative numbers which sum to unity, so it forms an N-simplex. Thus the set of stochastic matrices is equivalent to a Cartesian product of N simplexes Δ_N, so the estimates follow from section 5.1.3, as each column of T can be generated independently.

5.4. Bistochastic matrices.

The set $\mathcal{B}_N$ of bistochastic matrices of size N, called *Birkhoff polytope* and given by convex hull of all permutation matrices has dimensionality $d = (N-1)^2$. The radius of the out-sphere of $\mathcal{B}_N$, equal to the Hilbert–Schmidt distance between identity and the uniform matrix B_* containing all entries equal to $1/N$ reads $R = \sqrt{N-1}$. The radius of the inscribed sphere given by the distance between B_* and the matrix $B_0 = [NB_* - \mathbb{I}]/(N-1)$ is equal to $r = 1/\sqrt{N-1}$, which implies $\mu = 1/(N-1)$.

Consider the set $\mathcal{C}$ of all matrices of the form

$$C_{i\alpha\beta\gamma} = \begin{bmatrix} c_{11}=0 & \cdots & 0 & \cdots & c_{1N}=0 \\ \cdots & c_{i\alpha}=-1 & \cdots & c_{i\beta}=1 & \cdots \\ 0 & \cdots & 0 & \cdots & 0 \\ \cdots & c_{\gamma\alpha}=1 & \cdots & c_{\gamma\alpha}=-1 & \cdots \\ c_{N1}=0 & \cdots & 0 & \cdots & c_{NN}=0 \end{bmatrix}$$

for $i,\alpha,\beta,\gamma \in \{1,\dots,N\}$. The set $\mathcal{C}$ will play the role of $\mathbf{e} = \{e_1,\dots,e_l\}$. Obviously $l = N^2(N-1)^2$. It may be verified that the set $\mathcal{B}_N$ is $(N-1)^3$–accessible with respect to $\mathbf{e}$. To see it assume that $A = [a_{i,j}]_{1\le i,j\le N}$ is a bistochastic matrix with $a_{i,\alpha} > 1/N$. Let $\varepsilon = \min\{a_{i,\alpha} - 1/N, 1/N\}$. Observe that since $a_{i,\alpha} > 1/N$ and the matrix is bistochastic, there exists β such that $a_{i,\beta} < 1/N$. On the other hand, since $a_{i,\beta} < 1/N$, there exists γ such that $a_{\gamma,\beta} > 1/N$. Taking $A - \varepsilon C_{i\alpha\beta\gamma}$ we obtain the bistochastic matrix of the form

$$\begin{bmatrix} a_{11} & \cdots & \cdots & \cdots & a_{1N} \\ \cdots & a_{i\alpha}-\epsilon = \frac{1}{N} & \cdots & a_{i\beta}+\epsilon > 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_{\gamma\alpha}+\epsilon > 0 & \cdots & a_{\gamma\alpha}-\epsilon > 0 & \cdots \\ a_{N1} & \cdots & \cdots & \cdots & a_{NN} \end{bmatrix}.$$

Repeating this procedure at most $N-1$ times (possibly with different β and γ) we obtain a matrix with $a_{i,\alpha} = 1/N$. To obtain the matrix with all entries equal to $1/N$ we have to apply this procedure to at most $(N-1)^2$ entries and hence follows that $k = (N-1)^3$. Finally, we have $M = N(N-1)^2$ and $\theta = b_{(N-1)^2}(N-1)^{-4N(N-1)^2}$. Hence, in this case, one has

$$\alpha = (1-\theta)^{1/M} = (1 - b_{(N-1)^2}(N-1)^{-4N(N-1)^2})^{N(N-1)^{-2}}.$$

6. Concluding Remarks

The paper was mainly devoted to introducing the concept of accessibility of convex bodies. Our main result says that any convex body in $\mathbb{R}^d$ is accessible with some universal constant dependent only on d. But in the paper we also proposed a universal algorithm to generate random points inside an arbitrary compact set X in $\mathbb{R}^d$ according to the uniform measure. Any initial probability measure μ transformed by the corresponding Markov operator converges exponentially to the invariant measure μ_*, uniformly in X. Explicit estimations for the convergence rate are derived in terms of the ratio $\mu = r/R$ between the radii of the sphere inscribed inside X and the sphere outscribed on it and the number k determining the accessibility of the body with respect to a given orthogonal basis $\mathbf{e}$ in $\mathbb{R}^d$.

We hope that the algorithm presented here can be used in practice to generate, for instance, a sample of random quantum states. In the case of states of a composed quantum system, one can also generate a sequence of random states with positive partial transpose. Sampling random states satisfying a given condition and analyzing their statistical properties is relevant in the research on quantum entanglement and correlations in multi-partite quantum systems [16]. A standard approach of generating random points from the entire set of quantum states with respect to the flat measure [18] and checking a posteriori, whether the partial transpose of the state constructed is positive, becomes inefficient for large dimensions, as the relative volume of the set of PPT states becomes exponentially small [19].

Note that the notion of k–accessibility plays a crucial role in obtaining our estimations. Running the algorithm for the triangle ABC with the basis $\mathbf{e}$ (see Fig. 2.1b), with respect to which it is not finitely accessible, one would cover an open subset of the triangle (with two corners excluded). Although this set has the full measure of the triangle it is an open set, so the convergence will not be exponential.

In general, for any k–accessible set, the lower parameter k characterizing the accessibility is, the faster convergence of the Markov chain to the unique invariant measure μ_* one obtains.

Acknowledgements

The authors wish to thank David McDonald for many discussions. A part of this work emanates from TK's PhD thesis. Substantial progress was achieved during several peer visits of the authors to their co–authors' home institutions (and of TS and BC to Cambridge's INI). The hospitality of the aforementioned institutions is warmly acknowledged. It is a pleasure to thank R. Adamczak for constructive remarks and for drawing our attention to [17].

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