Isoentangled Mutually Unbiased Bases, Symmetric Quantum Measurements, and Mixed-State Designs

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Discrete structures in Hilbert space play a crucial role in finding optimal schemes for quantum measurements. We solve the problem of whether a complete set of five isoentangled mutually unbiased bases exists in dimension four, providing an explicit analytical construction. The reduced density matrices of these 20 pure states forming this generalized quantum measurement form a regular dodecahedron inscribed in a sphere of radius $\sqrt{3/20}$ located inside the Bloch ball of radius $1/2$. Such a set forms a mixed-state 2-design—a discrete set of quantum states with the property that the mean value of any quadratic function of density matrices is equal to the integral over the entire set of mixed states with respect to the flat Hilbert-Schmidt measure. We establish necessary and sufficient conditions mixed-state designs need to satisfy and present general methods to construct them. Furthermore, it is shown that partial traces of a projective design in a composite Hilbert space form a mixed-state design, while decoherence of elements of a projective design yields a design in the classical probability simplex. We identify a distinguished two-qubit orthogonal basis such that four reduced states are evenly distributed inside the Bloch ball and form a mixed-state 2-design.

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Introduction.—Recent progress of the theory of quantum information triggered renewed interest in foundations of quantum mechanics. Problems related to measurements of an unknown quantum state attract particular interest. The powerful technique of state tomography [1,2], allowing one to recover a density matrix, can be considered as a generalized quantum measurement, determined by a suitable set of pure quantum states of a fixed size d. Notable examples include symmetric informationally complete (SIC) measurements [3,4] consisting of d^2 pure states, which form a regular simplex inscribed inside the convex set $\Omega_d \subset \mathbb{R}^{d^2-1}$ of density matrices of size d, and complete sets of $(d+1)$ mutually unbiased bases (MUBs) [5] such that the overlap of any two vectors belonging to different bases is constant.

The above schemes are distinguished by the fact that they allow us to maximize the information obtained from a measurement and minimize the uncertainty of the results obtained under the presence of errors in both state preparation and measurement stages [4,6]. Interestingly, it is still unknown whether these configurations exist for an arbitrary dimension. In the case of SIC measurements analytical results were known in some dimensions up to $d = 48$; see Ref. [7] and references therein. More recently, a putative infinite family of SICs starting with dimensions $d = 4$, 8, 19, 48, 124, 323 has been constructed [8], while the general problem remains open. Nonetheless, numerical results suggest [7] that such configurations might exist in every finite dimension d. For MUBs, explicit constructions are known in every prime power dimension d [5], and it is uncertain whether such a solution exists otherwise, in particular [9,10] in dimension $d = 6$.

If the dimension is a square, $d = N^2$, the system can be considered as two subsystems of size N and the effects of quantum entanglement become relevant. It is possible to prove that the average entanglement of all bipartite states forming a SIC or a complete set of MUBs is fixed [11].

It is natural to ask whether there exists a particular configuration such that all the states forming the generalized measurements share the same amount of entanglement so that they are locally equivalent, $|\phi'\rangle = U_A \otimes U_B|\phi\rangle$. In the simplest case of $d = 4$, a set of 16 isoentangled vectors forming a SIC was analytically constructed by Zhu, Teo, and Englert [12]; thus, such a set can be obtained from a selected *fiducial* state $|\phi\rangle$ by local unitary operations. Further entanglement properties of SICs were studied in Refs. [13,14]. Although entanglement of the states forming MUBs in composite dimensions was analyzed [15–18], the analogous problem of finding a full set of isoentangled MUBs remained open till now even for a two-qubit system.

Collections of states forming a SIC measurement or a set of MUBs find numerous applications in the theory of quantum information [6,12,19,20]. They belong to the class of *projective designs*: finite sets of evenly distributed pure quantum states in a given dimension d such that the mean value of any function from a certain class is equal to the integral over the set of pure states with respect to the unitarily invariant Fubini-Study measure [3,21,22]. These discrete sets of pure quantum states, and analogous sets of unitary operators called *unitary designs* [23], proved to be useful for process tomography [24], construction of unitary codes [25], realization of quantum information protocols [26], derandomization of probabilistic constructions [27], and detection of entanglement [28].

A cognate notion of quantum conical design was recently proposed [29,30], which concerns operators of an arbitrary rank from the cone of mixed quantum states. However, these designs are not suitable to sample the set Ω_d of mixed states according to the flat, Hilbert-Schmidt (HS) measure. On the other hand, the general theory of *averaging sets* developed in Ref. [31] implies that such configurations of mixed quantum states do exist.

In this Letter we solve the long-standing problem of the existence of isoentangled MUBs in dimension four. Additionally, we introduce the notion of a quantum mixed-state design, such that mean values of selected functions over this discrete set of density matrices equals to the average value integrated over the set Ω_d, and provide a notable example with dodecahedral symmetry constructed from the constellation of isoentangled MUBs. Furthermore, we show that a projective t-design induces by the coarse graining map a t-design in the classical probability simplex and establish general links between the designs in the sets of classical and quantum states.

MUBs for bipartite systems.—The standard solution of 5 MUBs in dimension $d = 4$ consists of 12 separable states forming three bases and 8 maximally entangled states corresponding to the remaining two bases [6,16]. Thus the partial trace of these states yields a peculiar configuration inside the Bloch ball: 6 corners of a regular octahedron inscribed into the Bloch sphere, covered by two points each, correspond to 3 MUBs in $\mathcal{H}_2$. The other 8 points sit degenerated at the center of the ball representing the maximally mixed state $\mathbb{I}/2$. The total configuration thus consists of 7 points, at the expense of weighing the central point as four points at the surface. Note that the Schmidt vectors of the first 12 pure product states are $\lambda_{\rm sep} = (1, 0)$, while for the other eight states this vector reads $\lambda_{\rm ent} = (1/2, 1/2)$. As this set of vectors in $\mathcal{H}_4$ forms a projective 2-design, the average degree of entanglement measured by purity is fixed, $\langle \lambda_1^2 + \lambda_2^2 \rangle = 4/5$. For any dimension being a power of a prime, $d = p^k$, the standard solution of the MUB problem consist of $p + 1$ separable bases and $p^k - p$ maximally entangled bases [32]. In the case of $d = 9$, the set of MUBs consisting of 4 separable and 6 maximally entangled bases was studied by Lawrence [15].

Two-qubit isoentangled MUBs.—As the set of isoentangled vectors forming a SIC is known for two- [12] and three-qubit [33] systems, it is natural to ask whether there exists an analogous configuration of isoentangled MUBs. In other words, we wish to find a global unitary rotation $U \in U(4)$ acting on the standard constellation in such a way that the degeneracy of the configuration of 20 points is lifted and all of them become equally distant from the center of the Bloch ball. Then the corresponding vectors in $\mathcal{H}_4$ share the same degree of entanglement and can be obtained from a selected fiducial vector $|\phi_1\rangle$ by local unitaries, $|\phi_j\rangle = U_j \otimes W_j |\phi_1\rangle$, with $j = 2, \ldots, 20$.

We construct the desired set of five isoentangled MUBs in $\mathcal{H}_4$ making use of the fact that the group of local unitary operations is in this case isomorphic to the double cover of the alternating group A_5. It has two faithful irreducible representations of degree two and it admits a tensor product representation that allows us to construct the necessary local two-qubit gates $U_j \otimes W_j$.

As shown in Supplemental Material (SM) [34], the full analytic solution can be generated by local unitaries from the following fiducial state,

$$|\phi_1\rangle = \frac{1}{20}(a_+|00\rangle - 10i|01\rangle + (8i-6)|10\rangle + a_-|11\rangle), \quad (1)$$

where $a_\pm = -7 \pm 3\sqrt{5} + i(1 \pm \sqrt{5})$. Since the states forming five bases are isoentangled, their partial traces with respect to the first (or the second) subsystem share the same purity and belong to a sphere of radius $r = \sqrt{3/20}$, embedded inside the Bloch ball of radius $R = 1/2$. The set of 20 points enjoys a dodecahedral symmetry, shown in Fig. 1. Reductions of the four states stemming from each of the five bases in $\mathcal{H}_4$ form a regular tetrahedron in both reductions, so up to rescaling their Bloch vectors form a SIC for a single qubit. In both reductions the mixed states corresponding to all five bases form a five-tetrahedron compound with the same chirality, while their convex hull

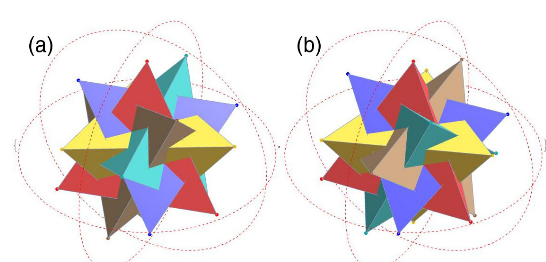

FIG. 1. One-qubit mixed-state design composed of 20 points inside the Bloch ball of radius $1/2$ obtained by partial trace of the 20 states in $\mathcal{H}_2 \otimes \mathcal{H}_2$, which form a set of isoentangled mutually unbiased bases for two qubits. Each basis is represented by the vertices of a regular tetrahedron inscribed in the sphere of radius $r = \sqrt{3/20}$. The reduced density matrices on both subsystems are shown in (a) and (b).

yields a regular dodecahedron. This configuration is not directly related to the arrangement of 20 pure states in dimension four forming the magic dodecahedron of Penrose [37], also studied in Refs. [38,39]. It differs also from the regular dodecahedron of Zimba [40], which describes a basis of five orthogonal anticoherent states in $\mathcal{H}_5$ in the stellar representation.

Projective and unitary designs.—Recall that a projective t-design is an ensemble of M pure states, $\{|\psi_j\rangle \in \mathcal{H}_d\}_{j=1}^M$, such that for any polynomial f_t of the state ψ of degree at most t, its average value is equal to the integral with respect to the unitarily invariant Fubini-Study measure $d\psi_{\mathrm{FS}}$ over the entire complex projective space of pure states, $\Xi_d = \mathbb{C}P^{d-1}$,

$$\frac{1}{M}\sum_{j=1}^{M} f_t(\psi_j) = \int_{\Xi_d} f_t(\psi) d\psi_{\mathrm{FS}}. \tag{2}$$

The notions of pure-state t-designs and unitary t-designs, consisting of matrices evenly distributed over the unitary group [23], found numerous applications in quantum information processing [24–27] and have been applied in experiments [20,28,41]. They can be considered as a special case of *averaging sets*, which are known to exist for arbitrary sets endowed with a probability measure [31]. Below we shall adopt this notion to the set of density matrices and show how such mixed-state designs can be constructed.

Mixed-state designs.—We shall start by introducing a formal definition of mixed-state t-designs with respect to the Hilbert-Schmidt measure in the space of density matrices.

Definition.—A collection of M density matrices $\{\rho_i \in \Omega_N\}_{i=1}^M$ is called a mixed-state t-design if for any polynomial g_t of the state ρ of degree t the average over the collection is equal to the mean value over the set Ω_N of mixed states in dimension N with respect to the normalized Hilbert-Schmidt measure $d\rho_{\mathrm{HS}}$:

$$\frac{1}{M}\sum_{i=1}^{M} g_t(\rho_i) = \int_{\Omega_N} g_t(\rho) d\rho_{\mathrm{HS}}. \tag{3}$$

The above condition, analogous to the definition of projective t-designs Eq. (2), is equivalent to the following relation:

$$\frac{1}{M}\sum_{i=1}^{M} \rho_i^{\otimes t} = \int_{\Omega_N} \rho^{\otimes t} d\rho_{\mathrm{HS}} =: \omega_{N,t}, \tag{4}$$

where the mean product state of a system consisting of t copies of a state ρ in dimension N averaged over the entire space Ω_N of mixed states is denoted by $\omega_{N,t}$. The measure $d\rho_{\mathrm{HS}}$ is defined by the requirement that each unit ball with respect to the Hilbert-Schmidt distance has the same volume.

Observe that for $t = 1$, definition (3) reduces to a resolution of the maximally mixed state, $(1/M)\sum_{i=1}^M \rho_i = (1/N)\mathbb{I}_N$, so any mixed-state design forms a generalized quantum measurement (also called POVM). Any set of positive semidefinite operators summing up to the identity defines a positive operator valued measure (POVM), the most general measurement allowed in quantum mechanics. To check whether a given configuration of density matrices forms a t-design, we establish the following necessary and sufficient condition.

Proposition 1.—A set consisting of M states from the set Ω_N of density matrices of size N forms a mixed-state t-design if and only if the following bound is saturated:

$$2\mathrm{Tr}\left(\omega_{N,t}\frac{1}{M}\sum_{i=1}^{M}\rho_i^{\otimes t}\right) - \frac{1}{M^2}\sum_{i,j=1}^{M}\mathrm{Tr}(\rho_i\rho_j)^t \le \gamma_{N,t}, \tag{5}$$

where $\gamma_{N,t} := \mathrm{Tr}(\omega_{N,t}^2)$, with $\omega_{N,t}$ defined by Eq. (4).

This condition, proved in SM [34], is closely related to saturation of the Welch bound [42] for projective and unitary designs [24]. Such a tool allows one to construct such designs by numerical minimization. Exact values of $\gamma_{N,t}$ for $t \le 5$ are given in SM [34].

Using the bound (5), we were able to find numerical lower bounds for the number M of states in a mixed 2-design: $M \ge 4$ for $N = 2$ and $M \ge 9$ for $N = 3$. In particular, for $N = 2$ the minimal mixed-state 2-design forms a tetrahedron inside the Bloch ball, an example of Platonic designs, equivalent to a single tetrahedron out of five plotted in Fig. 1; see SM [34].

Connection between pure- and mixed-state designs.—We will show that a mixed-state design for a single system of size N can be generated from a bipartite pure-state design of size $N \times N$. Since such constellations exist for all dimensions, the following result, proved in SM [34], implies that mixed-state t-designs exist for every N.

Proposition 2.—Any complex projective s-design $\{|\psi_j\rangle\}_{j=1}^M$ in the composite Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$ of dimension $d = N^2$ induces by partial trace a mixed-state t-design $\{\rho_j\}_{j=1}^M$ in Ω_N with $t \ge s$ and $\rho_j = \mathrm{Tr}_B|\psi_j\rangle\langle\psi_j|$. The same property holds also for the dual set $\{\rho_j' = \mathrm{Tr}_A|\psi_j\rangle\langle\psi_j|\}_{j=1}^M$.

In particular, Proposition 2 implies that taking partial trace of pure states forming a SIC in $\mathcal{H}_{N^2}$, or any other pure-state 2-design, one obtains a mixed-state 2-design in the set Ω_N of density matrices of size N. Interestingly, there exist distinguished cases for which the degree of the design increases, $t > s$: In SM we demonstrate that partial trace of any orthogonal basis, $t = 1$, of the five isoentangled MUBs yields a mixed-state 2-design, while the complete set of these MUBs, $t = 2$, leads to a mixed-state 3-design. Furthermore, the following one-to-one relation between a class of mixed-state 2-designs and projective 2-designs is proven in SM [34].

Proposition 3.—Any projective 2-design $\{|\psi_i\rangle\}_{i=1}^N$ of dimension N can be diluted into a mixed 2-design by taking projectors onto all states forming the projective 2-design with weight $p = (1 + N/1 + N^2)$ and the maximally mixed state $\mathbb{I}_N/N$ with weight $1 - p = (N^2 - N/1 + N^2)$.

Designs in classical probability simplex.—To construct one-qubit mixed-state designs one needs to determine the radial distribution of points inside the Bloch ball. It is related to an averaging set on the interval $[-1/2, 1/2]$ with respect to the Hilbert-Schmidt measure [43] determining the distribution of eigenvalues of a random mixed quantum state.

Returning to the general case of an arbitrary dimension N, consider any fixed probability measure $\mu(x)$ defined on the simplex Δ_N of N-point probability vectors. We wish to find an averaging set over the simplex, i.e., a sequence of M points $\{x_i : x_i \in \Delta_N\}_{i=1}^M$ which satisfy the condition analogous to t-designs, with respect to the integration measure $\mu(x)$:

$$\frac{1}{M}\sum_{i=1}^{M} f_t(x_i) = \int_{\Delta_N} f_t(x)\mu(x)dx, \tag{6}$$

where f_t denotes an arbitrary polynomial of order t.

Exemplary minimal solutions of this problem for low values of t and $N = 2$, so that the integration is done over the interval $\Delta_2 = [-1/2, 1/2]$, are presented in SM [34]. Here we shall concentrate on the cases of $t = 2$ for the Lebesgue and HS measure, as these results are linked to one-qubit pure- and mixed-state designs, respectively. An interval 2-design with respect to the flat Lebesgue measure gives coordinates of vertices of a tetrahedron inscribed in a unit sphere. The analogous design with respect to μ_{HS} provides the radius of a smaller sphere containing mixed-state 2-designs. An exemplary 2-design obtained by partial trace of 16 states forming a two-qubit isoentangled SIC POVM is shown in Fig. 2(d).

Positions of both points at the unit interval, which form 2-designs with respect to both measures, $x_\pm^L = \pm 1/2\sqrt{3}$ and $x_\pm^{\mathrm{HS}} = \pm\sqrt{3/20}$, can be thus related to the geometry of regular bodies inscribed into a sphere. Note that the design on [0, 1] with respect to the flat measure is formed by probabilities $p_i = |\langle i|\psi_j\rangle|^2$ related to projections of the states of the design onto the computational basis. This observation, corresponding to the decoherence of a quantum state to the classical probability vector, can be generalized for higher dimensions.

Proposition 4.—Any complex projective t-design $\{|\psi_j\rangle\}_{j=1}^M$ in the Hilbert space $\mathcal{H}_N$ induces, by the coarse graining map, $|\psi\rangle\langle\psi| \to \vec{p} \coloneqq \mathrm{diag}(|\psi\rangle\langle\psi|)$, a t-design in the N-point classical probability simplex Δ_N with respect to the flat measure μ_L.

To prove this fact it is sufficient to recall that the natural, unitarily invariant measure in the space of pure states induces, by decoherence, the flat measure μ_L in the probability simplex; see SM [34]. The notion of t-designs formulated for a probability simplex allows one to select classical states which are useful to approximate an integral over the entire set Δ_N. This also implies a simple, yet important observation that a mixed-state design in dimension $N = 2$ with $t > 3$ cannot be generated from isoentangled pure states in $\mathcal{H}_4$.

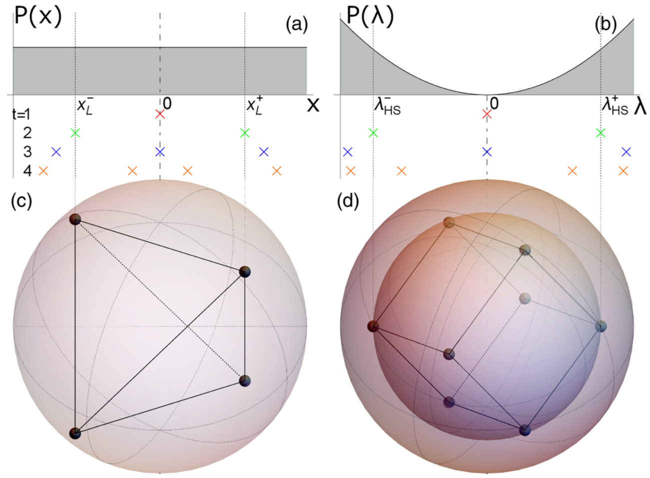

FIG. 2. Simplicial t-designs on $\Delta_2 = [-1/2, 1/2]$ for $t = 1$, 2, 3, 4 with respect to (a) flat measure and (b) Hilbert-Schmidt measure. (c) 2-design with respect to the flat measure μ_L corresponds to the x coordinates of a tetrahedron inscribed in a Bloch sphere, related to one-qubit projective 2-design produced by a SIC POVM in $\mathcal{H}_2$. (d) 2-design with respect to the HS measure corresponds to the radius of the sphere containing the mixed-state 2-design—the cube induced by the isoentangled SIC POVM in $\mathcal{H}_4$.

Furthermore, we suggest a general approach to obtain mixed designs of a product form. It will be convenient to use an asymmetric part $\tilde{\Delta}_N$ of the simplex Δ_N, which corresponds to ordering of eigenvalues, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N$.

Proposition 5.—Consider a t-design $\{\lambda_i\}_{i=1}^n$ in the simplex Δ_N with respect to the measure μ_{HS}, the corresponding set of diagonal matrices $\Lambda_i = \mathrm{diag}(\lambda_i)$, and any unitary t-design $\{U_j\}_{j=1}^m$. Let n' denote the number of points of the simplicial design belonging to the asymmetric part $\tilde{\Delta}_N$. Then the Cartesian product consisting of $n'm$ density matrices, $\rho_{ij} = U_j\Lambda_i U_j^\dagger$, $i = 1, \ldots, n'$ and $j = 1, \ldots, m$, forms a mixed-state t-design in Ω_N.

This statement, demonstrated in SM [34], allows us to construct Platonic mixed-state t-designs inside the Bloch ball: restricting the HS 2-design in Δ_2 to its half, $\tilde{\Delta}_2 = [0, 1/2]$, we arrive at a single point $x_+^{\mathrm{HS}} = \sqrt{3/20}$, which determines the radius of the sphere inside the Bloch ball. Taking the corresponding spectrum, $\Lambda = \mathrm{diag}(1/2 + x_+^{\mathrm{HS}}, 1/2 - x_+^{\mathrm{HS}})$, and rotating it by unitaries U_i from a unitary design in SU(2), we arrive at a mixed-state design. In the simplest case of the tetrahedral group, the mixed-state 2-design consists of four points forming one of the five tetrahedrons shown in Fig. 1, which arise by partial trace of the isoentangled bases listed in SM [34].

This example shows that there exist mixed-state t-designs which cannot be purified to a pure-state t-design.

Outlook and conclusions.—In this work we introduced the notion of mixed-state t-designs and established necessary and sufficient conditions for their existence. As any mixed-state 1-design forms a POVM, any design of a higher order t can be considered as a generalized measurement with additional symmetry properties [44]. From the physical perspective such a deterministic sequence of density matrices approximates a sample of random states and describes projective designs on a bipartite system AB, under the restriction that Alice receives no information from Bob. For potential applications concerning single-side measurements of bipartite systems, refer to SM [34].

Analyzing mixed-states designs, we solved the problem of existence of 20 locally equivalent two-qubit states which form a set of five MUBs. The obtained configuration defines a remarkable measurement scheme, useful for quantum state estimation [45] and for constructing symmetric entanglement witnesses based on MUBs [46,47], different from those analyzed earlier [48,49]. We analytically derived a two-qubit fiducial state, so that the other states forming the five bases were obtained by applying local unitaries. The partial trace of these two-qubit states forms a structure with dodecahedral symmetry, different from Penrose dodecahedron, inscribed into a sphere inside the Bloch ball [34]. This particular configuration consisting of five tetrahedrons, visualized in Fig. 1, leads to a notable example of a mixed-state 3-design. Each single tetrahedron, obtained by partial trace of a single basis, forms a 2-design.

This Letter establishes a direct link between designs in various sets which serve as a scene for quantum information processing: any projective t-design composed of pure states in dimension $d = N^2$ induces by partial trace a mixed-state design in the set of density matrices in dimension N, while by the decoherence channel it produces a design in the classical d-point probability simplex. A class of mixed-state designs can be constructed by the Cartesian product of a unitary design and a simplicial Hilbert-Schmidt design. These relations, based on transformations of measures, put the notion of designs in various spaces into a common framework, and show how to approximate averaging over continuous sets by discrete sums. Such an approach is not only of direct interest for theoretical work on foundations of quantum mechanics, but also for experimental realization of an approximate ensemble of random quantum or classical states [34].

We shall conclude the Letter with a brief list of open problems. (i) Find the minimal number of elements $M(N)$ forming a minimal mixed t-design in dimension N. (ii) Find minimal mixed-state t-designs, for which the variance of the purity of all the states is the smallest. (iii) Numerical calculations performed for $N = 3, 4, 5$ suggest that there exist orthogonal bases in $\mathcal{H}_N \otimes \mathcal{H}_N$ such that their partial trace gives a mixed-state 2-design in Ω_N. Determine whether this conjecture, proved here for $N = 2$, holds also for higher dimensions.

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