Trade-off relations for operation entropy of complementary quantum channels

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The entropy of a quantum operation, defined as the von Neumann entropy of the corresponding Choi–Jamiolkowski state, characterizes the coupling of the principal system with the environment. For any quantum channel acting on a state of a given size, one defines the complementary channel, which sends the input state into the state of the environment after the operation. Making use of subadditivity of entropy, we show that for any dimension the sum of both entropies is bounded from below. This result characterizes the trade-off between the information on the initial quantum state accessible to the principal system and the information leaking to the environment. For one qubit maps we describe the interpolating family of depolarizing maps, for which the sum of both entropies gives the lower boundary of the region allowed in the space spanned by both entropies.

Keywords: Quantum channels; entropy of an operation.

1. Introduction

Any time evolution of a density matrix $\rho$ can be described by a quantum operation $\Phi$, often called a quantum channel. It is defined by a completely positive, trace
preserving linear map, which sends the set of all quantum states into itself. Such a channel can be considered as a generalization of the unitary evolution of a density matrix that takes into account the interaction of the system with an environment or with the measurement apparatus. The action of such a map, following Stinespring’s dilation theorem, can also be interpreted as a unitary evolution of the joint system composed of the principal system and the environment, followed by the partial trace over the environment.

For any quantum operation $\Phi$ one defines the complementary operation $\tilde{\Phi}$, which maps the initial state $\rho$ into the final state of the environment. In the language of quantum communication, the state $\rho' = \Phi(\rho)$ describes the final state at the output of the channel, while the state $\rho'' = \tilde{\Phi}(\rho)$ describes the final state of the eavesdropper, who attempts to intercept the information transmitted in the state $\rho$.

To quantify the amount of information encoded in a classical or quantum state, various entropic measures (e.g. those based on von Neumann entropy) are often invoked. A similar entropic approach can also be used to describe the information flow induced by a quantum channel. In particular, the notions of Holevo quantity, coherent information and information exchange defined for a map $\Phi$ and an initial state $\rho$ are based on the von Neumann entropy. Entropies optimized over initial product states were also considered in a paper by Devetak et al.

To analyze the set of quantum operations, it is convenient to make use of the known Choi–Jamiolkowski isomorphism, which relates a quantum operation $\Phi$ acting on an $N$-dimensional state $\rho$ to an auxiliary state $\sigma_\Phi$ defined on an extended space of size $N^2$. If this state is pure, the corresponding map is unitary, $\Phi_U(\rho) = U\rho U^\dagger$, while the maximally mixed state for $\sigma_\Phi = \rho^\perp$ corresponds to the totally depolarizing channel. Thus, the degree of mixing of the Choi–Jamiolkowski state $\sigma_\Phi$, characterized by its von Neumann entropy, can be used to describe the degree of nonunitarity of the map $\Phi$ and the coupling with an environment.

More formally, for any channel $\Phi$, one defines its entropy as the von Neumann entropy of the corresponding Choi–Jamiolkowski state, $S_{\text{map}}(\Phi) := S(\sigma_\Phi)$. This quantity, also called the entropy of an operation or map entropy, yields an upper bound for the Holevo quantity $\chi$, associated with the transformation of the maximally mixed state $\rho^\perp$ by the operation $\Phi$.

The entropy of an operation is additive with respect to the tensor product, $S(\Phi \otimes \Psi) = S(\Phi) + S(\Psi)$. It is also known that for bistochastic channels, which preserve the maximally mixed state, the map entropy is subadditive with respect to concatenation. Furthermore, the entropy of an operation satisfies a trade-off relation with respect to the receiver entropy, which depends on singular values of the superoperator $\Phi$ and describes the receiver’s knowledge of the output state without any information on the input.

The aim of this work is to extend these results to establish a trade-off relation concerning the operation entropies of a given quantum channel and its complementary. The obtained lower bound for the sum of both entropies, $S(\Phi) + S(\tilde{\Phi})$, is
valid for an arbitrary system size $N$. We also analyze the distinguished channels, for which the above sum attains minimal values and yields the lower boundary of the allowed set in the plane $(S(\Phi), S(\bar{\Phi}))$. In the case of one-qubit maps, we identify the corresponding family of depolarizing channels and provide proof of extremity. Furthermore, we present families of channels in product dimensions saturating the obtained general bound and give conjecture concerning method of obtaining precise boundary of the allowed set of entropies in general dimension $N$.

2. Setting the Scene: Quantum Channels and Their Entropies

A quantum channel $\Phi$ denotes a trace preserving and completely positive linear map which maps a quantum state $\rho \in \mathcal{M}_N$ to another state of a possibly different dimension $M$, namely, $\rho' = \Phi(\rho) \in \mathcal{M}_M$. Any such channel, also called a quantum operation, can be represented by a set of $m$ Kraus operators $K_i$ 

$$\Phi(\rho) = \sum_{i=1}^{m} K_i \rho (K_i)^\dagger,$$

and this representation is not unique. In general, the number $m$ is arbitrary, but for any channel $\Phi$ one can find the canonical representation for which $m \leq N^2$ — see e.g., Ref. 11. Any map of the above form is completely positive, but to satisfy the trace preserving condition, $\text{Tr}(\Phi(\rho)) = \text{Tr}(\rho)$, the Kraus operators have to fulfill the identity resolution condition $\sum_i (K_i)^\dagger K_i = \mathbb{1}_N$.

Let us introduce the maximally entangled state on the extended system of dimensionality $N \times N$, $|\psi_+\rangle = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} |i\rangle \otimes |i\rangle$. For any map $\Phi$ taking states from $\mathcal{M}_N$ to states of the same dimensionality, it allows us to define the corresponding Choi–Jamiołkowski state, obtained by the action of an extended channel, $\mathbb{1} \otimes \Phi$, on the maximally entangled state, which can be written as a block matrix of linear dimension $N^2$, consisting of $N$ columns of $N \times N$ blocks

$$\sigma_\Phi := (\mathbb{1} \otimes \Phi) (|\psi_+\rangle \langle \psi_+|) = \begin{pmatrix} \Phi(|1\rangle \langle 1|) & \ldots & \Phi(|1\rangle \langle N|) \\ \vdots & \ddots & \vdots \\ \Phi(|N\rangle \langle 1|) & \ldots & \Phi(|N\rangle \langle N|) \end{pmatrix}. \quad (2)$$

Making use of this isomorphism, one defines the entropy of the channel as the von Neumann entropy $S(\rho) = -\text{Tr} \rho \log \rho$ of the corresponding Choi–Jamiołkowski state

$$S_{\text{map}}(\Phi) = S(\sigma_\Phi). \quad (3)$$

Since the Choi–Jamiołkowski state has dimension $N^2$, the entropy of a channel is bounded from above, $S_{\text{map}}(\Phi) \leq 2 \log N$. The entropy of a unitary channel is equal to zero, while the upper bound is saturated for the maximally depolarizing channel.11
If two quantum channels are close, so that the trace distance between the corresponding Choi–Jamiófkowski states is small, then due to the Fannes theorem, the entropies of both channels are similar. The entropy of a channel is easier to determine than the other entropic quantities, like the minimal output entropy, as no minimization is involved.

Several interesting properties of the entropy of a channel were obtained during the recent decade. However, to avoid misunderstanding, it is worth mentioning here that recently the notion of entropy of a channel was used in a similar spirit for a related but different quantity, calculation of which requires optimization.

Any quantum operation \( \Phi \) can also be represented in an environmental form using Stinespring dilation theorem, so that the initial state \( \rho \) is coupled to an environment of dimension \( M \) by a unitary operator \( U \) acting on the combined Hilbert space of dimension \( NM \)

\[
\Phi(\rho) = \text{Tr}_E[U(\rho \otimes |1\rangle\langle 1|)U^\dagger].
\]  

It can be assumed that the \( M \)-dimensional environment \( E \) is initially prepared in an arbitrary pure state \( \omega = |1\rangle\langle 1| \). The above expression is equivalent to the Kraus form (1), as the Kraus operators \( K^i \) are determined by the block-column of the matrix \( U \), namely, \( K^i_{jk} = U_{j+(i-1)N,k} \), and its unitarity imposes the trace preserving condition. The complementary operation \( \tilde{\Phi} \) is defined by an analogous formula with partial trace over the principal system \( S \)

\[
\tilde{\Phi}(\rho) = \text{Tr}_S[U(\rho \otimes |1\rangle\langle 1|)U^\dagger],
\]  

so it concerns the state of the environment after the operation.

The complementary channel \( \tilde{\Phi} \) can also be written with the use of the orthogonal SWAP operation, defined as \( O_{\text{SWAP}}(\rho \otimes \sigma)O_{\text{SWAP}} = \sigma \otimes \rho \), which exchanges the principal system with the environment

\[
\tilde{\Phi}(\rho) = \text{Tr}_E[O_{\text{SWAP}}U(\rho \otimes |1\rangle\langle 1|)U^\dagger O_{\text{SWAP}}].
\]  

These relations imply that the Kraus operators \( \tilde{K}_i \) forming the complementary operation \( \tilde{\Phi} \) can be obtained from Kraus operators \( K_i \) corresponding to the original channel by exchanging the rows of matrices

\[
(\tilde{K}_a)_{ij} = (K_i)_{aj},
\]  

where \( i, j = 1, \ldots, N, \alpha = 1, \ldots, M \) and Kraus operators that are not specified are assumed to be equal to zero.

3. Bounding the Sum of Two Entropies

In order to establish bounds for the entropy of a channel, we start by pointing out the relation between \( S(\Phi) \) and the entropy of the image of the maximally mixed state, \( \rho_* = \mathbb{I}/N \), with respect to the complementary channel.
Proposition 1. Consider a channel $\Phi$ acting on an $N$-dimensional system by coupling it with the environment of dimension $M$. The entropy of the channel is equal to the entropy of the image of the maximally mixed state $\rho_s = 1/N$ under the complementary channel $\tilde{\Phi}$. An analogous relation holds for the complementary channel $S_{\text{map}}(\Phi) = S(\tilde{\Phi}(\rho_s))$, $S_{\text{map}}(\tilde{\Phi}) = S(\Phi(\rho_s))$. \hfill (8)

Proof. The proof of this proposition is given in Appendix A. \hfill $\square$

To characterize the quantum information remaining in the initial state $\rho_s$ after an action of a given channel $\Phi$, one uses the coherent information $I_{\text{coh}}(\Phi, \rho)$ expressed by the difference between the output entropies of a channel and its complementary

$$I_{\text{coh}}(\Phi, \rho) = S(\Phi(\rho)) - S(\tilde{\Phi}(\rho)). \hfill (9)$$

Select now the initial state to be maximally mixed, $\rho = \rho_s$. Due to Proposition 1, the coherent information of $\Phi$ can be expressed in this case by the inversed difference of the entropy of the channel and its complementary

$$I_{\text{coh}}(\Phi, \rho_s) = S_{\text{map}}(\tilde{\Phi}) - S_{\text{map}}(\Phi). \hfill (10)$$

The more unitary the channel $\Phi$, the smaller its entropy $S_{\text{map}}(\Phi)$, and the less information leaks out of the initial state $\rho_s$ to the environment.

Proposition 1 allows us to demonstrate a general bound for the sum of entropies of a channel and its complementary.

Observation 1. The entropies of a given channel $\Phi$ and of its complementary $\tilde{\Phi}$ are bounded by the following inequality:

$$S_{\text{map}}(\Phi) + S_{\text{map}}(\tilde{\Phi}) = S_{\text{map}}(\Phi \otimes \tilde{\Phi}) \geq \log N. \hfill (11)$$

Proof. Observation 1 is easily proven by considering (8) and Proposition 8 in Ref. 15. An alternative proof is given in Appendix A. \hfill $\square$

Observation 2. The following two inequalities hold for any channel $\Phi$ acting on a space of dimension $N$ with $M$-dimensional environment:

$$S_{\text{map}}(\Phi) \leq \log M \quad \text{and} \quad S_{\text{map}}(\tilde{\Phi}) \leq \log N. \hfill (12)$$

Proof. Proof follows from direct inspection of (8). \hfill $\square$

Let us consider a channel $\Phi_U$ on an $N$-dimensional system with one Kraus operator $K_1 = U$ which is hence a unitary operator. The complementary channel $\tilde{\Phi}_U$ is given by a set of $N$ Kraus operators $\tilde{K}_i$ determined by successive rows of the Kraus operator $K_1$, $\tilde{K}_i = \sum_{j=1}^N U_{ij} |j\rangle \langle j|$. For the unitary channel $\Phi_U$, the Choi–Jamiołkowski state has the same entropy as the state related to the identity channel, thus $S_{\text{map}}(\Phi_U) = 0$. The Choi–Jamiołkowski state of the complementary
channel $\tilde{\Phi}$ is easily found to be composed of blocks $\sigma_{\mu\nu} = \rho_\xi \otimes |1\rangle\langle 1|$ of dimension $N$, thus $S^\text{map}(\tilde{\Phi}_U) = \log N$.Collecting the two entropies, we can see that for unitary channels of dimension $N$, the total entropy of channel and its complementary is given by

$$S^\text{map}(\Phi_U) + S^\text{map}(\tilde{\Phi}_U) = \log N,$$

(13)
and saturates the bound (11).

Another way of calculating (13) is through the use of Eq. (8). A unitary channel acting on a maximally mixed state $\rho_s = \mathbb{I}/N$ leaves it unchanged, $\Phi_U(\rho_s) = \rho_s$, from which $S(\Phi(\rho_s)) = \log N$. Furthermore, the complementary channel takes it to a projector state of the environment $\tilde{\Phi}_U(\rho_s) = |1\rangle\langle 1|$ so that $S(\tilde{\Phi}_U(\rho_s)) = 0$.

In the following sections, we aim to provide even more insight about the structure of the set of allowed operations in the plane of entropy of a channel and of its complementary.

4. Qubit Channels

The easiest system to consider is a qubit system coupled to a qubit environment. In such a case, the channel $\Phi$ and its complementary $\tilde{\Phi}$ can be both represented by two Kraus operators. In terms of entropies’ plane $(S^\text{map}(\Phi), S^\text{map}(\tilde{\Phi})) \equiv (S, \tilde{S})$, there are three points in the boundary of available region $A_2$, for which we identify the representative channels

(1) $(0, \log 2)$: Unitary channels $\Phi_U$, for which an exemplary channel is the identity channel with $K_1 = \mathbb{I}$;
(2) $(\log 2, 0)$ One-step emission channel $\Phi_E$ given by $K_1 = |1\rangle\langle 1|$ and $K_2 = |1\rangle\langle 2|$, complementary to identity channel, which sends any density matrix into the ground state $\Phi(\rho) = |1\rangle\langle 1|$;
(3) $(\log 2, \log 2)$: Coarse graining channel $\Phi_{\text{CG}} = \tilde{\Phi}_{\text{CG}}$, sending any quantum state into the diagonal matrix $\Phi(\rho) = \text{diag}(\rho)$, determined by Kraus operators $K_1 = |1\rangle\langle 1|$ and $K_2 = |2\rangle\langle 2|$.

In Fig. 1, we show the space available for qubit–qubit channels together with its boundary. The upper boundary for complementary channel entropy is given by a line segment $S = \ln 2$ and for any given value of $\tilde{S}$ there exists a channel, given by interpolation between coarse graining channel $\Phi_{\text{CG}}$ and unitary channel $\Phi_U$. Similarly, the upper boundary for channel entropy can be found by interpolating between $\Phi_{\text{CG}}$ and spontaneous emission channel $\Phi_E$. The third curve comprising the full boundary is given below.

**Proposition 2.** The lower boundary curve of the allowed set $A_2$ of one-qubit channels represented in the entropy plane, minimizing $S^\text{map}(\tilde{\Phi})$ for a given channel
The simplest way to obtain this parametric formula is to consider spontaneous emission channels interpolating between $\Phi_U$ and $\Phi_E$, given in terms of Kraus operators

$$
K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{x} \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \sqrt{1-x} \\ 0 & 0 \end{pmatrix},
$$

with $x = 2a - 1$. Entropy of this channel and its complementary follow the extreme curve in (14). It is important to note that these channels can be seen as those for entropy $S_{\text{map}}(\Phi)$ is given parametrically by

$$
S(a) = -a \log a - (1-a) \log(1-a)
$$

$$
\tilde{S}(a) = -\left(\frac{1}{2} - a\right) \log \left(\frac{1}{2} - a\right) - \left(\frac{1}{2} + a\right) \log \left(\frac{1}{2} + a\right),
$$

for $a \in [0, \frac{1}{2}]$.

**Proof.** A detailed proof of the extremity of this curve, labeled in Fig. 1 by letter ‘c’, is given in Appendix D.

The simplest way to obtain this parametric formula is to consider spontaneous emission channels interpolating between $\Phi_U$ and $\Phi_E$, given in terms of Kraus operators

$$
K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{x} \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & \sqrt{1-x} \\ 0 & 0 \end{pmatrix},
$$

with $x = 2a - 1$. Entropy of this channel and its complementary follow the extreme curve in (14). It is important to note that these channels can be seen as those for
which for constant entropy of the image of the maximally mixed state \( \Phi(\rho) \) the information escaping to the environment is minimized.

5. General Dimension

5.1. Emission channels and matrix \( L \)

Let us introduce a left upper triangular matrix \( L \) of dimension \( N \) with entries \( L_{ij} \in \{0, 1\} \) such that \( \sum_{i=1}^{N} L_{i,N+1-i} = 1 \). Any such matrix determines a valid quantum channel \( \Phi \) with Kraus operators

\[
K_i = \sum_{j=1}^{N+1-i} L_{ij} |j\rangle \langle j + i - 1|,
\]

with \( i = 1, \ldots, N \).

Its complementary channel \( \tilde{\Phi} \) is found by similar formula \( \tilde{K}_i = \sum_{j=1}^{N+1-i} L_{ji}^T |j\rangle \langle j + i - 1| \). We will call such channels emission channels.

The entropy for any emission channel represented by its matrix \( L \) is given by

\[
S_{\text{map}}(\Phi) = \frac{1}{N} \left( N \log N - \sum_{i=1}^{N} d_i \log d_i \right),
\]

where \( d_i = \sum_j L_{ij} \) is the number of ones in rows. Similarly, the entropy of the complementary channel \( \tilde{\Phi} \) can be given in terms of the numbers of ones \( \tilde{d}_i = \sum_j L_{ji} \) in columns of matrix \( L \)

\[
S_{\text{map}}(\tilde{\Phi}) = \frac{1}{N} \left( N \log N - \sum_{i=1}^{N} \tilde{d}_i \log \tilde{d}_i \right).
\]

Thus, the entropies may be calculated from the number of ones in columns and rows of the \( L \) matrix straight away. We can see that in terms of entropy, such channels may be denoted simply by an ordered pair of unordered sets \( \{d_i\}, \{\tilde{d}_i\} \).

To illustrate the point, we show below these triangular matrices for qubit one-step emission channel \( \Phi_{E} \) and identity channel \( \Phi_{I} \)

\[
L_{E} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad L_{I} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.
\]

Points in the entropy plane for these two channels are easily calculated from (17) and (18) as \( (0, \log 2) \) and \( (\log 2, 0) \), respectively. Interpolation between them can be denoted by a matrix \( A(L_{I}, L_{E}; x) \). The precise definition of channel defined by matrix \( A \) is given in Appendix B.
5.2. Boundary of the allowed set for $N = 3$

In the case of qutrits, the next easiest system to consider, there are four important points in the boundary of $A_3$, out of which three are similar to qubit channels already mentioned.

(1) $(0, \log 3)$: Unitary channels $\Phi_U$, exemplified by identity channel $K_1 = \mathbb{I}$, with matrix $L_U$:

$$L_U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix};$$

(2) $(\log 3, 0)$: Emission $\Phi_E$, with Kraus channels $K_i = |i\rangle\langle i|$ for $i = 1, 2, 3$, defined by a matrix $L_E$:

$$L_E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

(3) $(\log 3, \log 3)$: Coarse graining channel $\Phi_{CG}$, with Kraus operators $K_i = |i\rangle\langle i|$ for $i = 1, 2, 3$;

(4) $(\frac{1}{3} \log(\frac{27}{4}), \frac{1}{3} \log(\frac{27}{4}))$: Partial spontaneous emission channels, which can be exemplified by a channel $\Phi_4$ given by its matrix $L_4$:

$$L_4 = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (19)$$

With these four channels, we can characterize entire the boundary for qutrit channels with qutrit environment. Upper limits are analogous to ones for qubit–qubit channels: $(\log 3, \tilde{S})$ and $(S, \log 3)$ are trivially found by interpolating between coarse graining channel $\Phi_{CG}$ and either unitary channel $\Phi_U$ or spontaneous emission $\Phi_{SE}$, respectively. The lower boundary is conjectured below.

**Conjecture 1.** For qutrit channels with qutrit environment, the lower boundary of the allowed set $A_3$ is given by the parametric curve

$$S(a) = -a \log a - (1 - a) \log(1 - a)$$

$$\tilde{S}(a) = \log \frac{3}{3} - \left(\frac{1}{3} - a\right) \log \left(\frac{1}{3} - a\right) - \left(\frac{1}{3} + a\right) \log \left(\frac{1}{3} + a\right), \quad (20)$$

with $a \in (0, \frac{1}{3})$, together with reflection through line $S = \tilde{S}$, which contains all selfcomplementary channels $\Phi = \tilde{\Phi}$.

The parametric form of the curve $d$ shown in Fig. 1 can be obtained by considering entropy of the channels found from matrix $A(L_2, L_{\Phi_4}, 1 - 3a)$, as defined in Appendix B, which can be given explicitly in terms of two Kraus operators.
The curve $c$ can be found in similar manner by considering interpolation by $A(L_E, L_{\Phi_f}; 1 - 3a)$.

As we were unable to find any channels below the curve described above by extensive numerical probing of qutrit–qutrit channels by method described in Appendix C, we believe the conjecture should hold. Further checks through evolution via random Hamiltonians have been outlined in Appendix E, giving further evidence to the conjecture.

6. Bound Saturation in Product Dimensions

Let us consider an especially compelling example of a matrix $L$ of size 4

$$L = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
for which \( \{ d_i \} = \{ \hat{d}_i \} = \{ 2, 2 \} \). Using Eqs. (17) and (18), we find that the total entropy of such channel and its complementary yields \( S(\Phi) + S(\bar{\Phi}) = \log 4 \), which saturates the bound (11). This leads us to a more general statement.

**Proposition 3.** Consider a state of dimension \( N = N_A N_B \) and a unitary matrix, \( U \) of the same dimension. We define the channel \( \Phi \) by its Kraus operators

\[
(K_\alpha)_{ij} = \begin{cases} 
U_{i+(\alpha-1)N_A,j} & \text{for } i = 1, \ldots, N_A, \\
0 & \text{for } i > N_A.
\end{cases}
\]

With \( \alpha = 1, \ldots, N_B \) and \( j = 1, \ldots, N \), for such channels the bound (11) is saturated.

**Proof.** Using Eq. (8), we consider the action of channel on the maximally mixed state, given by

\[
\Phi(\rho_\alpha) = \sum_{\alpha=1}^{N_B} K_\alpha \frac{I_N}{N} K_\alpha^\dagger = \frac{N_B}{N} \sum_{j=1}^{N_A} |j\rangle\langle j|,
\]

which gives entropy \( S(\Phi(\rho_\alpha)) = S^{\text{map}}(\bar{\Phi}) = \log N_A \). Analogously, using relation (7) for the Kraus operators of the complementary channel, we find that \( S^{\text{map}}(\Phi) = \log N_B \), which completes the proof.

Noting that this family of channels is highly general, we formulate the following conjecture:

**Conjecture 2.** The number of distinct points saturating the bound (11) in the allowed set \( A_N \) is equal to the number of divisors of \( N \), including 1 and \( N \) itself.

Furthermore, extensive numerical searches for dimension \( N = 4, 5 \) suggest the following conjecture on the boundary:

**Conjecture 3.** For any dimension \( N \), the entire lower boundary of available region \( A_N \) in the entropy plane \( (S, \bar{S}) \) can be found as a family of curves attained by channels generated from matrices \( A(L_1, L_2; x) \).

In Appendix F, we provide the conjectured form of the boundary curve restricting the allowed region \( A_4 \) in the entropy plane for maps acting on four-level systems with the corresponding quantum channels.

7. Concluding Remarks

We established a general trade-off relation concerning the operation entropy of a given channel \( \Phi \) and its complementary \( \bar{\Phi} \). In this way, we bounded from below the sum of operation entropies, \( S^{\text{map}}(\Phi) + S^{\text{map}}(\bar{\Phi}) \), which characterizes the sum of information on the initial state \( \rho \) accessible to the receiver of the output state \( \rho' \) and the eavesdropper controlling the environment.

Furthermore, we provided an exact characterisation of the boundary of the allowed set \( A_2 \) describing all single-qubit quantum channels in the entropy plane.
\((S_{\text{map}}(\Phi), S_{\text{map}}(\bar{\Phi}))\). Similar results concerning one-qutrit channels are formulated as a conjecture that allows one to predict the general form of the boundary of the analogous sets \(A_N\) in the entropy plane for quantum operations acting on a system of arbitrary dimension \(N\). In this way, we identified a particular class of quantum channels which minimize the entropy \(S_{\text{map}}(\bar{\Phi})\) of the complementary channel among all the channels with fixed map entropy \(S_{\text{map}}(\Phi)\). Our results may find applications in quantum thermodynamics in situations where one wants to minimize entropy production in both a system and its environment.

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Appendix A. Proof of Proposition 1 and Observation 1

Proof. Let us consider an extended system \(ABC\) with subsystems \(A\) and \(B\) of dimensionality \(N\) and subsystem \(C\) of dimensionality \(M\) in a pure state

\[ \rho_{ABC} = |\psi_+\rangle \langle \psi_+|_{AB} \otimes |1\rangle \langle 1|_C, \]

where \(|\psi_+\rangle = \sum_{i=1}^N 1/\sqrt{N} |i\rangle \otimes |i\rangle\) is the \(N\)-dimensional Bell state. Assume now that the operation \(\Phi\) is induced by a unitary operation \(U_{BC}\), which couples the system \(B\) with \(M\)-dimensional environment \(C\) initially in a pure state, \(\Phi(\rho) = \text{Tr}_C U_{BC}(\rho \otimes |1\rangle \langle 1|) U_{BC}^\dagger\). Then, we can introduce a tri-partite unitary operation, \(W = 1_A \otimes U_{BC}\), and act with it on state the \(\rho_{ABC}\), obtaining another pure state

\[ \rho'_{ABC} = W\rho_{ABC}W^\dagger. \]

For any bipartite pure state, the entropies of both partial traces are equal

\[ S(\rho'_{AB}) = S(\rho'_{C}) \quad \text{and} \quad S(\rho'_{AC}) = S(\rho'_{B}), \]

which with use of the definition of the complementary channel \(\bar{\Phi}\) implies the desired relations (8).

\[ \Box \]

Proof. Let us again consider the state \(\rho'_{ABC}\) from (A.2) and its reductions. To prove inequality (11), we will consider Araki–Lieb inequality

\[ |S(\rho'_A) - S(\rho'_B)| \leq S(\rho'_{AB}). \]

Subsystem \(A\) does not change under the action of the unitary operation \(W\), so that we have \(S(\rho'_A) = S(\rho_*) = \log N\). Furthermore, the dimensions of systems \(A\) and \(B\) are

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the same, so that \( S(\rho'_B) \leq S(\rho'_A) \) and \( |S(\rho'_A) - S(\rho'_B)| = S(\rho'_A) - S(\rho'_B) \). Using (A.3), we can see that
\[
\log N \leq S(\rho'_{AB}) + S(\rho'_{AC}),
\]
which proves inequality (11).

\[\square\]

**Appendix B. Channels Determined by Matrix \( L \) and \( A \)**

For any matrix \( L \), the block structure of the corresponding Choi–Jamiołkowski state \( \sigma_\Phi \) is easily given in terms of the entries of the matrix \( L \)
\[
(\sigma_\Phi)_{\mu\nu} = \sum_{i=1}^{N} L_{i,\mu+1-i} L_{i,\nu+1-i} |\mu + 1 - i\rangle \langle \nu + 1 - i|,
\]
with \( L_{ij} \in \{0, 1\}, \mu, \nu = 1, \ldots, N \) and all projectors \(|i\rangle \langle j|\) referring to the system on which the channel \( \Phi \) acts, without the environment. We notice that blocks of the Choi–Jamiołkowski state \( \sigma_\Phi \) either remain as rank one projectors under summation or they are zeroed out. In general, every such channel generates a “double-block” structure of \( \sigma_\Phi \), where the size of the corresponding blocks can be read out from the number of ones in consecutive columns.

To illustrate the point, let us take a particular matrix \( L \) and construct the corresponding Choi–Jamiołkowski state.

\[
L = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \quad \sigma_\Phi = \frac{1}{3} \begin{pmatrix} |1\rangle \langle 1| & 0 & 0 \\ 0 & |1\rangle \langle 1| & |1\rangle \langle 2| \\ 0 & |2\rangle \langle 1| & |2\rangle \langle 2| \end{pmatrix}.
\]

To further simplify notation, we introduce a notion of interpolation between two channels represented by matrices \( L_1 \) and \( L_2 \). Matrix \( A \) is introduced in terms of its entries:
\[
(\sigma_\Phi)_{\mu\nu} = \sum_{i=1}^{N+1-i} A_{i,j} |i\rangle \langle j + i - 1|.
\]
It can be easily found that the Kraus operators obtained in this way fulfill the identity resolution if \( \sum_{i=1}^{N} (A_{i,N+1-i})^2 = 1 \).

As an example, let us consider the interpolation between channels given in terms of \( L \) matrices as
\[
L_1 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
which give $A$ matrix and corresponding Choi–Jamiołkowski states.

$$A(L_1, L_2; x) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{1 - x} & \sqrt{x} \end{pmatrix},$$

$$\rightarrow \sigma_\Phi = \frac{1}{3} \begin{pmatrix} |1\rangle\langle1| & 0 & 0 \\ 0 & |1\rangle\langle1| & \sqrt{1 - x}|1\rangle\langle2| \\ 0 & \sqrt{1 - x}|2\rangle\langle1| & x|1\rangle\langle1| + (1 - x)|2\rangle\langle2| \end{pmatrix}$$

$$\rightarrow \sigma_{\Phi} = \frac{1}{3} \begin{pmatrix} |1\rangle\langle1| & |1\rangle\langle2| & \sqrt{x}|1\rangle\langle3| \\ |2\rangle\langle1| & |2\rangle\langle2| & \sqrt{x}|2\rangle\langle3| \\ \sqrt{x}|3\rangle\langle1| & \sqrt{x}|3\rangle\langle2| & x|3\rangle\langle3| + (1 - x)|2\rangle\langle2| \end{pmatrix}. \quad (B.3)$$

Similar notion of upper-triangular matrix $A$ can be extended to arbitrary entries as long as they fulfill the condition imposed by the identity resolution, $\sum_{i=1}^{N} (A_{i,N+1-i})^2 = 1$.

**Appendix C. Generation of the Channels from the Allowed Set $A_N$ in the Entropy Plane**

Standard generation of Kraus operators for $N$-dimensional system with $M$-dimensional environment involves generating unitary matrix $U$ of dimension $N \cdot M$ with respect to flat measure and assuming that its first block column corresponds to the set of Kraus operators, i.e.

$$K^i_{jk} = U_{j+(i-1)N,k}.$$ \quad (C.1)

This, however, would yield underrepresentation of channels in certain regimes of entropy. In order to overcome this issue, we propose different ways of generating unitary matrices as follows:

1. From all $J$ sets of positive integers $\{n_i\}_{i=1}^J$ such that their sum is equal to the desired dimension of the unitary matrix, $\sum_i n_i = N \cdot M$, we take one at random with probability $1/J$.

2. We generate set of unitary matrices of dimensions defined by the chosen set of integers $\{U_i : \dim(U_i) = n_i\}_{i=1}^J$ and construct a block-diagonal matrix

$$U = \begin{pmatrix} U_1 & 0 & 0 \\ 0 & U_2 & 0 \\ 0 & 0 & \ddots \\ 0 & 0 & 0 & U_J \end{pmatrix}, \quad (C.2)$$
where zeroes are to be understood as matrices of dimension $n_i \times n_j$ filled with zeroes.

- From all possible permutation operations of size $N \cdot M$, we take two permutations $P_1$ and $P_2$ and define the unitary matrix

$$W = P_1 U P_2.$$  \hspace{1cm} (C.3)

The unitary operation $W$ allows us to probe the possible entropies of channels more uniformly than in the case of standard generation of Kraus operators.

**Appendix D. Proof of Proposition 2**

We start from the representation of the qubit channel with qubit environment by Kraus operators. By utilizing unitary freedom of pre- and postpreparation given by $K_i \rightarrow UK_i V$, we may transform first Kraus operator $K_1$ into a diagonal form with $a, b \in \mathbb{R}$ with use of standard procedure of singular value decomposition, thus getting Kraus operators of the form

$$K_1 = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}, \quad K_2 = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \hspace{1cm} (D.1)$$

Additional conditions imposed by decomposition of unity $\sum_i K_i^\dagger K_i = I$ can be rewritten as

$$a^2 + |\alpha|^2 + |\gamma|^2 = 1, \hspace{1cm} (D.2)$$
$$b^2 + |\beta|^2 + |\delta|^2 = 1, \hspace{1cm} (D.3)$$
$$\alpha \beta^* + \gamma \delta^* = 0. \hspace{1cm} (D.4)$$

Equations (D.2) and (D.3) allow us to introduce phased spherical coordinates

$$a = \cos \theta_1, \quad b = \cos \theta_2, \quad \alpha = \sin \theta_1 \cos \phi_1 e^{i\chi_1}, \quad \beta = \sin \theta_2 \cos \phi_2 e^{i\chi_2},$$
$$\gamma = \sin \theta_1 \sin \phi_1 e^{i\chi_2}, \quad \delta = \sin \theta_2 \sin \phi_2 e^{i\chi_2},$$

and in turn rewrite Eq. (D.1) as

$$K_1 = \begin{pmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{pmatrix}, \quad K_2 = \begin{pmatrix} \sin \theta_1 \cos \phi_1 e^{i\chi_1} & \sin \theta_2 \cos \phi_2 e^{i\chi_2} \\ \sin \theta_1 \sin \phi_1 e^{i\chi_2} & \sin \theta_2 \sin \phi_2 e^{i\chi_2} \end{pmatrix}, \hspace{1cm} (D.5)$$

with $\theta_1, \theta_2 \in (0, \pi)$ and $\phi_1, \phi_2, \chi_1, \chi_2, \chi_1', \chi_2' \in (0, 2\pi)$.

From here, we may inquire about subfamilies satisfying Eq. (D.4). We will do this in steps, eliminating possible cases one by one.

First, we consider family of channels with $\sin \theta_1 = 0$ given by Kraus operators of the form

$$K_1 = \begin{pmatrix} 1 & 0 \\ 0 & \cos \theta_2 \end{pmatrix}, \quad K_2 = \begin{pmatrix} \sin \theta_2 \cos \phi_2 e^{i\chi_2} \\ 0 \sin \theta_2 \sin \phi_2 e^{i\chi_2} \end{pmatrix},$$
with complementary channel defined accordingly
\[ \tilde{K}_1 = \begin{pmatrix} 1 & 0 \\ 0 & \sin \theta_2 \cos \phi_2 e^{i\chi_{21}} \end{pmatrix}, \quad \tilde{K}_2 = \begin{pmatrix} 0 & \cos \theta_2 \\ 0 & \sin \theta_2 \sin \phi_2 e^{i\chi_{22}} \end{pmatrix}. \]

Channels from this family, acting on the maximally mixed state \( \rho_s = I/2 \) give
\[ \rho_K = \frac{1}{2} \begin{pmatrix} \cos^2(\phi_2)\sin^2(\theta_2) + 1 & e^{i(\chi_{21} - \chi_{22})} \cos(\phi_2)\sin(\phi_2)\sin^2(\theta_2) \\ e^{-i(\chi_{21} - \chi_{22})} \cos(\phi_2)\sin(\phi_2)\sin^2(\theta_2) & 1 - \cos^2(\phi_2)\sin^2(\theta_2) \end{pmatrix}, \]
\[ \rho_{\tilde{K}} = \frac{1}{2} \begin{pmatrix} \cos^2(\theta_2) + 1 & e^{-i\chi_{22}} \cos(\phi_2)\sin(\phi_2)\sin(\theta_2) \\ e^{i\chi_{22}} \cos(\phi_2)\sin(\phi_2)\sin(\theta_2) & 1 - \cos^2(\theta_2) \end{pmatrix}. \]

The eigenvalues for these states read
\[ \lambda_{K,1} = \frac{1}{2} + \frac{1}{2} \cos(\phi_2)\sin^2(\theta_2), \]
\[ \lambda_{K,2} = \frac{1}{2} - \frac{1}{2} \cos(\phi_2)\sin^2(\theta_2), \]
\[ \lambda_{\tilde{K},1} = \frac{1}{2} + \frac{1}{8} \sqrt{2 \cos(4\theta_2)\cos^2(\phi_2) + 8 \cos(2\theta_2) - \cos(2\phi_2) + 7}, \]
\[ \lambda_{\tilde{K},2} = \frac{1}{2} - \frac{1}{8} \sqrt{2 \cos(4\theta_2)\cos^2(\phi_2) + 8 \cos(2\theta_2) - \cos(2\phi_2) + 7}. \]

In order to retrieve dependencies between the channel eigenvalues, we will set \( \lambda_{K,1} = a \in [0, \frac{1}{2}). \) Solving for \( \phi_2 \) we obtain two solutions
\[ \phi_2 = \pm \arccos \left( \frac{1 - 2a}{\sin^2 \theta_2} \right). \]  

This gives \( \lambda_{K,1} \) in terms of one variable \( \theta_2 \) and a constant \( a. \)

In case of qubits, minimization of von Neumann entropy \( S(\rho) = -\text{Tr}(\rho \log \rho) \) can be easily shown to be equivalent to minimization of linear entropy \( S_{\text{lin}} = 1 - \lambda_{K,1}^2 - \lambda_{\tilde{K},2}^2. \) Consider the differential of \( S(\rho) \) with eigenvalues \( (\lambda, 1 - \lambda). \) We find that
\[ dS = -(d(\lambda \log \lambda) + d[(1 - \lambda) \log(1 - \lambda)]) = (\log(1 - \lambda) - \log \lambda) d\lambda, \]
which is zero if \( \lambda = \frac{1}{2} \) or \( d\lambda = 0. \) Similarly, in case of linear entropy, we find that
\[ dS_{\text{lin}} = (2 - 4\lambda) d\lambda, \]
which is found to be zero under the same circumstances. Let us now consider the linear entropy of aforementioned \( \rho_{\tilde{K}} \) given as
\[ S_{\text{lin}} = 1 - \frac{1}{8} \left( \cos^2(\theta_2)(2\sin^2(\theta_2) \cos \left( 2 \arcsin \left( \frac{2a - 1}{\sin^2 \theta_2} \right) \right) + \cos(2\theta_2) + 3) + 4 \right). \]
In order to extremize this we need
\[
\frac{\partial S_{\text{lin}}}{\partial \theta_2} = 0,
\]
which leads to solutions \(\theta_2 = \pi/2, \theta_2 = \arccos(\pm \sqrt{2a})\) or \(\theta_2 = \arccos(\pm \sqrt{2(1-a)})\).

When \(\theta_2 = \pi/2\), the answer is independent from \(a\) and lies above the expected extremal curve. For the remaining solutions, they are equivalent up to permutation of eigenvalues between Kraus operators for channel or its complementary. For this reason, without loss of generality, we consider only one of them.

For \(\theta_2 = \arccos(\sqrt{2a})\) the eigenvalues read:
\[
\lambda_{K,1} = a, \quad \lambda_{K,2} = 1-a, \quad \lambda_{\tilde{K},1} = \frac{1}{2} - a, \quad \lambda_{\tilde{K},2} = \frac{1}{2} + a,
\]
and the solution is well defined for \(a \in [0, \frac{1}{2}]\). In this way we obtain a parametric form of the curve representing the boundary channels in terms of both entropies \(S\) and \(\tilde{S}\),
\[
S = S_{\text{map}}(\Phi) = -\left(a \log(a) + (1-a) \log(1-a)\right),
\]
\[
\tilde{S} = S_{\text{map}}(\tilde{\Phi}) = -\left(\left(\frac{1}{2} - a\right) \log \left(\frac{1}{2} - a\right) + \left(\frac{1}{2} + a\right) \log \left(\frac{1}{2} + a\right)\right).
\]
This is the curve given in Eq. (14). Analogous procedure, consisting in setting one eigenvalue for \(\tilde{\Phi}(\rho_s)\) equal to \(a\), extracting \(\phi_1(a)\) and calculating the extremas for linear entropy, leads to solution when \(\sin(\theta_2) = 0\). For this reason, we only conclude without presentation that it leads to analogous conclusions.

The next step is to consider the remaining case of (D.4), leading to nontrivial condition
\[
\cos \phi_1 \cos \phi_2 e^{i(\chi_{11} - \chi_{12})} + \sin \phi_1 \sin \phi_2 e^{i(\chi_{21} - \chi_{22})} = 0. \tag{D.12}
\]
Its consideration will be split into further substeps.

If \(\sin \phi_1 = 0\), the condition (D.12) is reduced to \(\pm \cos \phi_2 e^{i(\chi_{11} - \chi_{12})} = 0\) which is equivalent to \(\cos \phi_2 = 0\). Such statement reduces the Kraus operators to the form
\[
K_1 = \begin{pmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{pmatrix}, \quad K_2 = \begin{pmatrix} \pm \sin \theta_1 e^{i\chi_{11}} & 0 \\ 0 & \pm \sin \theta_2 e^{i\chi_{22}} \end{pmatrix}.
\]
Such channel, however, does not change the maximally mixed state \(\Phi(\rho_s) = \rho_s\), and so the entropy of its complementary \(S_{\text{map}}(\tilde{\Phi}) = \log 2\) is maximal and as such does not come into our interest. Similar reasoning applies to the case when \(\sin \phi_2 = 0\).

Now, if none of the trigonometric functions in condition (D.12) is zero, the condition can be rewritten in the form
\[
\tan \phi_1 \tan \phi_2 = -e^{i(\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22})}. \tag{D.13}
\]
Since \( \tan \phi_1, \tan \phi_2 \) are real-valued, the condition can be split into two separate ones

\[
e^{i(\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22})} = \pm 1, \quad \text{and} \quad \phi_1 = \arctan \left( \frac{\mp 1}{\tan \phi_2} \right) = \pm \phi_{2 \text{mod} \pi} \mp \pi/2. \quad (D.14)
\]

Without loss of generality, let us consider only the case with upper sign and \( \phi_2 \in (0, \pi) \), which gives

\[
e^{i(\chi_{11} - \chi_{12} - \chi_{21} + \chi_{22})} = 1, \quad \text{and} \quad \phi_1 = \phi_2 - \frac{\pi}{2}.
\]

This gives Kraus operators in the form

\[
K_1 = \begin{pmatrix} \cos \theta_1 & 0 \\ 0 & \cos \theta_2 \end{pmatrix},
\]

\[
K_2 = \begin{pmatrix} \sin \theta_1 \sin \phi_2 e^{i\chi_{11}} & \sin \theta_2 \cos \phi_2 e^{i\chi_{21}} \\ -\sin \theta_1 \cos \phi_2 e^{i\chi_{12}} & \sin \theta_2 \sin \phi_2 e^{i\chi_{22}} \end{pmatrix}.
\]

For this channel, the result \( \Phi(\rho) \) of action on the maximally mixed state has eigenvalues of the form

\[
\lambda_{1,2} = \frac{1}{2} \pm \frac{1}{4} \sqrt{\cos^2(\phi_2)(\cos(2\theta_1) - \cos(2\theta_2))^2} = \frac{1}{2} \pm \frac{1}{4} \cos(\phi_2)(\cos(2\theta_1) - \cos(2\theta_2)),
\]

where without loss of generality, we drop the absolute value. Now we may assume, once again, that \( \lambda_1 = a \in [0, \frac{1}{2}] \), which allows us to solve for \( \phi_2 \), which gives

\[
\phi_2 = \pm \arccos \left( \frac{2(2a - 1)}{\cos(2\theta_1) - \cos(2\theta_2)} \right). \quad (D.15)
\]

Given this, we consider the linear entropy for the complementary channel, which yields

\[
S_{\text{lin}} = 1 - \frac{1}{4} \left( -(1 - 2a)^2 \frac{\cos(\Delta_\chi)}{\sin^2(\theta_1 - \theta_2)} + (1 - 2a)^2 \frac{\cos(\Delta_\chi)}{\sin^2(\theta_1 + \theta_2)} + 8(a - 1)a + \sin(2\theta_1) \sin(2\theta_2) \cos(\Delta_\chi) + \cos(2\theta_1) \cos(2\theta_2) + 5 \right).
\]

First, we notice that the quantity is dependent only on the difference of the phases, \( \Delta_\chi = \chi_{11} - \chi_{22} \), which reduces the effective number of free parameters.

Considering the derivative with respect to it, we get

\[
\frac{\partial S_{\text{lin}}}{\partial \Delta_\chi} = \frac{1}{4} \sin(\Delta_\chi) \left( (1 - 2a)^2 \left( \frac{1}{\sin^2(\theta_1 - \theta_2)} - \frac{1}{\sin^2(\theta_1 + \theta_2)} \right) - \sin(2\theta_1) \sin(2\theta_2) \right) = 0.
\]
From the solution of this equation, we find that

$$\Delta_\chi = n\pi.$$  \hfill (D.16)

First, we consider only $\Delta_\chi = 0$. Next, we need to consider derivative with respect to $\theta_1$, which turns out to be dependent only on $\Delta_\theta = \theta_1 - \theta_2$,

$$\frac{\partial S_{\text{lin}}}{\partial \theta_1} = \frac{1}{4} \left( \frac{4(1 - 2a)^2}{\tan(\theta_1 - \theta_2) \sin^2(\theta_1 - \theta_2)} - 2 \sin(2(\theta_1 - \theta_2)) \right) = 0. \hfill (D.17)$$

The solutions of this equation are analogous to those of (D.10), yielding the same extremal entropy curve as earlier. For $\Delta_\chi = \pi$, the equation is slightly more complicated

$$\frac{\partial S_{\text{lin}}}{\partial \theta_1} = \frac{32a^2 - 32a + 4 \cos(2(\theta_1 + \theta_2)) - \cos(4(\theta_1 + \theta_2)) + 5}{\tan(\theta_1 + \theta_2) \sin^2(\theta_1 + \theta_2)} = 0, \hfill (D.18)$$

but again yields the solutions analogous to as (D.10).

This completes the proof, as the consideration exhausts, the set of possible qubit-qubit channels.

**Appendix E. Hamiltonian Evolution for Qutrit Boundary**

In order to solidify Conjecture 1 concerning the boundary of allowed region $A_3$, we performed numerical checks using evolution under random hamiltonians $H$ of dimension 9 drawn from GUE. Let us consider the channel given by Kraus operators

![Figure 3](image-url)  

*Fig. 3.* Evolution, described in Appendix E, of linear entropies of channels drawn from the conjectured boundary of allowed region $A_3$ for qutrit channels under fixed hamiltonian $H$ drawn from GUE. In both panels, each point on a time slice is connected by a line with the corresponding point on the next time slice, and shade-coding periodic in time is employed. In the left panel detailed evolution with $t \in [0, 1]$ and timestep 0.001 is shown, while in the right panel we can see evolution for $t \in [0, 100]$ with timestep 0.1. In neither of the two cases, channels outside of the conjectured boundary, given in black, have been found.

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$K_1(0), K_2(0), K_3(0)$. We can define its evolution by considering the block column formed by all the Kraus operators and transforming it under unitary generated by hamiltonian

$$\begin{pmatrix} K_1(t) \\ K_2(t) \\ K_3(t) \end{pmatrix} = \exp(iHt) \begin{pmatrix} K_1(0) \\ K_2(0) \\ K_3(0) \end{pmatrix}. \quad (E.1)$$

In Figs. 3 and 4, results of numerical checks are presented. For the ease of computation, instead of von Neumann entropy $S$, linear entropy $S_{lin}$ was used in the

![Diagrams showing evolution of linear entropies](image)

Fig. 4. Evolution of linear entropies, described in Appendix E, of channels taken from vicinity of cusp points in the boundary of allowed region $A_3$. Shade-coding has been employed here to distinguish evolutions originating from different points on the boundary. Black lines correspond to the region of origin of channels, whereas gray dashed extensions are further fragments of the boundary. Each of the panels includes evolutions with respect to 30 random hamiltonians for $t \in [0, 0.5]$ with timestep 0.01. For none of the 30 hamiltonians evolution beyond the boundary has been found.
computations. In order to justify this, consider the differential of linear entropy in any dimension, given by
\[
dS_{\text{lin}} = d \left( 1 - \sum_{i=1}^{N} \lambda_i^2 \right) = -2 \sum_{i=1}^{N-1} \lambda_i \, d \lambda_i - 2 \left( \sum_{j=1}^{N-1} \lambda_j \right) \, d \left( 1 - \sum_{j=1}^{N-1} \lambda_j \right)
\]
\[
= -2 \sum_{i=1}^{N-1} \left( \lambda_i + \sum_{j=1}^{N-1} \lambda_j - 1 \right) \, d \lambda_i,
\]
which is zero when \( d \lambda_i = 0 \) for all \( i \) or for \( \lambda_i = \frac{1}{N} \), which can be calculated by direct solving of the system of equations of the form \( \lambda_i + \sum_{j=1}^{N-1} \lambda_j - 1 = 0 \).

Similarly, consider the differential of von Neumann entropy, that is given by
\[
dS = -d \left( \sum_{i=1}^{N} \lambda_i \log \lambda_i \right) = -\sum_{i=1}^{N-1} \log \lambda_i \, d \lambda_i - \log \left( 1 - \sum_{j=1}^{N-1} \lambda_j \right) \, d \left( 1 - \sum_{j=1}^{N-1} \lambda_j \right)
\]
\[
= \sum_{i=1}^{N-1} \log \left( \frac{1 - \sum_{j=1}^{N-1} \lambda_j}{\lambda_i} \right) \, d \lambda_i.
\]
This is zero either when \( d \lambda_i = 0 \) for every \( i \) or when the system of equations of the form \( \frac{1 - \sum_{j=1}^{N-1} \lambda_j}{\lambda_i} = 1 \) is satisfied. However, by elementary manipulation it is seen that they are equivalent to conditions for extrema of linear entropy. Thus, the equivalence is established. In terms of visual appearance, the difference between the plots presented in Figs. 3 and 4 and the plot in Fig. 2 is a nonuniform rescaling.

**Appendix F. Boundary for \( N = 4 \)**

In order to obtain the full lower boundary of the allowed set \( A_4 \), we define three emission channels by their matrices \( L \).

\[
L_1 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
\end{pmatrix}, \\
L_2 = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}, \\
L_3 = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix},
\]

among which \( L_1 \) is in fact corresponding to the identity channel \( \Phi(\rho) = \rho \).

Using these, we define two interpolation channels as described in Appendix B:

1. \( A(L_1, L_2; x) \) with \( x = 1 - 4a \), giving a parametric curve

\[
S(a) = -a \log a - (1 - a) \log(1 - a)
\]

\[
\tilde{S}(a) = -\left( \frac{1}{2} - a \right) \log \left( \frac{1}{2} - a \right) - \left( \frac{1}{2} + a \right) \log \left( \frac{1}{2} + a \right),
\]

for \( a \in (0, \frac{1}{4}) \).
(2) \( A(L_2, L_3; x) \) with \( x = 4a - 2 \), giving a parametric curve

\[
p(a) = \left( \log \frac{4}{a} - (1 - a) \log(1 - a) - \left( a - \frac{1}{2} \right) \log \left( a - \frac{1}{2} \right), \right.
\]

\[
- a \log a - (1 - a) \log(1 - a) \left),
\]

for \( a \in (\frac{1}{2}, \frac{3}{4}) \).

Remaining part of the boundary may be given as reflection through the line \( S = \bar{S} \), containing the self-complementary channels \( \Phi = \bar{\Phi} \).

**Appendix G. Boundaries for Qubit and Qutrit Systems with Maximally Extended Boundaries**

To further extend the analysis, in Fig. 5, we provide full boundaries for qubit and qutrit systems with environment extended to dimension \( N^2 \) in order to cover all the available region in the entropy plane. The results given in this paper allow us to form more precise boundaries than in a paper by Roga et al. Moreover, in case of qubits, the bounds in Proposition 2 are proven to be tight.

![Fig. 5](image)

**Fig. 5.** Possible entropies of channels and their complementaries for qubits and qutrits in the respective panels. Black dots are channels generated according to the method given in Appendix C. Solid triangles are the boundaries obtained from Observation 1, Proposition 1 and subadditivity for von Neuman entropy. Dashed lines are the proposed tight lower bounds, proved for qubits in Proposition 2 and conjectured for qutrits in Conjecture 1. As channels corresponding to the vertices are well known, the empty region on the right side of qutrit plot should be regarded as an artifact of the chosen method of channel generation.

**References**