J. Phys. A: Math. Gen. **30** (1997) L117–L123. Printed in the UK PII: S0305-4470(97)79780-4

LETTER TO THE EDITOR

Time-reversal symmetry and random polynomials

Daniel Braun†, Marek Kuś†‡ and Karol Życzkowski†§

† Fachbereich Physik, Universität-GH Essen, 45117 Essen, Germany

‡ Center for Theoretical Physics, Polish Academy of Sciences, Warsaw, Poland

§ Instytut Fizyki, Uniwersytet Jagielloński, Kraków, Poland

Received 21 November 1996

Abstract. We analyse the density of roots of random polynomials where each complex coefficient is constructed of a random modulus and a fixed, deterministic phase. The density of roots is shown to possess a singular component only in the case for which the phases increase linearly with the index of coefficients. This means that, contrary to earlier belief, eigenvectors of a typical quantum chaotic system with some antiunitary symmetry will *not* display a clustering curve in the stellar representation. Moreover, a class of time-reverse invariant quantum systems is shown, for which spectra display fluctuations characteristic of orthogonal ensemble, while eigenvectors confer to predictions of unitary ensemble.

The distribution of roots of polynomials of high degree with random coefficients was investigated recently in connection with properties of quantum chaotic systems [1–4]. In particular, the authors of the cited references considered the coherent state representation of eigenstates of a quantum mechanical spin system with the total spin S. The polynomials in question have the form

$$P(z) = \sum_{k=0}^{N} \sqrt{C_N^k} a_k z^k \qquad N = 2S \tag{1}$$

where C_N^k stand for binomial coefficients and a_k are components of an eigenvector. The complex variable z is connected to the Bloch sphere angular variables θ, ϕ *via* $z = \tan(\theta/2)\exp(\mathrm{i}\phi)$. It was shown by Lebœuf and Voros [1], that for large values of S when the quantum system in question is chaotic the distribution of the roots is given by

$$\rho(z) = \frac{N}{\pi} \frac{1}{1+|z|^2} \tag{2}$$

corresponding to the uniform distribution of the roots over the Bloch sphere. This is the consequence of the fact that in the semiclassical limit $N \to \infty$ the components with respect to a 'generic basis' of the eigenvectors of a chaotic system are independently normally distributed (see [5] and references therein).

The details of the distribution of the components a_k depend on symmetries of the system in question. For systems which are not time-reversal invariant the eigenvector components are complex, with independently, normally distributed real and imaginary parts, whereas for time-reversal invariant systems the eigenvectors can be made real (also with normally distributed components). In the latter situation the uniform distribution (2) is modified. In particular, the roots tend to concentrate on the real line $\mathrm{Im}\, z = 0$, which is a symmetry line

0305-4470/97/060117+07$19.50

for the roots (if z_0 is a root then its complex conjugate z_0^* is also a root [2, 3], see also below). When projected back on the sphere the symmetry line is the great circle $\phi = 0$.

This simplest situation corresponds to the case when the time-reversal operator is represented by the complex conjugation operator. On the other hand, it is known that generalized time-reversal symmetries, represented by the complex conjugation supplemented by a unitary transformation, influence statistical properties of eigenvector components in the same way as the conventional time-reversal symmetry [5]. As an illustration the authors of [2, 3] considered various models of the so-called kicked top system [6], which is described by the one-step evolution operator of the form $U = \exp(-\mathrm{i}f_1)\exp(-\mathrm{i}f_2)\exp(-\mathrm{i}f_3)$ with $f_i = f_i(S_x, S_y, S_z)$, $i = 1, 2, 3$ polynomial functions of the components of the spin operator $\boldsymbol{S} = (S_x, S_y, S_z)$. The simplest case displaying chaotic dynamics in the classical limit is obtained by choosing $U_0 = \exp(-\mathrm{i}\mu S_x)\exp(-\mathrm{i}pS_z^2/2S)$ with appropriate values of the parameters μ and p. It has two generalized time-reversal symmetries $T_1 = \exp(-\mathrm{i}\mu S_x)K$ and $T_2 = \exp(-\mathrm{i}\mu S_x)\exp(\mathrm{i}\pi S_y)\exp(\mathrm{i}\pi S_z)K$, $T_i U T_i^{-1} = U^\dagger$ both being compositions of linear rotations with the complex conjugation operator K. The rotations shift the symmetry line from the great circle $\phi = 0$ to other ones, the phenomenon exhibited by the numerical investigations performed by the authors of [2, 3].

A non-homogeneous distribution of zeros of Husimi functions is linked to statistical properties of coherent states expanded in the eigenbasis of the Floquet operator. In particular, the number of relevant eigenstates [7] and the entropy of coherent states [8] was found for this model to be smaller than average along the symmetry lines T_i. A smaller number of significantly occupied eigenstates denotes a larger number of weakly occupied states, in consistency with investigated clustering of zeros of eigenstates in Husimi representation along the symmetry curves. Moreover, the distribution of expansion coefficients of a coherent state localized sufficiently far away from the symmetry lines is statistically indistinguishable from properties of a generic coherent state of a system without any antiunitary symmetry [8]. This corresponds to the recent result of Prosen [9], who showed that the densities of zeros of random polynomials with real and complex coefficient are equal sufficiently far away from the real axis.

In order to break the generalized time-reversal symmetry, the original model U_0 was supplemented by a nonlinear rotation $f_1 = qS_y^2/2S$ (in [2]) or $f_1' = qS_z^2/2S$ (in [3]) instead of $f_1 = 0$. In their numerical investigations Bogomolny *et al* observed vanishing of the concentration of the roots which they attributed to the breaking of the time-reversal symmetry. In what follows we will argue that the concentration of the roots on the symmetry lines happens in the case of generalized time invariance only exceptionally and as such cannot be treated as a criterion discriminating between the time-reversal invariant and non-invariant systems. In particular, the kicked tops $U_1 = \exp(-\mathrm{i}qS_y^2)U_0$ and $U_2 = \exp(-\mathrm{i}qS_z^2)U_0$ differ with respect to the statistical properties of the spectra for generic values of the parameter q. The additional rotation term breaks all generalized time-reversal symmetries for the first top and U_1 pertains to the circular unitary ensemble (CUE), while the second still possess such a symmetry

$$T' = \exp(-\mathrm{i}qS_z^2)\exp(-\mathrm{i}\mu S_x)\exp(-\mathrm{i}qS_z^2)K \tag{3}$$

and its spectrum is typical to circular orthogonal ensemble (COE), irrespective of the value of q. Note that the above operator is constructed of a nonlinear unitary rotation (quadratic term S_z^2 in the exponent), in contrast to the operators T_1 and T_2.

Inasmuch as level statistics reveals directly the symmetry properties of quantum systems, special care has to be taken interpreting the statistical properties of eigenvectors, since their distribution depends on the basis chosen. For example, the distribution of eigenvectors of

U_0 in the S_z basis does not confer to COE predictions. The agreement with random matrices is recovered in S_x basis: the geometric symmetry of the top manifests itself in the structure of operator U_0. It splits into two parities of size S and $S+1$, which have to be treated separately to achieve results according to random matrices. In earlier papers [10–12] the variables of the top were exchanged $x \leftrightarrow z$, which gives the same effect.

The distribution of eigenvectors can be characterized by their mean entropy H, which for random matrices of size N is equal to $H(N,\beta) = \Psi(N\beta/2+1) - \Psi(\beta/2+1)$, where Ψ stands for the digamma function and $\beta = 1$ for COE and $\beta = 2$ for CUE [13]. Figure 1 presents the entropy of eigenvectors relative to the entropy of CUE for two tops U_1 and U_2 as a function of the control parameter q. Observe similar behaviour for 'unitary' top U_1 and the 'orthogonal' top U_2! The dips in the data for unitary top at $q = 0$ and $q = p = 6.0$ correspond to transitions to the orthogonal class, while U_2 pertains to COE for any value of q due to the symmetry (3). This difference is visualized in level spacing distribution $P(s)$ displayed in the inset. An explanation of this fact is simple: out of any 'orthogonal' spectrum D_1 by a generic unitary rotation W one can produce an operator $U_W = WD_1W^\dagger$ which enjoys COE-like properties of the spectrum and CUE-like properties of the eigenvectors. This is exactly the case of the top U_2, for which the operator $\exp(-\mathrm{i}qS_z^2)$ plays the role of W. Observe that U_2 is similar to the orthogonal top $U_2' = \exp(-\mathrm{i}\mu S_x)\exp[-\mathrm{i}(p+q)S_z^2/2S]$.

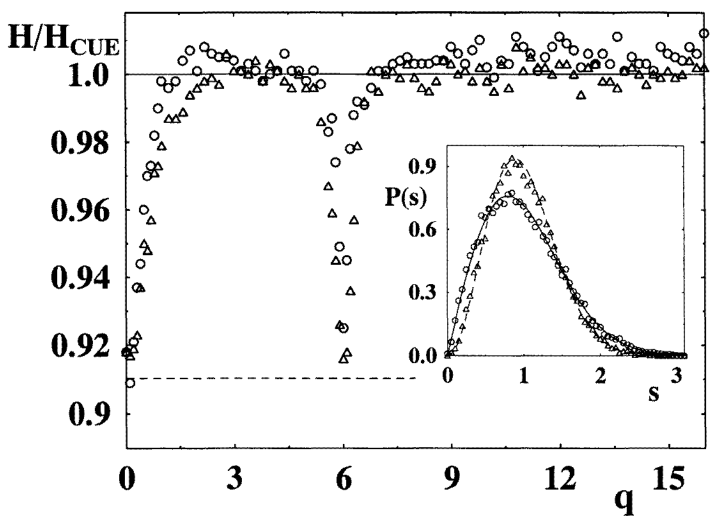

Figure 1. Mean entropy of eigenvectors compared with the entropy H_{CUE} of the unitary ensemble drawn as a function of the perturbation parameter q for two models: 'unitary' top $U_1(\triangle)$ and 'orthogonal' top $U_2(\circ)$ with $\mu = 1.7$, $p = 6.0$ and spin length $S = 40$. The dashed line represents the value $H_{\mathrm{COE}}/H_{\mathrm{CUE}} \approx 0.91$. The inset shows the cumulative level spacing distribution $P(s)$ obtained for both models out of 100 operators U with fixed $q = 2.0$ and p varying from 6.0 to 12.0 and compared to the Wigner surmises for both universality classes.

A similar effect is visible in the distribution of zeros of Husimi function representing eigenvectors: both tops show lack of roots concentration lines as shown for U_1 in [2] and for U_2 in [3], even though they belong to different universality classes.

In order to understand the above announced results let us derive the density of roots ρ of a polynomial (1), where a_k are Gaussian distributed random quantities *with fixed* but arbitrary phases φ_k:

$$a_k = r_k \, \mathrm{e}^{\mathrm{i}\varphi_k} \tag{4}$$

the r_k being distributed according to

$$P(r_k) = \frac{1}{\sqrt{2\pi}} \, \mathrm{e}^{-r_k^2/2}. \tag{5}$$

We will make use of the same technique employed in [3], namely representing $\rho(r, \varphi)$ by the Kac formula,

$$\rho(z) = \delta[P(z)] \left| \frac{\mathrm{d}P(z)}{\mathrm{d}z} \right|^2 \tag{6}$$

and then expressing the delta functions for the real and imaginary parts of $P(z)$ as Fourier integrals. We then get, in full analogy with equation (C6) of [3],

$$\rho(r, \varphi) = \frac{1}{(2\pi)^2} \int \mathrm{d}\xi_1 \int \mathrm{d}\xi_2 \left\{ \sum_{k=0}^{N} k^2 C_N^k r_k^2 r^{2(k-1)} + \sum_{k \neq l = 0}^{N} kl \sqrt{C_N^k C_N^l} r_k r_l \, r^{k+l-2} \, \mathrm{e}^{\mathrm{i}(\varphi_k - \varphi_l)} \, \mathrm{e}^{\mathrm{i}(k-l)\varphi} \right\} \times \exp \sum_{k=0}^{N} r_n (\alpha_n \cos\varphi_n + \beta_n \sin\varphi_n) \tag{7}$$

where

$$\alpha_n = \mathrm{i} r^n \sqrt{C_N^n} (\cos(n\varphi)\xi_1 + \sin(n\varphi)\xi_2) \tag{8}$$

$$\beta_n = \mathrm{i} r^n \sqrt{C_N^n} (\cos(n\varphi)\xi_2 - \sin(n\varphi)\xi_1) \tag{9}$$

and $z = r\,\mathrm{e}^{\mathrm{i}\varphi}$. Averaging over the random coefficients r_k amounts now to simple Gaussian integrations. The resulting average density can be cast in the following form:

$$\langle \rho(r, \varphi) \rangle = \frac{1}{(2\pi)^2} \int \mathrm{d}\xi_1 \int \mathrm{d}\xi_2 (A + B\xi_1\xi_2 + C\xi_1^2 + D\xi_2^2) \exp(-a\xi_1^2 - b\xi_2^2 - 2c\xi_1\xi_2) \tag{10}$$

where

$$A = \sum_{k=0}^{N} k^2 r^{2(k-1)} C_N^k \tag{11}$$

$$B = -\sum_{k,l}^{N} h_{kl} \sin(\varphi_k + k\varphi + \varphi_l + l\varphi) \tag{12}$$

$$C = -\sum_{k,l}^{N} h_{kl} \cos(\varphi_k + k\varphi) \cos(\varphi_l + l\varphi) \tag{13}$$

$$D = -\sum_{k,l}^{N} h_{kl} \sin(\varphi_k + k\varphi) \sin(\varphi_l + l\varphi) \tag{14}$$

and $h_{kl} = klr^{2(k+l-1)} \sqrt{C_N^k C_N^l} \cos(\varphi_k + k\varphi - \varphi_l - l\varphi)$. The coefficients of the quadratic form in the exponential are given by

$$a = \frac{1}{2} \sum_{n=0}^{N} C_N^n r^{2n} \cos^2(\varphi_n + n\varphi) \tag{15}$$

$$b = \frac{1}{2}\sum_{n=0}^{N} C_N^n r^{2n} \sin^2(\varphi_n + n\varphi) \tag{16}$$

$$c = \frac{1}{4}\sum_{n=0}^{N} C_N^n r^{2n} \sin(2(\varphi_n + n\varphi)). \tag{17}$$

It has the eigenvalues

$$\lambda_{1,2} = (a + b \pm \sqrt{(a-b)^2 + 4c^2})/2 \tag{18}$$

and can be diagonalized by a rotation in the (ξ_1, ξ_2) plane by an angle $\gamma = -\arctan((a - b - \sqrt{(a-b)^2 + 4c^2})/2c)$. The ξ-integrals are Gaussian again and lead to the following explicit expression for the mean density of roots:

$$\langle \rho(r,\varphi)\rangle = \frac{1}{4\pi}\left[\frac{A}{\sqrt{\lambda_1\lambda_2}} + \frac{1}{2}\left(\frac{K_1}{\sqrt{\lambda_1^3\lambda_2}} + \frac{K_2}{\sqrt{\lambda_1\lambda_2^3}}\right) + \frac{1}{4\pi}\frac{K_3}{\lambda_1\lambda_2}\right] \tag{19}$$

with the coefficients

$$K_1 = -B\cos\gamma\sin\gamma + C\cos^2\gamma + D\sin^2\gamma \tag{20}$$

$$K_2 = +B\cos\gamma\sin\gamma + C\sin^2\gamma + D\cos^2\gamma \tag{21}$$

$$K_3 = +B\cos 2\gamma + (C - D)\sin 2\gamma. \tag{22}$$

Obviously, $\langle\rho(r,\varphi)\rangle$ can only be singular if at least one of the two eigenvalues λ_1 or λ_2 is zero. This condition leads to $ab = c^2$. After some straightforward manipulations it can be written in the form $Q(r,\varphi) = 0$ with

$$Q(r,\varphi) = \sum_{n<m=0}^{N} C_N^n C_N^m \sin^2(\varphi_m + m\varphi - \varphi_n - n\varphi) r^{2(n+m)}. \tag{23}$$

Thus, the points (r,φ) for which the average density of roots diverges are the zeros of the polynomial in equation (23). However, $Q(r,\varphi)$ is positive semi-definite. The only possibility of $Q(r,\varphi) = 0$ is given by $r = 0$ (which is always a solution and thus always a point of singular density), or by simultaneous vanishing of all coefficients: $\sin(\varphi_m + m\varphi - \varphi_n - n\varphi) = 0$ for all m, n. In the latter case $Q(r,\varphi)$ will be zero for all r, implying immediately that lines of singular density can only be straight lines in the z-plane. On the other hand, assuming that φ_k is a differentiable function of the index k, one finds that $\varphi_k = -k\varphi +$ constant with a k-independent constant. Since the phases φ_k were chosen as constants, the only way to fulfil this equation is by $\varphi_k = k\alpha + \beta$, $\varphi = -\alpha$. For any other choice of the k-dependence of the φ_k, lines with more or less pronounced maxima of $\rho(r,\varphi)$ may still exist, but the singular character of the density is lost—with the exception of the origin.

The above reasoning proves our claim that curves of singular density are only possible if the phases φ_k increase linearly with the index k. This is exactly the case of the top U_0, for which the symmetries T_1 and T_2 manifest themselves as singularities along straight lines on the complex plane, which correspond to great circles on the sphere.

On the other hand, all the deviations from the above form result in a blurring of the sharp lines seen when plotting numerically obtained roots of random polynomials, irrespective of whether a particular symmetry of the possibly underlying physical system is still preserved or not. To demonstrate this effect we have analysed random polynomials (1) with coefficients (4) given by $\varphi_k = qk^2/N$. This assumption corresponds to the problem induced by the generalized time-reversal symmetry (3) of the top U_2. For $q = 0$ (real coefficients a_k) the

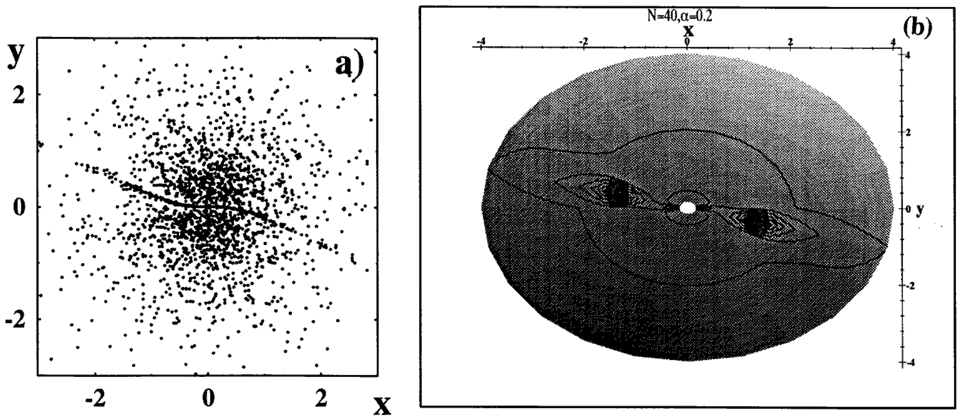

Figure 2. The distribution of roots of 50 random polynomials with quadratically increasing phases ($q = 0.2$) shown in (a) follows the analytically obtained density shown in the contour plot in (b). The concentration line of the zeros deviates from the real axis and is no longer a line of singular density.

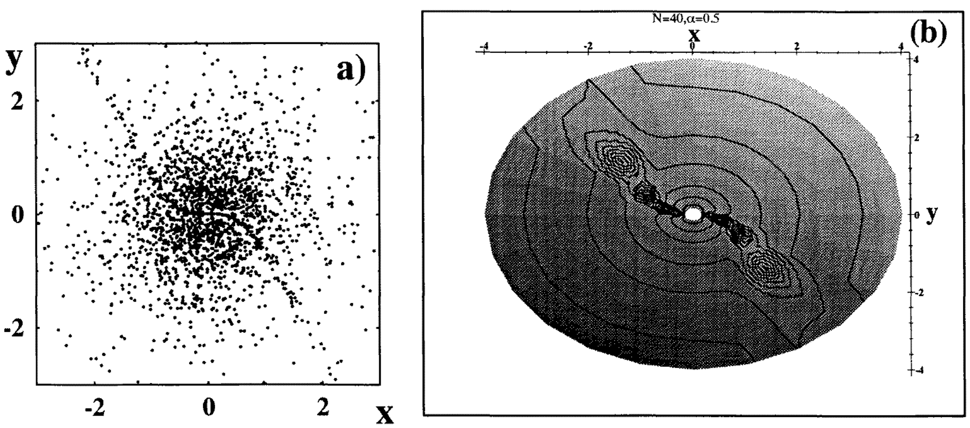

Figure 3. As in figure 2 for $q = 0.5$. The concentration line of the zeros is even more blurred than for $q = 0.2$.

distribution of zeros suffers a singularity along the real axis, while for larger value of q the clustering curve twists and acquires a finite width. This is visible in figures 2 and 3 where we plotted on a complex plane zeros of 50 random polynomials with $N = 40$ (a) and the density of zeros obtained according to equation (19) (b). For $q = 0.2$ the symmetry line already deviates from the real axis. For $q = 0.5$ a ridge in the density of zeros is still observed, at $q \sim 1$ the distribution of zeros is almost homogeneous. Interestingly, the qualitative character of the density does not change much with N.

Let us mention here that the density of zeros of random polynomials (1) with fixed phases can be obtained using slightly different techniques proposed by Edelman and Kostlan [14], Shepp and Vanderbei [15] or Prosen [9]. Moreover, the density of roots of some generalized random polynomials was recently discussed in [16].

It is a pleasure to thank P Lebœuf and T Prosen for stimulating discussions and for providing us with their preprints prior to publication. MK and KZ are grateful to F Haake for hospitality during their stays in Essen, where this work has been initiated. This work was partially supported by the Sonderforschungsbereich 'Unordnung und große Fluktuationen' der Deutschen Forschungsgemeinschaft and partially by Polish Committee of Scientific Research under grant No 2P03B 03810.

References

[1] Lebœuf P and Voros A 1990 *J. Phys. A: Math. Gen.* **23** 1765
[2] Bogomolny E, Bohigas O and Lebœuf P 1992 *Phys. Rev. Lett.* **68** 2726
[3] Bogomolny E, Bohigas O and Lebœuf P 1996 *J. Stat. Phys.* **85** 639
[4] Lebœuf P and P Shukla 1996 *J. Phys. A: Math. Gen.* **29** 4827
[5] Haake F 1991 *Quantum Signatures of Chaos* (Berlin: Springer)
[6] M Kuś, Scharf R and Haake F 1987 *Z. Phys.* B **66** 129
[7] Haake F, Kuś M and Scharf R 1986 *Z. Phys.* B **65** 381
[8] Życzkowski K 1991 *Proc. of Adriatico Research Conference on Quantum Chaos, (Trieste 1990)* ed H A Cerdeira (Singapore: World Scientific) p 153
[9] Prosen T 1996 *J. Phys. A: Math. Gen.* **29** 4417
[10] Kuś M, Mostowski J and Haake F 1988 *J. Phys. A: Math. Gen.* **21** L1073
[11] Haake F and Życzkowski K 1990 *Phys. Rev.* A **42** 1013
[12] G Lenz and Życzkowski K 1992 *J. Phys. A: Math. Gen.* **25** 5539
[13] Jones K R W 1990 *J. Phys. A: Math. Gen.* **23** L1247
[14] Edelman A and Kostlan E 1995 *Bull. Am. Math. Soc.* **32** 1
[15] Shepp L A and Vanderbei R J 1995 *Trans. Am. Math. Soc.* **347** 4365
[16] Mezincescu A, Bessis D, Fournier J-D, Mantica G and Aaron F D 1997 *J. Stat. Phys.* **86** 675