Asymptotic entropic uncertainty relations

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We analyze entropic uncertainty relations for two orthogonal measurements on a N-dimensional Hilbert space, performed in two generic bases. It is assumed that the unitary matrix U relating both bases is distributed according to the Haar measure on the unitary group. We provide lower bounds on the average Shannon entropy of probability distributions related to both measurements. The bounds are stronger than those obtained with use of the entropic uncertainty relation by Maassen and Uffink, and they are optimal up to additive constants. We also analyze the case of a large number of measurements and obtain strong entropic uncertainty relations, which hold with high probability with respect to the random choice of bases. The lower bounds we obtain are optimal up to additive constants and allow us to prove a conjecture by Wehner and Winter on the asymptotic behavior of constants in entropic uncertainty relations as the dimension tends to infinity. As a tool we develop estimates on the maximum operator norm of a submatrix of a fixed size of a random unitary matrix distributed according to the Haar measure, which are of independent interest. © *2016 AIP Publishing LLC.* [http://dx.doi.org/10.1063/1.4944425]

I. INTRODUCTION

Uncertainty relations belong to key features of quantum theory. In the original approach of Heisenberg,[1] Kennard[2] and Robertson[3] one considers the product of variances characterizing measurements of two non-commuting observables. In a later complementary approach one studies entropies of probability vectors associated with both measurements and derives lower bounds for the sum of the two entropies.[4]

State independent bounds for any two orthogonal measurements performed on a state from a Hilbert space $\mathcal{H}_N$ of a finite dimension N were obtained first by Deutsch[5] and later improved by Maassen and Uffink.[6] The problem is entirely specified by the unitary matrix U defining the transition from one measurement basis to the other one. The bounds of Refs. 5 and 6 are both expressed in terms of the absolute value of the largest entry of U. More information on entropic uncertainty relations can be found in review articles,[7,8] while some of their numerous applications in the theory of quantum information are discussed in Refs. 9–12. Certain improvements with respect to the result of Maassen and Uffink have been recently obtained in Refs. 13–18.

Although usually one aims to obtain bounds for two measurements in bases related by a specific unitary matrix U, alternatively one may benchmark the quality of a given bound by averaging it over the set of all unitaries with respect to the Haar measure on the unitary group $U(N)$. Such an approach was advocated in the papers of Hayden *et al.*[20] and of Wehner and Winter,[7] in which the authors considered the special case, where the number L of measurements taken was a function of the dimension N of the Hilbert space.

Following their approach we analyze entropic uncertainty principles for a fixed number of measurements in bases related by random unitary matrices (throughout the article all random unitary matrices we consider are distributed according to the Haar measure on the unitary group) and provide lower bounds on the sum of entropies which hold with high probability and differ from the best possible only by an additive, dimension independent constant.

Our goals and motivation depend on the number of measurements we consider. For $L = 2$ measurements, many uncertainty relations are known, including the Maassen–Uffink bound,[6] the majorization bounds,[13,14] strong majorization relation of Ref. 16 or a recent result by Coles and Piani[15]—see also a recent review.[19] While in general such inequalities complement each other, it is of interest to verify how they perform on typical measurements, i.e., on measurements related by a random unitary matrix. To answer this question we derive an optimal entropic uncertainty relation for generic measurements — see Theorem 13 in Section IV.

A question of entropic uncertainty relations for a large number L of generic bases in N dimensional Hilbert spaces was posed by Wehner and Winter.[7] In this work we prove that, with high probability, the average entropy is bounded from below by $\frac{L-1}{L}\log N - c$, where c is an additive constant independent of N and L. This allows us to give the affirmative answer to a strong form of a conjecture by Wehner and Winter[7]—see Theorem 16 and Corollary 18 in Section V. Asymptotic uncertainty relations derived in this work improve estimations on the quality of the information locking protocols recently obtained by Fawzi *et al.*[21]

Our approach is based on the Schur concavity of entropy which together with the approach proposed in Ref. 16 allows us to reduce the problem of finding lower bounds on the sums of Shannon entropies to the problem of finding upper bounds on norms of submatrices of a random unitary matrix. The latter can be then obtained by employing the concentration of measure phenomenon on the unitary group. We believe that estimates of maximum norms of a submatrix of fixed size of a random unitary matrix in high dimensions are of independent interest as similar quantities have previously appeared in the context of asymptotic geometric analysis and compressed sensing.

At a technical level it may be noted that the Schur concavity of entropy allows to reduce the analysis to functions whose Lipschitz constants behave better (as the dimension increases) then the Lipschitz constant of the entropy itself, thus allowing us to obtain the right balance between the complexity of approximation and available tail bounds.

This work is organized as follows. In Section II we briefly recall the Maassen–Uffink relations and their improvements. Bounds for the norms of submatrices of random unitary matrices, also called their truncations,[22] are presented in Section III. Asymptotic entropic uncertainty relations are analyzed in Section IV for the case of two measurements, while the case of several measurements is discussed in Section V. The presentation and discussion of the results are concluded in Section VI while the proofs of some lemmas are deferred to the Appendices A and B.

II. ENTROPIC UNCERTAINTY RELATIONS

In this section we present entropic uncertainty relations we are going to study in the asymptotic case. The most important bound for the sum of entropies is due to Maassen and Uffink.[6]

Consider a normalized vector $|\psi\rangle$ belonging to a N-dimensional complex Hilbert space $\mathcal{H}_N$ and a non-degenerate observable A, whose eigenstates $|a_i\rangle$, $i = 1,\dots,N$, form an orthonormal basis of $\mathcal{H}_N$. The probability that this observable measured in the state $|\psi\rangle$ gives the ith outcome is given by $p_i^\psi = |\langle a_i|\psi\rangle|^2$. Clearly $\sum_{i=1}^N p_i^\psi = 1$, so the vector $p^\psi = (p_1^\psi,\dots,p_N^\psi)$ can be identified with a probability distribution on the set $\{1,\dots,N\}$. The uncertainty associated to the measurement A can be then described by the Shannon entropy of p^ψ, defined as

$$H(p^\psi) = -\sum_{i=1}^N p_i^\psi \ln p_i^\psi.$$

Consider now another observable B and let $|b_i\rangle$, $i = 1,\dots,N$, be its eigenstates. Let q^ψ be the probability distribution associated with B, i.e., $q = (q_1^\psi,\dots,q_N^\psi)$, where $q_j^\psi = |\langle b_j|\psi\rangle|^2$. The uncertainty corresponding to B can be quantified by the corresponding Shannon's entropy $H(q^\psi)$. If the observables A and B do not share an eigenvector, then the sum of both entropies for any state $|\psi\rangle$

is bounded from below, and (as one can easily see) the bound depends only on the unitary matrix $U = (U_{ij})_{i,j=1}^N$, where $U_{ij} = \langle a_i|b_j\rangle$.

In 1988 Maassen and Uffink[6] obtained the result of the form

$$H(p^\psi) + H(q^\psi) \geq -\ln c^2 \equiv B_{MU}, \tag{1}$$

where $c = \max_{ij} |U_{ij}|$. The Maassen–Uffink bound has been recently improved in the whole range of the parameter c by Coles and Piani[15] who provided a state independent bound

$$H\left(p^\psi\right) + H\left(q^\psi\right) \geq -\ln c^2 + \left(\frac{1}{2} - \frac{c}{2}\right) \ln \frac{c^2}{c_2^2} \equiv B_{\mathrm{CP}}, \tag{2}$$

with c_2 being the second largest value among $\left|U_{ij}\right|$, $1 \leq i, j \leq N$. Since $c_2 \leq c$, the second term in (2) is a non-negative correction to (1).

Let us now pass to uncertainty relations based on the Schur concavity of the Shannon entropy. They will take into account not only the largest or the two largest elements of the transition matrix U, but the behavior of the operator norms of all submatrices of U.

Let us first introduce some auxiliary notation related to matrices. By $U(N)$ we will denote the unitary group of $N \times N$ unitary matrices. For $U = (U_{ij})_{i,j=1}^N \in U(N)$ and nonempty sets $I, J \subset \{1, \ldots, N\}$, let $U(I, J) = (U_{ij})_{i \in I, j \in J}$, i.e., $U(I, J)$ is the matrix obtained from U by restricting to rows and columns corresponding to the elements of I and J, respectively. For a matrix M, by $\|M\|$ we denote its operator norm, equal to its largest singular value, $\sigma_{\max}(M)$. Finally, for $1 \leq n, m \leq N$ we define

$$\|\widehat{U}^{(n,m)}\| = \max\left\{\|U(I,J)\| \colon I, J \subset \{1,\ldots,N\}, |I| = n, |J| = m\right\}, \tag{3}$$

i.e., $\|\widehat{U}^{(n,m)}\|$ is the maximal norm of a submatrix of U of size $n \times m$.

For any fixed matrix U we shall introduce a set of N coefficients

$$s_k \coloneqq \max\left\{\|\widehat{U}^{(1,k)}\|, \|\widehat{U}^{(2,k-1)}\|, \ldots, \|\widehat{U}^{(k,1)}\|\right\}, \ k = 1, \ldots, N. \tag{4}$$

In the next step we define coefficients

$$R_k = \left(\frac{1 + s_k}{2}\right)^2, \ k = 1, \ldots, N, \tag{5}$$

so that $\left(\frac{1+c}{2}\right)^2 = R_1 \leq R_2 \leq \cdots \leq R_N = 1$.

Recall also that if $x, y \in \mathbb{R}^N$ have nonnegative coordinates then we say that x is majorized by y (which we denote by $x \prec y$) if for $k = 1, \ldots, N$, $\sum_{i=1}^k x_i^\downarrow \leq \sum_{i=1}^k y_i^\downarrow$ and $\sum_{i=1}^N x_i^\downarrow = \sum_{i=1}^N y_i^\downarrow$, where $x_i^\downarrow$ is the non-increasing rearrangement of the numbers x_i. We can extend the above relation to vectors which are of unequal dimension by padding the vector of lower dimension with zeros. We say that a function $f \colon \mathbb{R}_+^N \to \mathbb{R}$ is Schur concave if $f(x) \geq f(y)$ whenever $x \prec y$. It is well known that the function $f(x) = \sum_{i=1}^N -x_i \ln x_i$ is Schur concave (see e.g. Ref. 23).

We are now ready to formulate a result proved in Ref. 14.

Theorem 1. *Let $(|a_i\rangle)_{i=1}^N$ and $(|b_i\rangle)_{i=1}^N$ be two orthonormal bases in $\mathcal{H}_N$ and $U = (\langle a_i|b_j\rangle)_{i,j=1}^N$ be the corresponding transition matrix. Define $Q = (R_1, R_2 - R_1, R_3 - R_2, \ldots, R_N - R_{N-1})$, where the coefficients R_i are given above. Then for any pure state $|\psi\rangle \in \mathcal{H}_N$, the probability vectors $p^\psi = (p_1^\psi, \ldots, p_N^\psi)$, $q^\psi = (q_1^\psi, \ldots, q_N^\psi)$ with $p_i^\psi = |\langle a_i|\psi\rangle|^2$, $q_i^\psi = |\langle b_i|\psi\rangle|^2$, satisfy $p^\psi \otimes q^\psi \prec Q$.*

Notice that from the above theorem and the Schur concavity of the Shannon's entropy we obtain directly the following corollary.

Corollary 2. In the setting of Theorem 1,

$$\min_\psi \left(H(p^\psi) + H(q^\psi)\right) \ \geq \ H\left(Q\right), \tag{6}$$

where the minimum is taken over the set of all pure states $|\psi\rangle \in \mathcal{H}_N$.

Recently an improved version of majorization entropic uncertainty relations was derived in Ref. 16.

Theorem 3. *In the setting of Theorem 1, we define the numbers* $x_i, i = 1,\dots,2N$ *by the equality* $p^{\psi} \oplus q^{\psi} = (x_1,\dots,x_{2N})$. *Then for* $k = 1,\dots,2N$,

$$\sum_{i=1}^{k} x_i^{\downarrow} \leq 1 + s_{k-1}, \tag{7}$$

where we additionally set $s_0 = 0$. *As a consequence,*

$$p^{\psi} \oplus q^{\psi} \prec (1, s_1, s_2 - s_1, s_3 - s_2, \dots, s_N - s_{N-1}). \tag{8}$$

The majorization relation of Theorem 3 implies the following uncertainty relation

$$\min_{\psi} \Big(H(p) + H(q) \Big) \geq H((s_1, s_2 - s_1, s_3 - s_2, \dots, s_N - s_{N-1})), \tag{9}$$

where as usual the minimum is taken over the set of all pure states.

Note that in this case we apply majorization techniques working with positive vectors which are not normalized to unity. In paper[16] it has been shown that bound (9) based on the direct sum is not weaker than bound (6) based on the tensor product of probability vectors.

Another result proved in Ref. 16 is an uncertainty relation for many measurements, which we now recall. For $L \geq 2$ consider $N \times N$ unitary matrices $U_1,\dots,U_L$ and let $|u_j^{(i)}\rangle$ be the jth column of U_i. Consider the probability distributions $p^{(i)}$, $i = 1,\dots,L$ given by $p_j^{(i)} = |\langle u_j^{(i)}|\psi\rangle|^2$, $i = 1,\dots,L$, $j = 1,\dots,N$. Note that to simplify the notation we suppress here the dependence on $|\psi\rangle$.

Let finally U be the concatenation of matrices $U_1,\dots,U_L$ and for a set $I \subset \{1,\dots,LN\}$ with $|I| = k$, let U_I be the $N \times k$ matrix obtained from U by selecting the columns of U corresponding to the set I. Define for $k = 0,\dots,NL-1$,

$$\mathcal{S}_k = \max\{\|U_I\|^2 \colon I \subset \{1,\dots,LN\}, |I| = k+1\}. \tag{10}$$

Note that $\mathcal{S}_0 = 1$, independently of the choice of unitary matrices U_i.

The following theorem was proved in Ref. 16.

Theorem 4. *In the setting described above, define the coefficients* $x_1,\dots,x_{NL}$ *by the equality* $p^{(1)} \oplus \cdots \oplus p^{(L)} = (x_1,\dots,x_{NL})$. *Then for* $k \leq NL$, $\sum_{i=1}^{k} x_i^{\downarrow} \leq \mathcal{S}_{k-1}$. *As a consequence,* $p^{(1)} \oplus \cdots \oplus p^{(L)} \prec (\mathcal{S}_0, \mathcal{S}_1 - \mathcal{S}_0, \mathcal{S}_2 - \mathcal{S}_1, \dots, \mathcal{S}_{LN-1} - \mathcal{S}_{LN-2})$ *and*

$$\sum_{i=1}^{L} H(p^{(i)}) \geq -\sum_{i=1}^{LN-1} (\mathcal{S}_i - \mathcal{S}_{i-1}) \ln(\mathcal{S}_i - \mathcal{S}_{i-1}). \tag{11}$$

We note that for general $L > 2$, it is an open problem to construct deterministic unitary $N \times N$ matrices $U_1,\dots,U_L$ such that for all pure states $|\psi\rangle$,

$$\sum_{i=1}^{L} H(p^{(i)}) \geq (L-1)\ln N - CL, \tag{12}$$

where C is a constant independent of N. A deterministic construction is known only for $L = N+1$, in which case it was proved by Ivanovic[24] and Sánchez[25] that the above bound holds for unitary matrices corresponding to a maximal set of mutually unbiased basis. On the other hand, as shown in Ref. 26, if N is an even power of a prime number and $L \leq \sqrt{N} + 1$, then there exist L mutually unbiased bases such that for some state $|\psi\rangle$,

$$\sum_{i=1}^{L} H(p^{(i)}) = \frac{1}{2} L \ln N. \tag{13}$$

In particular this shows that the approach of Refs. 24 and 25 cannot be generalized to arbitrary L.

To the best of our knowledge, for a "small" number of measurements the only available constructions of bases satisfying (12) are given by the random choice of bases and work for

$L \geq \ln^4 N$.[20] We will discuss them in Section V (together with related work[21]), where we show that random bases provide strong uncertainty relations also for a smaller number of measurements.

III. NORMS OF TRUNCATIONS OF RANDOM UNITARIES

In this section, we will provide estimates for the operator norms of submatrices of a random unitary matrix, which as seen in Sec. II, appear in majorization entropic uncertainty principles. These estimates will become crucial in the proofs of entropic uncertainty principles for random unitaries. We emphasize that although from the point of view of uncertainty principles, bounds on norms of submatrices are simply a tool, we have decided to state them in a separate section as we believe that they may be of independent interest, especially from the perspective of random matrix theory or asymptotic geometric analysis.

Before stating our results let us recall some basic notions related to random unitary matrices. As is well known, the unitary group $U(N)$ of all $N \times N$ unitary matrices admits a unique probability measure invariant under left and right multiplications, i.e., the Haar measure. In what follows by a $N \times N$ random unitary matrix we will always mean a random element of the group $U(N)$ distributed according to the Haar measure. Usually we will denote such a random matrix by U, suppressing the dependence on N, as it is customary in the random matrix theory literature.

Motivated by the result of Maassen and Uffink, for $U = (U_{ij})_{i,j=1}^N$, we denote

$$c(U) = \max_{1 \leq i,j \leq N} |U_{ij}|. \tag{14}$$

The behavior of $c(U)$ for random unitaries was studied by Jiang,[27] who obtained the following.

Theorem 5. *If U is a $N \times N$ random unitary matrix, then for all $\varepsilon > 0$,*

$$\mathbb{P}\left((1-\varepsilon)\sqrt{\frac{2}{N}\ln N} \leq c(U) \leq (1+\varepsilon)\sqrt{\frac{2}{N}\ln N}\right) \to 1 \text{ as } N \to \infty. \tag{15}$$

Theorems 6–9, presented below, provide estimates for the operator norms of submatrices of a random unitary matrix. These results are new with proofs relegated to Appendix A.

The next theorem is a generalization of the result obtained by Jiang to the maximal norm of submatrices of a random unitary matrix (as defined by (3)).

Theorem 6. *For any fixed positive integers n,m and any $\varepsilon > 0$, if U is a $N \times N$ random unitary matrix, then*

$$\mathbb{P}\Big((1-\varepsilon)\sqrt{\frac{n+m}{N}\ln N} \leq \|\widehat{U}^{(n,m)}\| \leq (1+\varepsilon)\sqrt{\frac{n+m}{N}\ln N}\Big) \to 1 \text{ as } N \to \infty. \tag{16}$$

The above theorem works for fixed n,m, independent of the dimension N. Its proof is based on the following result, which provides an estimate on the maximal norm $\|\widehat{U}^{(n,m)}\|$ for arbitrary $n,m \leq N$. Before we formulate the theorem, let us recall also that $o(1)$ denotes any sequence which converges to zero as $N \to \infty$, in particular for non-vanishing sequences of real numbers a_N and b_N, we have $a_N = (1+o(1))b_N$ if and only if $\lim_{N\to\infty} \frac{a_N}{b_N} = 1$.

Theorem 7. *Let U be a $N \times N$ random unitary matrix. Then*

$$\mathbb{P}\left(\Big|\|\widehat{U}^{(n,m)}\| - \mathbb{E}\|\widehat{U}^{(n,m)}\|\Big| \geq t\right) \leq 2\exp\Big(-\frac{Nt^2}{12}\Big) \quad \text{for } t \geq 0. \tag{17}$$

Moreover, for any $0 < \varepsilon < 1/3$,

$$\mathbb{E}\|\widehat{U}^{(n,m)}\| \leq \frac{1}{1-2\varepsilon-\varepsilon^2}\sqrt{\frac{2}{2N-1}}\left(m\ln\frac{eN}{m} + n\ln\frac{eN}{n} + 2(n+m)\ln\left(1+\frac{2}{\varepsilon}\right)\right)^{1/2}. \tag{18}$$

In particular for any fixed n,m and $N \to \infty$,

$$\mathbb{E}\|\widehat{U}^{(n,m)}\| \leq (1+o(1))\sqrt{\frac{m+n}{N}\ln N}. \tag{19}$$

In the special case, when one of the parameters n, m equals to one, more precise estimates are provided by subsequent theorems. The first one relies on a geometric argument, exploiting the fact that in the special situation when $n = 1$ or $m = 1$, the norms we consider are Euclidean.

Theorem 8. *If U is a $N \times N$ random unitary matrix, then for all $\varepsilon > 0$,*

$$\min_{1\leq n\leq N} \mathbb{P}\left(\frac{n}{N}(1 + H_N - H_n) - \varepsilon \leq \|\widehat{U}^{(n,1)}\|^2 \leq \frac{n}{N}(1 + H_N - H_n) + \varepsilon\right) \to 1 \text{ as } N \to \infty, \quad (20)$$

where $H_m = \sum_{j=1}^m 1/j$ denotes the mth harmonic number.

The next theorem provides a complete characterization of the behavior of $\|\widehat{U}^{(n,1)}\|$ for large random unitary matrices. Its proof relies on a combination of Theorem 6, which allows to handle the case of 'small' n and Theorem 8 which provides good estimates for large values of n.

Theorem 9. *Let U be a $N \times N$ random unitary matrix. For all $\varepsilon > 0$,*

$$\mathbb{P}\left(\forall_{1\leq n\leq N}\ (1-\varepsilon)\sqrt{\frac{n+1}{N}\left(1+\ln\left(\frac{N}{n}\right)\right)} \leq \|\widehat{U}^{(n,1)}\| \leq (1+\varepsilon)\sqrt{\frac{n+1}{N}\left(1+\ln\left(\frac{N}{n}\right)\right)}\right) \to 1 \quad (21)$$

as $N \to \infty$.

Observe, that Theorem 9 provides a complete description of asymptotic behavior of the whole sequence $\|\hat{U}^{(n,1)}\|$, $n = 1, 2, \ldots, N-1$, while by setting $m = 1$ in Theorem 7 one obtains non-trivial bounds on the norm only for $n \leq N/\ln N - 1$.

Proofs of all the results described in this section are deferred to Appendix A.

IV. ASYMPTOTIC ENTROPIC UNCERTAINTY RELATIONS

In this section we assume that $N \gg 1$ and analyze the asymptotic behavior of entropic uncertainty relations for random unitary matrices. We consider two orthogonal von Neumann measurements with respect to two bases related by a random unitary matrix U distributed according to the Haar measure on the unitary group. We note that as mentioned in Section II, if $(|a_i\rangle)_{i=1}^N$, $(|b_i\rangle)_{i=1}^N$ are two orthonormal bases in $\mathcal{H}_N$ and for a pure state $|\psi\rangle$ in $\mathcal{H}_N$, the vectors $p^\psi = (p_1^\psi, \ldots, p_N^\psi)$, $q^\psi = (q_1^\psi, \ldots, q_N^\psi)$ are given by

$$p_i^\psi = |\langle a_i|\psi\rangle|^2,\ q_i^\psi = |\langle b_i|\psi\rangle|^2,$$

then the quantity

$$\min_\psi \left(H(p^\psi) + H(q^\psi)\right)$$

depends only on the unitary transition matrix $U = (U_{ij})_{i,j=1}^N$, given by $U_{ij} = \langle a_i|b_j\rangle$. Therefore, when U is a $N \times N$ random unitary matrix, we can speak about probabilities of the form

$$\mathbb{P}\Big(\min_\psi \left(H(p^\psi) + H(q^\psi)\right) \geq r\Big).$$

There is clearly a slight abuse of notation in this convention since to define p^ψ or q^ψ one has to choose the bases $(|a_i\rangle)_{i=1}^N$, $(|b_i\rangle)_{i=1}^N$, but it should not lead to ambiguity. Alternatively, to give definite meaning to p^ψ and q^ψ one can decide (without loss of generality) that $\mathcal{H}_N = \mathbb{C}^N$, $(|a_i\rangle)_{i=1}^N$ is some fixed basis of $\mathcal{H}_N$ (e.g., the standard one) and $|b_i\rangle = U|a_i\rangle$.

Let us start our study of uniform uncertainty principles in the random setting by evaluating the typical behavior of deterministic bounds of Section II. We emphasize that this part of our analysis will follow easily from known bounds on maximal entries of random unitary matrices. The more challenging part will be to obtain optimal bounds, given in Theorem 13 below, which will allow us to conclude that in generic situations bounds of Maassen–Uffink type give only sub-optimal results.

The first proposition evaluates the performance of the Maassen–Uffink entropic uncertainty relation.

Proposition 10. Let U be a $N \times N$ random unitary matrix and let $B_{MU} = -\ln c^2(U)$. Then for any $\varepsilon > 0$,

$$\mathbb{P}\left(\ln N - \ln\ln N - \ln 2 - \varepsilon \leq B_{MU} \leq \ln N - \ln\ln N - \ln 2 + \varepsilon\right) \to 1 \tag{22}$$

as $N \to \infty$.

The interpretation of this result in the context of entropic uncertainty relations is that in sufficiently large dimension N the lower bound obtained by an application of the Maassen–Uffink inequality to a typical (i.e., related by a random unitary matrix) pair of orthogonal von Neumann measurements is (with probability close to one)

$$\min_{\psi}\left(H(p^{\psi}) + H(q^{\psi})\right) \geq \ln N - \ln\ln N - \ln 2 - o(1). \tag{23}$$

As we will see in Theorem 13, this bound is off by the term of the order $\ln\ln N$. This shows that while in the extreme situation (e.g., when the measurements are related by a Hadamard matrix), the Maassen–Uffink bound cannot be improved, its typical performance is sub-optimal.

Proof of Proposition 10. Plugging the estimation from Theorem 5 to the Maassen–Uffink relation, we obtain that with probability tending to one as $N \to \infty$,

$$-\ln c^2(U) = -\ln\left((1 + o(1))\frac{2}{N}\ln N\right) = \ln N - \ln\ln N - \ln 2 + o(1). \tag{24}$$

□

In view of the discussion above, one may wonder whether in typical situations it is possible to obtain a significant gain by employing Coles and Piani relation (2) instead of the Maassen–Uffink bound. It turns out however that this will not provide a notable improvement, since for large N with high probability we have $c(U) \simeq c_2(U)$ (recall that $c_2(U)$ is the second largest number among $|U_{ij}|$, $1 \leq i, j \leq N$). This is formalized in the following proposition.

Proposition 11. Let U be a random $N \times N$ unitary matrix and let

$$B_{CP}(U) = -\ln c^2(U) + (1 - c(U))\ln\frac{c(U)}{c_2(U)}, \tag{25}$$

where $c(U)$ and $c_2(U)$ denote, respectively, the largest and second largest absolute value of an entry of U. Then for every $\varepsilon > 0$,

$$\mathbb{P}\left(B_{CP} \leq \ln N - \ln\ln N - \frac{1}{2}\ln 2 + \varepsilon\right) \to 1 \tag{26}$$

as $N \to \infty$.

As one can see from the above proposition, in typical situations the gain obtained from the Coles and Piani relation with respect to the Maassen–Uffink bound is just $2^{-1}\ln 2$.

Proof of Proposition 11. Note that $c^2(U) + c_2^2(U) \geq \|\widehat{U}^{(1,2)}\|^2$. Thus, by Theorem 6, we obtain that for all $\varepsilon > 0$,

$$\mathbb{P}\left(c_2^2 \geq (1 - \varepsilon)\frac{\ln N}{N}\right) \to 1 \text{ as } N \to \infty. \tag{27}$$

In particular (again by Theorem 6), with probability tending to one as $N \to \infty$, $c(U)/c_2(U) \leq 3$. Therefore, with probability tending to one as $N \to \infty$,

$$B_{\mathrm{CP}} = -\ln c^2(U) + \left(\frac{1}{2} - \frac{c(U)}{2}\right)\ln\frac{c^2(U)}{c_2^2(U)} = -\frac{1}{2}(\ln c^2(U) + \ln c_2^2(U)) + o(1). \tag{28}$$

To prove (26) it is now enough to combine the above estimate with (24) and (27). □

Let us now turn to the majorization entropic uncertainty relation discussed in Section II. As shown in Ref. 14 in many cases it provides a tighter bound than Maassen–Uffink relation (1), however it turns out that this is not the case for typical measurements in high dimension as we have the following proposition.

Proposition 12. Assume that $N \geq 4$ and let U be any $N \times N$ unitary matrix and let

$$Q = (R_1, R_2 - R_1, R_3 - R_2, \ldots, R_N - R_{N-1}), \tag{29}$$

where the coefficients R_i are given by formula (5). Then

$$H(Q) \leq \frac{3}{4}\ln(N-1) + H\left(\frac{1}{4}, \frac{3}{4}\right) = \frac{3}{4}\ln(N-1) + \frac{1}{4}\ln 4 + \frac{3}{4}\ln\frac{4}{3}. \tag{30}$$

Let us note that in the above proposition the matrix U is not random, it can be any $N \times N$ unitary matrix. Together with examples presented in Ref. 14, the inequality (30) shows that entropy estimates based on tensor product majorization of Theorem 1 do not perform well in typical or extremal situations, even though they can still outperform the classical Maassen-Uffink bound when the entropy is small. This is intuitively clear, since the probability distribution Q has an atom R_1 of size at least $\frac{1}{4}$.

Proof of Proposition 12. Denote $q_1 = R_1$, $q_i = R_i - R_{i-1}$ for $i = 2, \ldots, N$. We have $q_1 \geq \frac{1}{4}$. If $q_1 = 1$, then $H(Q) = 0$, otherwise we have with $I = \{2 \leq i \leq N : q_i > 0\}$,

$$\begin{aligned} H(Q) &= -q_1 \ln q_1 + \sum_{i \in I} -q_i \ln q_i && (31)\\ &= -q_1 \ln q_1 + (1-q_1) \sum_{i \in I} \frac{q_i}{1-q_1} \ln \frac{1}{q_i} && (32)\\ &\leq -q_1 \ln q_1 + (1-q_1) \ln \left(\sum_{i \in I} \frac{q_i}{1-q_1} \frac{1}{q_i} \right) && (33)\\ &= -q_1 \ln q_1 - (1-q_1)\ln(1-q_1) + (1-q_1)\ln|I| && (34)\\ &\leq H(q_1, 1-q_1) + (1-q_1)\ln(N-1). && (35) \end{aligned}$$

where in the first inequality we used concavity of the logarithm and the fact that $\sum_{i \in I} q_i = 1 - q_1$. Now, expression (35) decreases for $q_1 \in [\frac{1}{N}, 1]$. In particular for $N \geq 4$ this implies (30). □

We will now pass to the first main result of the article, i.e., to optimal (up to a universal additive constant) entropic uncertainty principles for typical measurements, which hold with high probability on the unitary group. We emphasize that the method of proof will rely heavily on strong (direct sum) majorization of Theorem 3, more specifically on bound (7), combined with the results of Section III. Our result shows in particular that strong majorization techniques of Ref. 16 perform in typical high-dimensional scenarios in an almost optimal way. We refer the reader to Ref. 16 for a comparison of the inequality of Theorem 3 with result (6) and with the Maassen–Uffink bound in deterministic, low-dimensional situations and here we just mention that Theorem 3 is stronger than (6) and in general incomparable with the relation of Maassen and Uffink (one can construct examples in which any of the bounds outperforms the other one).

The following theorem provides optimal uncertainty relations for generic measurements.

Theorem 13. *Let U be a $N \times N$ random unitary matrix and let $C_1 = 3.49$. Then*

$$\mathbb{P}\Big(\min_{\psi} (H(p^{\psi}) + H(q^{\psi})) \geq \ln N - C_1 \Big) \to 1 \tag{36}$$

as $N \to \infty$.

Recall the definition of the parameters s_k given in (4). The proof of Theorem 13 will be based on the following proposition.

Proposition 14. With probability tending to 1 as $N \to \infty$,

$$s_k \leq \sqrt{C_2 \frac{k+1}{N}\left(1+\ln\left(\frac{2N}{k+1}\right)\right)} \quad \text{for } 1 \leq k \leq N, \tag{37}$$

where $C_2 = 4.18$.

Proof of Proposition 14. Note that if for any n, m, such that $n + m = k + 1$, we have

$$\mathbb{E}\|\widehat{U}^{(n,m)}\| \leq \sqrt{D\frac{k+1}{N}\left(1+\ln\left(\frac{2N}{k+1}\right)\right)}, \tag{38}$$

then (37) holds with $C_2 = D + \delta$, for any $\delta > 0$, since by Theorem 19 and (A6) we get

$$\begin{aligned}
&\mathbb{P}\left(\exists_{k\leq N}\, s_k > \sqrt{C_2\frac{k+1}{N}\left(1+\ln\left(\frac{2N}{k+1}\right)\right)}\right)\\
&\quad\leq \sum_{k=1}^{N-1}\sum_{n=1}^{k}\mathbb{P}\left(\|\widehat{U}^{(n,k+1-n)}\| > \sqrt{C_2\frac{k+1}{N}\left(1+\ln\left(\frac{2N}{k+1}\right)\right)}\right)\\
&\quad\leq \sum_{k=1}^{N} k\exp\left(-c_{D,\delta}N\frac{k+1}{N}\left(1+\ln\left(\frac{2N}{k+1}\right)\right)\right) \to 0 \text{ as } N\to\infty.
\end{aligned} \tag{39}$$

By Theorem 7 we get

$$\mathbb{E}\|\widehat{U}^{(n,m)}\| \leq \frac{1}{1-2\varepsilon-\varepsilon^2}\sqrt{\frac{2}{2N-1}}\left(m\ln\frac{eN}{m}+n\ln\frac{eN}{n}+2(n+m)\ln\left(1+\frac{2}{\varepsilon}\right)\right)^{1/2} \text{ for } \varepsilon < 1/3. \tag{40}$$

Note that when $k + 1 > N/D$, the right hand side of (38) exceeds 1, so the inequality is satisfied trivially. We can therefore assume that $k + 1 \leq N/D$. We maximize the right hand side of (40) under the constraint $n + m = k + 1$ and get

$$\begin{aligned}
\mathbb{E}\|\widehat{U}^{(n,m)}\| &\leq \frac{1}{1-2\varepsilon-\varepsilon^2}\sqrt{\frac{2}{2N-1}}\left((k+1)\ln\frac{2eN}{k+1}+2(k+1)\ln\left(1+\frac{2}{\varepsilon}\right)\right)^{1/2}\\
&\leq \sqrt{\frac{2}{2N-1}}\left((k+1)\ln\frac{2eN}{k+1}\right)^{1/2}\left(\frac{1}{(1-2\varepsilon-\varepsilon^2)^2}\left(1+\frac{2\ln\left(1+\frac{2}{\varepsilon}\right)}{\ln(2eD)}\right)\right)^{1/2},
\end{aligned} \tag{41}$$

where we used the assumption $N/(k + 1) \geq D$.

Now we set $D = 4.175$ and perform a minimization with respect to $\varepsilon \in (0, 1/3)$ of the expression

$$\frac{1}{(1-2\varepsilon-\varepsilon^2)^2}\left(1+\frac{2\ln(1+\frac{2}{\varepsilon})}{\ln(2eD)}\right). \tag{42}$$

The numerical value of the minimum is approximately $4.172 \leq 4.175$ (obtained for $\varepsilon = 0.039$). This shows (38) with $D = 4.175$ and thus the proposition holds true with $C_2 = 4.18$. □

Now we are in position to prove the Theorem 13,

Proof of Theorem 13. Let as fix a unitary vector $|\psi\rangle$ and let $p := p^{\psi}$, $q := q^{\psi}$. Recall that $C_2 = 4.18$. We define the sequence m_i as

$$m_1 = \sqrt{C_2\frac{2}{N}(1+\ln(N))}, \tag{43}$$

and for $2 \leq i \leq 2N - 1$,

$$m_i = \sqrt{C_2\frac{i+1}{N}\left(1+\ln\left(\frac{2N}{i+1}\right)\right)} - \sqrt{C_2\frac{i}{N}\left(1+\ln\left(\frac{2N}{i}\right)\right)} > 0, \tag{44}$$

which we can rewrite as

$$m_i = \sqrt{2C_2}\left(f\left(\frac{i+1}{2N}\right) - f\left(\frac{i}{2N}\right)\right), \tag{45}$$

where $f\colon (0,e) \to \mathbb{R}$ is given by $f(x) = \sqrt{x\,(1-\ln x)}$. The function f is concave, which can be verified by simple calculations, i.e.,

$$\frac{d}{dx} f(x) = \frac{-\ln x}{2\sqrt{x\,(1-\ln x)}} \tag{46}$$

and

$$\frac{d^2}{dx^2} f(x) = \frac{-(1-\ln x)^2 - 1}{4(x(1-\ln x))^{3/2}} < 0. \tag{47}$$

From concavity we obtain that for $2 \le i \le 2N-1$,

$$m_i \le \sqrt{2C_2}\frac{1}{2N}\frac{d}{dx} f\left(\frac{i}{2N}\right) = \frac{1}{N}\frac{\sqrt{C_2}\ln(\frac{2N}{i})}{2\sqrt{\frac{i}{N}(\ln(\frac{2N}{i})+1)}}. \tag{48}$$

Note that

$$m_1 + \sum_{i=2}^{N} \frac{1}{N}\frac{\sqrt{C_2}\ln(\frac{2N}{i})}{2\sqrt{\frac{i}{N}(\ln(\frac{2N}{i})+1)}} \ge \sum_{i=1}^{N} m_i = \sqrt{C_2\frac{N+1}{N}\left(1+\ln\left(\frac{2N}{N+1}\right)\right)} > 1. \tag{49}$$

Let N_0 be the greatest integer not exceeding N, such that

$$m_1 + \sum_{i=2}^{N_0} \frac{1}{N}\frac{\sqrt{C_2}\ln(\frac{2N}{i})}{2\sqrt{\frac{i}{N}(\ln(\frac{2N}{i})+1)}} \le 1 \tag{50}$$

and define a vector $r = (r_1, \ldots, r_{N_0+1}) \in \mathbb{R}^{N_0+1}$ by specifying its coordinates as follows. Set $r_1 = m_1$ and

$$r_i = \frac{1}{N}\frac{\sqrt{C_2}\ln(\frac{2N}{i})}{2\sqrt{\frac{i}{N}(\ln(\frac{2N}{i})+1)}} \tag{51}$$

for $i = 2, \ldots, N_0$. As the last coordinate set $r_{N_0+1} = 1 - \sum_{i=1}^{N_0} r_i$, so r is a probability vector. Note that

$$r_{N_0+1} \le \frac{1}{N}\frac{\sqrt{C_2}\ln(\frac{2N}{N_0+1})}{2\sqrt{\frac{N_0+1}{N}(\ln(\frac{2N}{N_0+1})+1)}} \le \sqrt{\frac{C_2\ln N}{N}}. \tag{52}$$

Let z be the non-increasing rearrangement of $p \oplus q$. For $k \le N_0 + 1$, Theorem 3 and Proposition 14 give

$$\begin{aligned} z_1 + \cdots + z_k &\le 1 + s_{k-1} \le 1 + \sqrt{C_2\frac{k}{N}\left(\ln\left(\frac{2N}{k}\right)+1\right)} \\ &= 1 + m_1 + \cdots + m_{k-1} \le 1 + r_1 + \cdots + r_{k-1}. \end{aligned} \tag{53}$$

Obviously we also have $z_1 + \cdots + z_k \le 2 = 1 + r_1 + \cdots + r_{N_0+1}$ for $k > N_0 + 1$, and so $z \prec 1 \oplus r$. As a consequence,

$$H(p) + H(q) \ge H(r). \tag{54}$$

We will now bound from below the entropy of the vector r. We have

$$\begin{aligned} H(r) &\geq -\sum_{i=2}^{N_0} r_i \ln r_i = -\sum_{i=2}^{N_0} r_i \ln\left(\frac{1}{N}\frac{\sqrt{C_2}\ln(\frac{2N}{i})}{2\sqrt{\frac{i}{N}(\ln(\frac{2N}{i})+1)}}\right) \\ &= \sum_{i=2}^{N_0} r_i \ln N - \sum_{i=2}^{N_0} \frac{1}{N}\frac{\sqrt{C_2}\ln(\frac{2N}{i})}{2\sqrt{\frac{i}{N}(\ln(\frac{2N}{i})+1)}} \ln\left(\frac{\sqrt{C_2}\ln(\frac{2N}{i})}{2\sqrt{\frac{i}{N}(\ln(\frac{2N}{i})+1)}}\right) \\ &= \ln N - (m_1 + r_{N_0+1})\ln N - A_N = \ln N - O\Big(\frac{\ln^{3/2} N}{\sqrt{N}}\Big) - A_N, \end{aligned} \tag{55}$$

where

$$A_N = \sum_{i=2}^{N_0} \frac{1}{N}\frac{\sqrt{C_2}\ln(\frac{2N}{i})}{2\sqrt{\frac{i}{N}(\ln(\frac{2N}{i})+1)}} \ln\left(\frac{\sqrt{C_2}\ln(\frac{2N}{i})}{2\sqrt{\frac{i}{N}(\ln(\frac{2N}{i})+1)}}\right). \tag{56}$$

Above we used (52) and the estimate $m_1 = O(\sqrt{\frac{\ln N}{N}})$.

Let us now bound N_0/N from above. Since

$$m_1 + \sum_{i=2}^{k} \frac{1}{N}\frac{\sqrt{C_2}\ln(\frac{2N}{i})}{2\sqrt{\frac{i}{N}(\ln(\frac{2N}{i})+1)}} \geq \sum_{i=1}^{k} m_i = \sqrt{C_2\frac{k+1}{N}\left(1+\ln\Big(\frac{2N}{k+1}\Big)\right)}, \tag{57}$$

we have $N_0 \leq N_1$, where N_1 is the largest integer smaller than N, such that

$$\sqrt{C_2\frac{N_1+1}{N}\left(1+\ln\Big(\frac{2N}{N_1+1}\Big)\right)} \leq 1. \tag{58}$$

We have $N_1/N \to x^*$, where x^* is the unique solution of

$$C_2 x^*(1+\ln(2/x^*)) = 1. \tag{59}$$

Since $C_2 = 4.18$ we can evaluate numerically that $x^* \simeq 0.051$ and so we can write

$$\limsup A_N < \int_0^{0.052} \frac{\sqrt{C_2}\ln\left(\frac{2}{x}\right)}{2\sqrt{x\ln\left(\frac{2e}{x}\right)}} \ln\left(\frac{\sqrt{C_2}\ln\left(\frac{2}{x}\right)}{2\sqrt{x\ln\left(\frac{2e}{x}\right)}}\right) dx \simeq 3.488, \tag{60}$$

which ends the proof (note that the integrand above is positive on the interval of integration). □

The state independent lower bound $\ln N - C_1$ on the sum of entropies is clearly stronger than all the bounds derived from known entropic uncertainty relations that we have analyzed above. Also it differs from the best possible one by at most C_1, since by choosing ψ to be a member of one of the bases related to measurements we can enforce the equality $H(p^\psi) = 0$, whereas trivially $H(q^\psi) \leq \ln N$. In fact by taking the randomness into account one can show that the gap between the result of Theorem 13 and the optimal one is even smaller, since for random U and fixed ψ, the quantity $H(q^\psi)$ can be interpreted as the entropy of a random state. An estimation for the mean entropy of a random state follows from the work[28] by Jones. Let $|\psi\rangle$ and $|\phi\rangle$ be N-dimensional normalized vectors in $\mathbb{C}^N$ and $\mathrm{d}\Omega_\phi$ be the unique, normalized unitary invariant measure $d\Omega_\phi$ upon the set of pure quantum states. Jones analyzed the mean value of the following entropy:

$$H(1,1) = -N\int |\langle\psi|\phi\rangle|^2 \ln(|\langle\psi|\phi\rangle|^2)\mathrm{d}\Omega_\phi \tag{61}$$

and derived its asymptotic behavior

$$H(1,1) = \Psi(N+1) - \Psi(2) \sim_{N\to\infty} \ln(N) - \Psi(2) + o(1). \tag{62}$$

Here $\Psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ denotes the digamma function, and $\Psi(2) = 1 - \gamma \simeq 0.42$, where $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ is the Gamma function and γ is the Euler constant. Note that $H(1,1)$ is also the mean value of the entropy of a probability vector $p_i = |\langle\psi|U|i\rangle|^2$, $i = 1,2,\ldots,N$ describing von Neumann measurement of a fixed pure state $|\psi\rangle$ with respect to a basis related to a random unitary matrix U, or equivalently entropy of a pure random state with respect to a fixed basis. Since it is known that the Shannon's entropy of a pure random state concentrates strongly around the expectation (see Appendix B.2. of Ref. 20), by combining the above result with Theorem 13 we arrive at a sandwich relation described by the following theorem.

Theorem 15. *Let U be a $N \times N$ random unitary matrix. Let C_0 be any real number smaller than $1 - \gamma \simeq 0.42$ and let $C_1 = 3.49$. Then*

$$\mathbb{P}\left(\ln N - C_0 \geq \min_{\psi}\left(H(p^{\psi}) + H(q^{\psi})\right) \geq \ln N - C_1\right) \to 1, \tag{63}$$

as $N \to \infty$.

V. SEVERAL MEASUREMENTS

Here we will consider the case of an arbitrary number L of orthogonal measurements. Assume that the measurement bases are determined by independent random unitary matrices $U_1,\ldots,U_L$ of size N. For a state $|\psi\rangle$, let $p^{(\psi,i)} = (p_1^{(i)},\ldots,p_N^{(i)})$, with $p_j^{(i)} = |\langle u_j^{(i)}|\psi\rangle|^2$, where $u_j^{(i)}$ is the jth column of U_i.

Uncertainty relations for random unitaries were studied in Refs. 20 and 21. In the special case as the number L of measurements grows with the dimension N as $\alpha \ln^4 N$ for a certain numerical constant α the following asymptotic bound for the average entropy was derived[20]

$$\mathbb{P}\Big(\min_{\psi}\frac{1}{L}\sum_{i=1}^{L} H(p^{(\psi,i)}) \geq \ln N - \alpha\Big) \to 1. \tag{64}$$

In our work we improve the above result and relax the assumption that the number of measurements L and the dimensionality of the system N are related. The second main result of this paper shows that uniform unitaries satisfy strong uncertainty relations for an arbitrary number of measurements.

Theorem 16. *There exists a universal constant C_3 such that if $U_1,\ldots,U_L$ are independent $N \times N$ random unitary matrices, then*

$$\mathbb{P}\Big(\min_{\psi}\frac{1}{L}\sum_{i=1}^{L} H(p^{(\psi,i)}) \geq \frac{L-1}{L}\ln N - C_3\Big) \to 1 \tag{65}$$

as $N \to \infty$. Moreover, the convergence is uniform in $L \geq 2$.

Note that for $L \gg \ln N$ we have $\frac{L-1}{L}\ln N = \ln N - o(1)$, so in particular our result recovers (64).

Before providing the proof we will present a few comments concerning our approach and emphasize the differences with arguments in Ref. 20 or 21. We rely on strong majorization relations obtained in Ref. 16, which we combine with estimates of norms of submatrices of a random unitary matrix presented in Section III. The main probabilistic ingredient of our proof is the concentration of measure phenomenon combined with discretization, also used in Refs. 20 and 21. The advantage of the majorization approach stems from the fact that it reduces the problem to the analysis of norms of matrices, which are 1-Lipschitz functions of the matrix (with respect to the Hilbert-Schmidt norms), whereas the Lipschitz constant of the Shannon's entropy as a function of a state increases with the dimension. A better Lipschitz constant yields stronger concentration results which gives more freedom in choosing appropriate approximating sets and as a consequence allows to find the right balance between the complexity of the problem in dimension N and available probabilistic bounds.

In the proof of Theorem 16 we will use the following technical lemma, which will be proved in Appendix B. Recall the definition of $\mathcal{S}_k$ given in formula (10) and the notation used therein: U is the concatenation of matrices $U_1,\dots,U_L$ and for a set $I \subset \{1,\dots,LN\}$ with $|I| = k$, we define U_I to be the $N \times k$ matrix obtained from U by selecting the columns of U corresponding to the set I.

Lemma 17. In the setting of Theorem 16, with probability tending to 1 as $N \to \infty$ (uniformly in $L \geq 2$), for all $k \leq LN - 1$,

$$\sqrt{\mathcal{S}_k} = \max_{|I|=k+1} \|U_I\| \leq 1 + \sqrt{C_4 \frac{k+1}{N} \ln\Big(\frac{eNL}{k+1}\Big)}, \tag{66}$$

where C_4 is a universal constant.

Proof of Theorem 16. Let C_4 be the constant from Lemma 17. Define $M_1 = \Big(1 + \sqrt{C_4 \frac{2}{N} \ln\Big(\frac{eLN}{2}\Big)}\Big)^2 - 1$ and for $2 \leq i \leq NL - 1$,

$$M_i = \Big(1 + \sqrt{C_4 \frac{i+1}{N} \ln\Big(\frac{eLN}{i+1}\Big)}\Big)^2 - \Big(1 + \sqrt{C_4 \frac{i}{N} \ln\Big(\frac{eLN}{i}\Big)}\Big)^2. \tag{67}$$

We have for $2 \leq i \leq NL - 1$,

$$\begin{aligned} M_i &= 2\left(\sqrt{C_4 \frac{i+1}{N} \ln\Big(\frac{eLN}{i+1}\Big)} - \sqrt{C_4 \frac{i}{N} \ln\Big(\frac{eLN}{i}\Big)}\right) + \Big(C_4 \frac{i+1}{N} \ln\Big(\frac{eLN}{i+1}\Big) - C_4 \frac{i}{N} \ln\Big(\frac{eLN}{i}\Big)\Big) \\ &= 2\sqrt{C_4 L}\left(f\left(\frac{i+1}{LN}\right) - f\left(\frac{i}{LN}\right)\right) + C_4 L\left(g\left(\frac{i+1}{LN}\right) - g\left(\frac{i}{LN}\right)\right), \end{aligned} \tag{68}$$

where $f, g\colon [0,e] \to \mathbb{R}$ are given by $f(x) = \sqrt{x \ln(e/x)}$ and $g(x) = x \ln(e/x)$.

Both f and g are concave and thus

$$\begin{aligned} M_i &\leq 2\sqrt{C_4 L} \frac{1}{LN} \frac{d}{dx} f\left(\frac{i}{LN}\right) + C_4 L \frac{1}{LN} \frac{d}{dx} g\left(\frac{i}{LN}\right) \\ &= 2\sqrt{C_4 L} \frac{1}{LN} \frac{\ln(\frac{LN}{i})}{2\sqrt{\frac{i}{LN} \ln(\frac{eLN}{i})}} + \frac{C_4 L}{LN} \ln\left(\frac{LN}{i}\right) \\ &= \frac{1}{N}\left(\frac{\sqrt{C_4} \ln(\frac{LN}{i})}{\sqrt{L}\sqrt{\frac{i}{LN} \ln\left(\frac{eLN}{i}\right)}} + C_4 \ln\Big(\frac{LN}{i}\Big)\right) =: \tilde{M}_i. \end{aligned} \tag{69}$$

Since

$$M_1 + \sum_{i=2}^{LN-1} \tilde{M}_i \geq \sum_{i=1}^{LN-1} M_i = \Big(1 + \sqrt{C_4 \frac{LN}{N} \ln\Big(\frac{eLN}{LN}\Big)}\Big)^2 - 1 > L - 1, \tag{70}$$

there exists maximum $N_0 < LN - 1$ such that

$$M_1 + \sum_{i=2}^{N_0} \tilde{M}_i \leq L - 1 \tag{71}$$

(note that for N sufficiently large, independent of L, $M_1 \leq L - 1$).

Define a vector $W \in \mathbb{R}^{N_0+1}$ by $W_1 = M_1$,

$$W_i = \tilde{M}_i, \tag{72}$$

for $i = 2,\dots,N_0$ and $W_{N_0+1} = L - 1 - \sum_{i=1}^{N_0} W_i$.

Let z_i, $i = 1,\dots,NL$ be the non-increasing rearrangement of $p_1 \oplus \cdots \oplus p_L$. Using Theorem 4 and Lemma 17 we get that with probability tending to one as $N \to \infty$, for $k \leq N_0 + 1$,

$$\begin{aligned} z_1 + \cdots + z_k \leq \mathcal{S}_{k-1} &\leq \Big(1 + \sqrt{C_4 \frac{k}{N} \ln\Big(\frac{eNL}{k}\Big)}\Big)^2 \\ &= 1 + M_1 + \cdots + M_{k-1} \leq 1 + W_1 + \cdots + W_{k-1}. \end{aligned} \tag{73}$$

Also for $k > N_0 + 1$ we have $z_1 + \cdots + z_k \leq L = 1 + W_1 + \cdots + W_{N_0+1}$, so $z \prec 1 \oplus W$ and as a consequence

$$\sum_{i=1}^{L} H(p_i) \geq -\sum_{i=1}^{N_0+1} W_i \ln W_i. \tag{74}$$

Now, using the definition of M_1 and $\tilde{M}_i$, it is easy to see that $M_1 + W_{N_0+1} \leq C_5\left(\sqrt{\frac{\ln(LN)}{N}} + \frac{\ln(LN)}{N}\right)$. In particular for large N (uniformly in $L \geq 2$) we have $M_1 \ln M_1 + W_{N_0+1} \ln W_{N_0+1} \leq C_6 \ln^2 L$ and so

$$\begin{aligned}
-\sum_{i=1}^{N_0+1} W_i \ln W_i &\geq \sum_{i=1}^{N_0} -W_i \ln W_i \\
&\geq -C_6 \ln^2 L + \left(\sum_{i=2}^{N_0} W_i\right) \ln N \\
&\quad - \sum_{i=2}^{N_0} \frac{1}{N} \left(\frac{\sqrt{C_4} \ln(\frac{LN}{i})}{\sqrt{L}\sqrt{\frac{i}{LN} \ln(\frac{eLN}{i})}} + C_4 \ln\left(\frac{LN}{i}\right) \right) \ln \left(\frac{\sqrt{C_4} \ln(\frac{LN}{i})}{\sqrt{L}\sqrt{\frac{i}{LN} \ln(\frac{eLN}{i})}} + C_4 \ln\left(\frac{LN}{i}\right) \right) \\
&= (L-1) \ln N - (M_1 + W_{N_0+1}) \ln N - C_6 \ln^2 L - B_N \\
&\geq (L-1) \ln N - C_6 \ln^2 L - C_5 \left(\sqrt{\frac{\ln(LN)}{N}} + \frac{\ln(LN)}{N} \right) \ln N - B_N,
\end{aligned} \tag{75}$$

where

$$B_N = \sum_{i=2}^{N_0} \frac{1}{N} \left(\frac{\sqrt{C_4} \ln(\frac{LN}{i})}{\sqrt{L}\sqrt{\frac{i}{LN} \ln(\frac{eLN}{i})}} + C_4 \ln\left(\frac{LN}{i}\right) \right) \ln \left(\frac{\sqrt{C_4} \ln(\frac{LN}{i})}{\sqrt{L}\sqrt{\frac{i}{LN} \ln(\frac{eLN}{i})}} + C_4 \ln\left(\frac{LN}{i}\right) \right). \tag{76}$$

Now, the following holds for a sufficiently large absolute constant C_7. If $i < LN/C_7$, then $\ln(\frac{eLN}{i}) \leq (\frac{LN}{i})^{1/4}$ and using the inequality $L \geq 2$, we get

$$\left(\frac{\sqrt{C_4} \ln(\frac{LN}{i})}{\sqrt{L}\sqrt{\frac{i}{LN} \ln(\frac{eLN}{i})}} + C_4 \ln\left(\frac{LN}{i}\right) \right) \ln \left(\frac{\sqrt{C_4} \ln(\frac{LN}{i})}{\sqrt{L}\sqrt{\frac{i}{LN} \ln(\frac{eLN}{i})}} + C \ln\left(\frac{LN}{i}\right) \right) \leq C_8 \left(\frac{LN}{i}\right)^{7/8}, \tag{77}$$

for some absolute constant C_8.

On the other hand if $i > LN/C_7$, then

$$\left(\frac{\sqrt{C_4} \ln(\frac{LN}{i})}{\sqrt{L}\sqrt{\frac{i}{LN} \ln(\frac{eLN}{i})}} + C_4 \ln\left(\frac{LN}{i}\right) \right) \ln \left(\frac{\sqrt{C_4} \ln(\frac{LN}{i})}{\sqrt{L}\sqrt{\frac{i}{LN} \ln(\frac{eLN}{i})}} + C_4 \ln\left(\frac{LN}{i}\right) \right) \leq C_9, \tag{78}$$

where C_9 is another absolute constant.

Thus, using $N_0 \leq LN$, we get

$$\begin{aligned}
B_N &\leq \frac{C_9}{N}(N_0 - LN/C_7)_+ + \frac{C_8 (LN)^{7/8}}{N} \sum_{i=2}^{\lfloor LN/C_7 \rfloor} \frac{1}{i^{7/8}} \\
&\leq C_9 L + \frac{C_8 (LN)^{7/8}}{N} C_{10} \left(\frac{LN}{C_7}\right)^{1/8} \leq C_{11} L.
\end{aligned} \tag{79}$$

It remains to bound from above the term $C_6 \ln^2 L + C_5 \left(\sqrt{\frac{\ln(LN)}{N}} + \frac{\ln(LN)}{N} \right) \ln N$ appearing in (75). It is easy to see that for sufficiently large N (uniformly in $L \geq 2$) it is bounded by $C_{12} L$.

Combining this estimate with (75) and (79) gives

$$-\sum_{i=1}^{N_0+1} W_i \ln W_i \geq (L-1)\ln N - C_3 L, \tag{80}$$

with $C_3 = C_{11} + C_{12}$. By (74) this ends the proof of the theorem. □

To relate the above result with earlier literature recall that Wehner and Winter[7] defined a function

$$h(L) = \lim_{N\to\infty} \max_{U_1,\ldots,U_L \in U(N)} \frac{1}{\ln N} \min_{\psi} \frac{1}{L} \sum_{i=1}^{L} H(p^{(\psi,i)}). \tag{81}$$

They ask whether $h(L) = 1 - \frac{1}{L}$ for all $L \geq 2$. A related weaker question is whether there exists an increasing function $f\colon \mathbb{N} \to [0,\infty)$, such that $\lim_{L\to\infty} f(L) = \infty$ and $h(L) \geq 1 - \frac{1}{f(L)}$. Clearly any such function must be bounded from above by L. Theorem 16 immediately yields the following corollary.

Corollary 18. The conjecture by Wehner and Winter holds true, i.e., for every L the limit h(L) in (81) exists and

$$h(L) = 1 - \frac{1}{L}. \tag{82}$$

In Ref. 21 Fawzi *et al.* showed by probabilistic methods that for any $L \geq 2$ and $N > 2$ there exist L unitary matrices $U_1, \ldots, U_L$ such that

$$\min_{\psi} \frac{1}{L} \sum_{i=1}^{L} H(p^{(\psi,i)}) \geq \Big(1 - \sqrt{\frac{c' \ln L}{L}}\Big) \ln N - \ln\Big(\frac{18L}{c' \ln L}\Big) - H\Big(\sqrt{\frac{c' \ln L}{L}}, 1 - \sqrt{\frac{c' \ln L}{L}}\Big) \tag{83}$$

for some universal constant c'. In particular this proves the weak form of the Wehner and Winter conjecture with $f(L) = \sqrt{\frac{L}{c' \ln N}}$. We remark that Fawzi *et al.*[21] obtained also more explicit constructions of matrices satisfying entropic uncertainty principles for N being a power of 2, as their constructions can be efficiently performed by quantum circuits. However the number of measurements L in their scheme is of polynomial order in $\ln N$.

Following the strategy of Ref. 21, our Theorem 16 can be directly applied to protocols of locking of the classical information in quantum states.[29,30] Our bounds for the average entropy, valid for large dimension N and an arbitrary number of measurements L, provide more precise estimations concerning the information leaked by a measurement from a quantum system used in an information locking scheme.

VI. CONCLUDING REMARKS

In this work we analyzed truncations of $N \times N$ random unitary matrices and obtained estimations (16) and (19)–(21) for their norms. These results allowed us to study various entropic uncertainty relations providing the bounds for the sum of entropies describing information gained in two orthogonal measurements of any N-dimensional pure quantum state.

Our analysis reveals in particular that classical relations, known to be optimal in extremal settings, do not perform well in generic situations. For instance, Maassen–Uffink bound (1) averaged with the Haar measure over the unitary group behaves asymptotically as $\ln N - \ln\ln N - \ln 2$. As the largest element of a random orthogonal matrix is typically larger by a factor of $\sqrt{2}$,[27] the same bound averaged over the orthogonal group gives $\ln N - \ln\ln N - 2\ln 2$. These results can be compared with implications of the strong entropic uncertainty relation which averaged over the unitary group gives a lower bound $\ln N - C$, which is close to the best possible one. Although the exact value of the optimal constant C is still unknown, sandwich form (63) implies that $C \in (0.42, 3.49)$.

It is natural to conjecture that if U is drawn from the Haar measure on the unitary group $U(N)$ and $D_N = \min_\psi(H(p^\psi) + H(q^\psi))$, then there exists a limit

$$\lim_{N\to\infty}(\ln N - \mathbb{E}D_N). \tag{84}$$

Strong majorization entropic uncertainty relations can be also formulated for L orthogonal measurements, determined by a collection of L unitary matrices of order N. Making use of bounds for the norms of their submatrices we established estimate (65), which implies that the sum of L entropies behaves asymptotically as $(L-1)\ln N - \text{const}$. This result, holding for an arbitrary number L of measurements, is up to an additive constant compatible with estimate (63) valid for $L = 2$. In particular it allows us to answer completely an open question by Wehner and Winter on asymptotic behavior of constants in entropic uncertainty relations for many measurements as the dimension of the underlying Hilbert space tends to infinity. Furthermore, these bounds can be used to quantify the information leaked due to measurements from a quantum system, in which information locking protocol is applied.[21]

A natural open question is to find more precise estimations for these additive constants determining the typical behavior of entropic uncertainty relations. To get tighter bounds for averaged relation (9) one would need to improve the bounds for the average norms of the leading truncations of random unitaries. Note that bounds (20) and (19) derived in this work can be considered as complementary: The former one holds for $m = 1$ and an arbitrary $n \in [1, N]$, while the latter one works for any sizes n and m of the submatrix, but provides non-trivial estimates if n is small with respect to the matrix size N. Therefore, it is tempting to believe that establishing a new family of bounds for the norms $\|\widehat{U}^{(n,m)}\|$, which share advantages of both known results, would allow one to improve the quality of the asymptotic entropic uncertainty relations. We also mention that obtaining optimal bounds on norms of submatrices of a random unitary matrix seems to be an interesting problem in its own rights, with potential applications in random matrix theory or asymptotic geometric analysis.

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APPENDIX A: PROOFS OF ESTIMATES FOR NORMS OF SUBMATRICES

1. Notation

Before we proceed with the proofs let us gather here some (rather standard) notation we are going to use.

For $|x\rangle \in \mathbb{C}^n$, by $\|x\|$, we will denote its standard Euclidean norm, i.e., $\|x\| = \sqrt{\langle x|x\rangle}$. In the course of the proof we will often encounter the Euclidean norm of $A|x\rangle$, where A is a matrix and $|x\rangle \in \mathbb{C}^n$. To shorten the notation we will denote it by $\|Ax\|$, i.e.,

$$\|Ax\| = \sqrt{\langle x|A^\dagger A|x\rangle}. \tag{A1}$$

Recall also that if A is a matrix, by $\|A\|$ we denote the operator norm of A. We will also use the Hilbert-Schmidt norm of A, defined as $\|A\|_{HS} = \sqrt{\mathrm{Tr}AA^\dagger}$. By $|I|$ we will denote the cardinality of a finite set I. For a positive integer n by S^{n-1} we will denote the unit sphere in $\mathbb{R}^n$ equipped with the standard Euclidean norm, while $S_{\mathbb{C}}^{n-1}$ will denote the unit sphere in $\mathbb{C}^n$. Clearly $S_{\mathbb{C}}^{n-1}$ is isometric to

S^{2n-1}. By Δ_{n-1} we will denote the standard $(n-1)$-dimensional simplex in $\mathbb{R}^n$, i.e.,

$$\Delta_{n-1} = \Big\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon \forall_{1 \le i \le n}\ x_i \ge 0 \text{ and } \sum_{i=1}^{n} x_i = 1\Big\}. \tag{A2}$$

We will sometimes use the O notation. For two sequences $(a_N)_{N\ge 1}$ and $(b_N)_{N\ge 1}$ we will write $a_N = O(b_N)$ if there exists a constant K such that for all $N \ge 1$, $a_N \le K b_N$. We recall that by $\prec$ we denote the majorization relation defined in Section II after formula (5).

2. Proof of Theorem 7

Recall that a probability measure μ on a metric space (X,d) satisfies a log-Sobolev inequality with constant C if for any locally Lipschitz function f

$$\int f^2 \ln f^2 d\mu - \int f^2 d\mu \ln \int f^2 d\mu \le 2C \int |\nabla f|^2 d\mu, \tag{A3}$$

where $|\nabla f|$ is the length of gradient with respect to the metric d, i.e.,

$$|\nabla f|(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(x,y)} \tag{A4}$$

(see e.g., Chapter 3.1. of Ref. 31 or the Appendix of Ref. 32). For any such measure and any L-Lipschitz function F we then have (cf., Ref. 31 [Section 5.1])

$$\int \exp\Big(\lambda\Big(F - \int F d\mu\Big)\Big) d\mu \le \exp\Big(\frac{CL^2}{2}\lambda^2\Big) \quad \text{for all } \lambda \in \mathbb{R} \tag{A5}$$

and

$$\mu\Big(F \ge \int F d\mu + t\Big) \le \exp\Big(-\frac{t^2}{2CL^2}\Big) \quad \text{for } t \ge 0. \tag{A6}$$

We will use the following estimate of the log-Sobolev constant for the unitary group (cf., Ref. 32 [Theorem 15]).

Theorem 19. *The Haar measure on the unitary group $U(N)$ satisfies a log-Sobolev inequality with constant $6/N$ with respect to the Hilbert-Schmidt distance.*

We recall, that for a $N \times N$ matrix $U = (U_{i,j})_{i,j=1}^N$ by $\|\widehat{U^{(n,m)}}\|$ we denote the maximal norm of its $n \times m$ submatrices, i.e.,

$$\|\widehat{U^{(n,m)}}\| := \max_{|I|=n, |J|=m} \|U(I,J)\|, \tag{A7}$$

where $U(I,J) := (U_{i,j})_{i\in I, j\in J}$.

Proof of Theorem 7. The function $U \mapsto \|\widehat{U^{(m,n)}}\|$ is 1-Lipschitz with respect to the Hilbert-Schmidt norm. Therefore estimate (17) immediately follows by (A6) and Theorem 19.

Observe that for any $|x\rangle \in \mathbb{C}^N$ with $\|x\| = 1$, the random variable $U|x\rangle$ is uniformly distributed on $S_{\mathbb{C}}^{N-1} \simeq S^{2N-1}$. It is well known that for any l, the uniform distribution on S^l satisfies a log-Sobolev inequality with constant $1/l$, (cf., formula (5.7) in Ref. 31). For any $|y\rangle \in \mathbb{C}^N$ the function $z \mapsto \mathbf{Re}\langle y|z\rangle$ is $\||y\rangle\|$-Lipschitz on S^{N-1}. Therefore, using the fact that $\mathbb{E}\langle y|U|x\rangle = 0$, we get

$$\mathbb{E}e^{\lambda \mathbf{Re}\langle y|U|x\rangle} \le \exp\Big(\frac{1}{2(2N-1)}\lambda^2\Big) \quad \text{for all } \lambda \in \mathbb{R},\ |x\rangle, |y\rangle \in S_{\mathbb{C}}^{N-1}. \tag{A8}$$

Now suppose that we have a finite set $E \subset S_{\mathbb{C}}^{N-1} \times S_{\mathbb{C}}^{N-1}$. Then

$$\mathbb{E} \max_{(|x\rangle,|y\rangle)\in E} \mathbf{Re}\langle y|U|x\rangle \le \sqrt{\frac{2\ln|E|}{2N-1}}. \tag{A9}$$

Indeed we have for $\lambda > 0$,

$$\mathbb{E}\exp\Big(\lambda \max_{(|x\rangle,|y\rangle)\in E} \mathbf{Re}\langle y|U|x\rangle\Big) \le \mathbb{E}\sum_{(|x\rangle,|y\rangle)\in E} e^{\lambda \mathbf{Re}\langle y|U|x\rangle} \le |E|\exp\Big(\frac{1}{2(2N-1)}\lambda^2\Big). \tag{A10}$$

Jensen's inequality gives

$$\mathbb{E}\exp\Big(\lambda \max_{(|x\rangle,|y\rangle)\in E} \mathbf{Re}\langle y|U|x\rangle\Big) \ge \exp\Big(\lambda \mathbb{E}\max_{(|x\rangle,|y\rangle)\in E} \mathbf{Re}\langle y|U|x\rangle\Big), \tag{A11}$$

hence

$$\mathbb{E}\max_{(|x\rangle,|y\rangle)\in E} \mathbf{Re}\langle y|U|x\rangle \le \inf_{\lambda>0}\frac{1}{\lambda}\Big(\ln|E| + \frac{1}{2(2N-1)}\lambda^2\Big) = \sqrt{\frac{2\ln|E|}{2N-1}}. \tag{A12}$$

Let us now estimate $\mathbb{E}\|\widehat{U}^{(m,n)}\|$. For any $\emptyset \neq I \subset \{1,\dots,N\}$ consider the $(|I|\text{-}1)$-dimensional unit sphere

$$S_I := \{|x\rangle = (x_1,\dots,x_N) \in S_{\mathbb{C}}^{N-1} \colon x_i = 0 \text{ for } i \notin I\} \tag{A13}$$

and choose an ε-net E_I in S_I of cardinality at most $(1+2/\varepsilon)^{2|I|}$ (such a net exists by standard volumetric estimates, see e.g., Ref. 33). Let $E_l := \bigcup_{|I|=l} E_I$, then for any $1 \le l \le N$,

$$|E_l| \le \binom{N}{l}\Big(1+\frac{2}{\varepsilon}\Big)^{2l} \le \Big(\frac{eN}{l}\Big)^l\Big(1+\frac{2}{\varepsilon}\Big)^{2l}. \tag{A14}$$

Estimate (A9) gives

$$\mathbb{E}\max_{|x\rangle\in E_n,|y\rangle\in E_m} \mathbf{Re}\langle y|U|x\rangle \le \sqrt{\frac{2}{2N-1}(\ln|E_n| + \ln|E_m|)}. \tag{A15}$$

Finally it is not hard to see that

$$\|\widehat{U}^{(m,n)}\| \le \frac{1}{1-2\varepsilon-\varepsilon^2}\max_{|x\rangle\in E_n,|y\rangle\in E_m} \mathbf{Re}\langle y|U|x\rangle. \tag{A16}$$

Inequality (18) follows now easily by the three last estimates. Bound (19) follows from (18) by elementary calculations. □

3. Proof of Theorem 6

Note that the upper bound on $\|\widehat{U}^{(n,m)}\|$ follows from the already proven Theorem 7. To complete the proof it is thus enough to show that for all fixed positive integers n, m and $\varepsilon > 0$,

$$\mathbb{P}\Big(\|\widehat{U}^{(n,m)}\| \le (1-\varepsilon)\sqrt{(n+m)\frac{\ln N}{N}}\Big) \to 0, \tag{A17}$$

as $N \to \infty$.

Let $\Gamma = (\Gamma_{ij})_{i,j=1}^N$ be a $N \times N$ matrix whose entries are i.i.d. standard complex Gaussian variables (i.e., their real and imaginary parts are independent, with Gaussian distribution of mean zero and variance $1/2$, or equivalently with the density $g(x) = \frac{1}{\sqrt{\pi}}e^{-x^2}$ with respect to the Lebesgue measure on $\mathbb{R}$).

Set $M = M_N = N/\ln^2 N$. By Theorem 6 in Ref. 27 (applied with $m = M_N$, $r = 1/\ln N$, $s = \ln N/(\ln\ln N)^{1/2}$, $t = \sqrt{\ln N/\ln\ln N}$, cf., formula (2.10) in Ref. 27) we can assume that

$$\max_{i\le N, j\le M_N} |\sqrt{N}U_{ij} - \Gamma_{ij}| \le C_{13}\sqrt{\ln N/\ln\ln N}, \tag{A18}$$

with probability at least $1 - C_{14}\exp(-\ln^{3/2}N)$.

By (A18) it is enough to show that with probability tending to 1,

$$\|\widehat{\Gamma'}^{(n,m)}\| \ge (1-\varepsilon)\sqrt{(n+m)\ln N}, \tag{A19}$$

where Γ' is the $M_N \times M_N$ principal submatrix of Γ.

Since $\frac{\ln M_N}{\ln N} \to 1$ as $N \to \infty$, (A17) will follow if we prove the following.

Proposition 20. For any positive integers n, m and any $\varepsilon > 0$,

$$\mathbb{P}\Big(\|\widehat{\Gamma}^{(n,m)}\| \leq (1-\varepsilon)\sqrt{(n+m)\ln N}\Big) \to 0. \tag{A20}$$

Proof. First note that by the concentration property of Gaussian measures (see, e.g., Ref. 31) and the fact that $\|\widehat{\Gamma}^{(n,m)}\|$ is 1-Lipschitz with respect to the Hilbert-Schmidt norm, we have

$$\mathbb{P}\Big(\Big|\|\widehat{\Gamma}^{(n,m)}\| - \mathbb{E}\|\widehat{\Gamma}^{(n,m)}\|\Big| \geq t\Big) \leq 2\exp(-t^2). \tag{A21}$$

Thus to prove the proposition it is enough to show that for every $\varepsilon > 0$, and N large enough $\mathbb{E}\|\widehat{\Gamma}^{(n,m)}\| \geq (1-\varepsilon)\sqrt{(n+m)\ln N}$. Assume that $\mathbb{E}\|\widehat{\Gamma}^{(n,m)}\| < (1-\varepsilon)\sqrt{(n+m)\ln N}$. Then, again by concentration $\mathbb{P}(\|\widehat{\Gamma}^{(n,m)}\| \geq (1-\varepsilon/2)\sqrt{(n+m)\ln N}) \leq 1/N^{(n+m)\varepsilon^2/4} \to 0$ as $N \to \infty$. Therefore, to prove (A20) it is enough to show that for every $\varepsilon > 0$, there exists $d > 0$ such that for N large enough, we have

$$\mathbb{P}\Big(\|\widehat{\Gamma}^{(n,m)}\| \geq (1-\varepsilon)\sqrt{(n+m)\ln N}\Big) > d. \tag{A22}$$

It is well known that $|\Gamma_{ij}|^2$ are standard exponential variables (i.e., they have a density $g(x) = e^{-x}\mathbf{1}_{[0,\infty)}(x)$), therefore

$$\mathbb{P}\Big(|\Gamma_{ij}|^2 \geq \frac{n+m}{nm}\ln N\Big) = \frac{1}{N^{\frac{n+m}{nm}}}. \tag{A23}$$

Moreover Γ_{ij} are rotationally invariant, so for any $\delta \in (0,1)$,

$$\mathbb{P}\Big(|\Gamma_{ij}|^2 \geq \frac{n+m}{nm}\ln N, \operatorname{Arg}\Gamma_{ij} \in [0, 2\pi\delta)\Big) = \frac{\delta}{N^{\frac{n+m}{nm}}}. \tag{A24}$$

Consider any $I, J \subset \{1,\dots,N\}$ with $|I| = n$, $|J| = m$ and define the event

$$\mathcal{E}(I,J) = \Big\{\forall_{i\in I, j\in J}\ |\Gamma_{ij}|^2 \geq \frac{n+m}{nm}\ln N, \operatorname{Arg}\Gamma_{ij} \in [0, 2\pi\delta)\Big\}. \tag{A25}$$

Note that for δ small enough, depending on ε, on the event $\mathcal{E}(I,J)$ we have

$$\|\Gamma(I,J)\| \geq (1-\varepsilon)\sqrt{(n+m)\ln N}, \tag{A26}$$

where $\Gamma(I,J) = (\Gamma_{ij})_{i\in I, j\in J}$.

Indeed for the unit vector $|z\rangle = m^{-1/2}(1,\dots,1) \in \mathbb{C}^m$ we have (recall our notation introduced in (A1))

$$\|\Gamma(I,J)z\|^2 = \langle z|\Gamma(I,J)^\dagger\Gamma(I,J)|z\rangle = \sum_{i\in I}\frac{1}{m}\Big|\sum_{j\in J}\Gamma_{ij}\Big|^2. \tag{A27}$$

Now for δ small enough,

$$\begin{aligned}\Big|\sum_{j\in J}\Gamma_{ij}\Big|^2 &= \sum_{j,j'\in J}|\Gamma_{ij}||\Gamma_{ij'}|\cos\left(\operatorname{Arg}\Gamma_{ij}\overline{\Gamma_{i,j'}}\right)\\ &\geq \sum_{j,j'\in J}|\Gamma_{ij}||\Gamma_{ij'}|\cos(2\pi\delta) \geq (1-\varepsilon)^2 m^2\frac{n+m}{nm}\ln N\end{aligned} \tag{A28}$$

and thus

$$\|\Gamma(I,J)\|^2 \geq \|\Gamma(I,J)z\|^2 \geq (1-\varepsilon)^2(n+m)\ln N, \tag{A29}$$

which proves (A26).

Thus to prove the proposition it is enough to show that for N large enough,

$$\mathbb{P}\Big(\bigcup_{|I|=n, |J|=m}\mathcal{E}(I,J)\Big) \geq d. \tag{A30}$$

By the Bonferroni inequality we have

$$\mathbb{P}\Big(\bigcup_{|I|=n,|J|=m}\mathcal{E}(I,J)\Big) \geq \sum_{|I|=n,|J|=m}\mathbb{P}(\mathcal{E}(I,J)) - \sum_{\substack{|I|=|I'|=n,|J|=|J'|=m\\(I,J)\neq(I',J')}}\mathbb{P}(\mathcal{E}(I,J)\cap\mathcal{E}(I',J')) \quad (A31)$$

$$=: A - B.$$

By (A24) and independence of the entries of Γ,

$$A = \binom{N}{m}\binom{N}{n}\frac{\delta^{mn}}{N^{n+m}} \to \frac{\delta^{nm}}{n!m!}, \quad (A32)$$

as $N\to\infty$.

Now we group the summands in B, depending on the cardinality of $I\cup I'$ and $J\cup J'$ and obtain

$$B = \sum_{\substack{n\leq r\leq 2n, m\leq s\leq 2m\\ r+s>n+m}}\ \sum_{\substack{|I|=|I'|=n,|J|=|J'|=m\\|I\cup I'|=r,|J\cup J'|=s}}\mathbb{P}(\mathcal{E}(I,J)\cap\mathcal{E}(I',J')). \quad (A33)$$

For fixed r,s there are at most $C_{rs}N^{r+s}$ pairs $(I,J),(I',J')$ such that $|I|=|I'|=n$, $|J|=|J'|=m$, $|I\cup I'|=r, |J\cup J'|=s$ where C_{rs} is a constant depending only on r and s. For each such pair the event $\mathcal{E}(I,J)\cap\mathcal{E}(I',J')$ is the intersection of $rs-2(r-n)(s-m)$ independent events of form (A24). Therefore,

$$\mathbb{P}(\mathcal{E}(I,J)\cap\mathcal{E}(I',J')) = \delta^{rs-2(r-n)(s-m)}N^{-(rs-2(r-n)(s-m))(n+m)/(nm)} \quad (A34)$$

and as a consequence

$$\sum_{\substack{|I|=|I'|=n,|J|=|J'|=m\\|I\cup I'|=r,|J\cup J'|=s}}\mathbb{P}(\mathcal{E}(I,J)\cap\mathcal{E}(I',J')) \leq C_{rs}\delta^{rs-2(r-n)(s-m)}N^{r+s-(rs-2(r-n)(s-m))(n+m)/(nm)} \quad (A35)$$

$$= C_{rs}\delta^{rs-2(r-n)(s-m)}N^{(s-2m)(r-n)/n+(r-2n)(s-m)/m}.$$

One can see that if $r\neq 2n$ or $s\neq 2m$ then $\frac{(s-2m)(r-n)}{n}+\frac{(r-2n)(s-m)}{m}<0$ and so the contribution to (A33) from such pairs converges to 0 as $N\to\infty$. Therefore, for δ small enough and large N,

$$\mathbb{P}(\bigcup_{|I|=n,|J|=k}\mathcal{E}(I,J)) \geq \frac{\delta^{nm}}{2n!m!} - C_{2n,2m}\delta^{2nm} > \frac{\delta^{nm}}{4n!m!}. \quad (A36)$$

Thus (A30) holds with $d=\frac{\delta^{nm}}{4n!m!}$, which ends the proof of the proposition. □

4. Proof of Theorems 8 and 9

Proof of Theorem 8. Note that the first column $(U_{1i})_{i=1}^N$ (or any other column or row of U) is uniformly distributed on $S_{\mathbb{C}}^{N-1}\simeq S^{2N-1}$. Hence the squares of the moduli of its entries, $q_i=|U_{1i}|^2$, $i=1,\ldots,N$, form a random probability vector uniformly distributed on the simplex $\Delta_{N-1}\subset\mathbb{R}^N$ (this observation seems to be a part of the folklore, it can be easily obtained by (1) expressing the uniform measure on $S_{\mathbb{C}}^{N-1}$ in terms of normalized complex Gaussian vectors, (2) using the fact that the square of the absolute value of a standard complex Gaussian variable has standard exponential distribution, (3) invoking the well known fact that a self-normalized vector with i.i.d. standard exponential coordinates is distributed uniformly on Δ_{N-1}, see e.g., Ref. 34).

To look for the largest component of the vector we order $q_1,\ldots,q_N$ in a weakly decreasing order, $q_1^{\downarrow}\geq q_2^{\downarrow}\geq\cdots\geq q_N^{\downarrow}$. It is not hard to notice that the random vector $q^{\downarrow}=(q_1^{\downarrow},q_2^{\downarrow},\ldots,q_N^{\downarrow})$ is uniformly distributed on the simplex $\tilde{\Delta}_{N-1}$ with vertices $(1,0,\ldots,0)$, $\frac{1}{2}(1,1,0,\ldots,0),\ldots,\frac{1}{N}(1,1,\ldots,1)$.

Thus the mean value of $q^{\downarrow}$ is the barycenter of $\tilde{\Delta}_{N-1}$. Its coordinates can be expressed in terms of the *harmonic* numbers $H_m:=\sum_{j=1}^m 1/j$, which asymptotically behave as $\ln m+\gamma$, where $\gamma\approx 0.5772$ denotes the Euler constant. Namely,

$$\mathbb{E}q_m^{\downarrow} = \frac{1}{N}\sum_{j=m}^N\frac{1}{j} = \frac{1}{N}(H_N - H_{m-1}). \quad (A37)$$

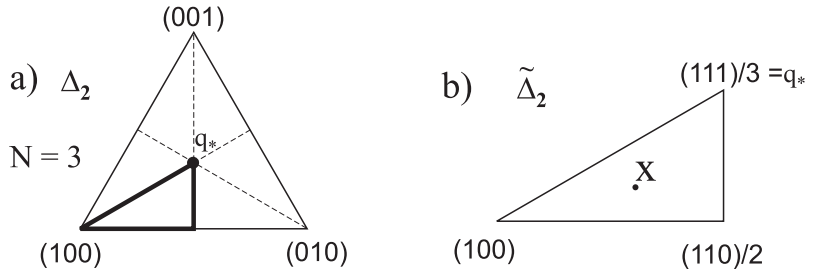

FIG. 1. (a) Simplex Δ_2 for $N=3$ centered at $q^*=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ and (b) its asymmetric part $\tilde{\Delta}_2$. The barycenter X of $\tilde{\Delta}_2$ with components $\left(\frac{11}{18},\frac{5}{18},\frac{2}{18}\right)$ represents the averaged ordered vector $\mathbb{E}q^{\downarrow}$.

Denote by $X_{n,i}$ $(i=1,\dots,N)$ the maximum norm of a subvector of dimension $n\le N$ of the ith column of U. The average of $X_{n,i}^2$ is equal to the sum of the first n components of the ordered vector $q^{\downarrow}$, averaged over the simplex $\tilde{\Delta}_{N-1}$ (Fig. 1 presents such a simplex for the case $N=3$),

$$\mathbb{E}X_{n,i}^2=\mathbb{E}\sum_{m=1}^{n}q_m^{\downarrow}=\frac{1}{N}\sum_{m=1}^{n}\sum_{j=m}^{N}\frac{1}{j}\,. \tag{A38}$$

To evaluate this sum we divide the summation region in the (m,j) plane into a triangle and a rectangle and change the summation order,

$$\mathbb{E}X_{n,i}^2=\frac{1}{N}\Big[\sum_{j=1}^{n}\frac{1}{j}\sum_{m=1}^{j}1+\sum_{j=n+1}^{N}\frac{1}{j}\sum_{m=1}^{n}1\Big]=\frac{n}{N}\Big[1+\sum_{j=n+1}^{N}\frac{1}{j}\Big]=\frac{n}{N}\Big[1+H_N-H_n\Big]. \tag{A39}$$

We can now easily finish the proof, since we have

$$\|\widehat{U}^{(n,1)}\|=\max_{i\le N}X_{n,i},$$

and (20) follows by the concentration of measure (recall that the uniform distribution on $S_{\mathbb{C}}^{N-1}$ satisfies the log-Sobolev inequality with constant $1/(2N-1)$). □

Proof of Theorem 9. Let us fix $\varepsilon>0$ and let $n_0=n_0(\varepsilon)$ be a sufficiently large constant depending on ε, to be chosen later on. By Theorem 6, with probability tending to 1 as $N\to\infty$, we have for all $n\le n_0$

$$(1-\varepsilon)\sqrt{\frac{n+1}{N}\Big(1+\ln\Big(\frac{N}{n}\Big)\Big)}\le\|\widehat{U}^{(n,1)}\|\le(1+\varepsilon)\sqrt{\frac{n+1}{N}\Big(1+\ln\Big(\frac{N}{n}\Big)\Big)} \tag{A40}$$

(note that in this range of n, $(1+\ln(N/n))=(1+o(1))\ln N$ as $N\to\infty$).

Consider any $n\ge n_0$. As in the proof of Theorem 8, denote by $X_{n,i}$ $(i=1,\dots,N)$ the maximum norm of a $n\times 1$ submatrix of the ith column of U.

Using (A39) we get $\mathbb{E}X_{n,i}^2=\frac{n}{N}(1+H_N-H_n)$ and so

$$(1-\varepsilon/8)^2\frac{n+1}{N}(1+\ln(N/n))\le\mathbb{E}X_{n,i}^2\le(1+\varepsilon/8)^2\frac{n+1}{N}(1+\ln(N/n)), \tag{A41}$$

where we used the fact that $n>n_0$.

Now, by integration by parts and (A6) it is easy to see that for large N,

$$\mathbb{E}X_{n,i}\ge\sqrt{\mathbb{E}X_{n,i}^2}-O(1/\sqrt{N})\ge(1-\varepsilon/8)\sqrt{\mathbb{E}X_{n,i}^2}, \tag{A42}$$

where the second inequality holds for $n>n_0$ and n_0 large enough. Thus

$$(1-\varepsilon/2)\sqrt{\frac{n+1}{N}(1+\ln(N/n))}\le\mathbb{E}X_{n,i}\le(1+\varepsilon/2)\sqrt{\frac{n+1}{N}(1+\ln(N/n))}. \tag{A43}$$

Now, using again (A6) together with the union bound we get

$$(1-\varepsilon)\sqrt{\frac{n+1}{N}(1+\ln(N/n))} \le X_{n,i} \le (1+\varepsilon)\sqrt{\frac{n+1}{N}(1+\ln(N/n))}, \tag{A44}$$

for all $n > n_0$ and $i \le N$, with probability at least

$$1 - N\sum_{n=n_0+1}^{N} \exp\Big(-\frac{\varepsilon^2}{24}(n+1)\ln(eN/n)\Big), \tag{A45}$$

which can be made arbitrarily close to one for $N \to \infty$ if one chooses $n_0(\varepsilon)$ sufficiently large (as can be easily seen by splitting the second term into two separate sums over $n_0 < n < \sqrt{N}$ and $\sqrt{N} \le n \le N$, respectively). The proof is concluded by combining (A40) and (A44). □

APPENDIX B: PROOF OF LEMMA 17

Proof of Lemma 17. Recall the notation introduced in Equation (A1). Let us note that by Theorem 19 and the tensorization property of entropy, U satisfies the log-Sobolev inequality with parameter $6/N$ with respect to the Hilbert-Schmidt metric. In particular, since for any unit vector $|x\rangle = (x_1,\dots,x_{NL}) \in \mathbb{C}^{NL}$, the map $U \mapsto \|Ux\|$ is 1-Lipschitz, we get

$$\mathbb{P}(\|Ux\| \ge \mathbb{E}\|Ux\| + t) \le 2e^{-Nt^2/12}. \tag{B1}$$

Denote the columns of U by $|Y_i\rangle$, $i = 1,\dots,NL$. We also have

$$\mathbb{E}\|Ux\|^2 = \mathbb{E}\langle x|U^\dagger U|x\rangle = \sum_{i=1}^{NL}|x_i|^2\mathbb{E}\||Y_i\|^2 + \sum_{i\ne j} x_i\bar{x}_j\mathbb{E}\langle Y_i|Y_j\rangle = 1, \tag{B2}$$

where we used the fact that for each $i \ne j$, $|Y_i\rangle$, and $|Y_j\rangle$ are of mean zero and either stochastically independent or orthogonal with probability one. Thus $\mathbb{E}\|Ux\| \le 1$. Moreover, by (B1) and integration by parts

$$1 = \sqrt{\mathbb{E}\|Ux\|^2} \le \mathbb{E}\|Ux\| + \sqrt{\frac{24}{N}}. \tag{B3}$$

Consider now a fixed set $I \subset \{1,\dots,NL\}$ with $|I| = k+1$ and let $\mathcal{N}_I$ be a 1/4-net in the unit ball of $\mathbb{C}^I = \{|x\rangle = (x_1,\dots,x_{NL}) \in \mathbb{C}^{NL}\colon x_i = 0 \text{ for } i \notin I\}$ of cardinality $10^{2(k+1)}$ (it exists by standard volumetric estimates, see Ref. 33). If C_{15} is a sufficiently large absolute constant, then by the union bound, with probability at least

$$1 - 10^{2(k+1)}\binom{LN}{k+1}e^{-C_{15}(k+1)\ln(eNL/(k+1))/13} \ge 1 - e^{-(k+1)\ln(eNL/(k+1))}, \tag{B4}$$

we have

$$1 - \sqrt{C_{15}\frac{k+1}{N}\ln\Big(\frac{eNL}{k+1}\Big)} \le \|Ux\| \le 1 + \sqrt{C_{15}\frac{k+1}{N}\ln\Big(\frac{eNL}{k+1}\Big)}, \tag{B5}$$

for all I with $|I| = k+1$ and $|x\rangle \in \mathcal{N}_I$.

Let $\delta = \sqrt{C_{15}\frac{k+1}{N}\ln\left(\frac{eNL}{k+1}\right)}$. If $\delta > 1$, then the second inequality in (B5) implies that

$$\|U_I\| \le \frac{4}{3}(1+\delta) \le 1 + \sqrt{C_{16}\frac{k+1}{N}\ln\Big(\frac{eNL}{k+1}\Big)}, \tag{B6}$$

for C_{16} sufficiently large (depending only on C_{15}).

If $\delta < 1$, then on the event where (B5) holds, we have for I with $|I| = k+1$ and $|x\rangle \in \mathcal{N}_I$,

$$1 - 2\delta \le \|U_I x\|^2 = \langle x|U_I^\dagger U_I|x\rangle \le 1 + 3\delta, \tag{B7}$$

which implies that the operator $A = U_I^\dagger U_I - \mathrm{Id}$ on $\mathbb{C}^I$ (where Id is the identity matrix), satisfies

$$|\langle x|A|x\rangle| \leq 3\delta \quad \text{for } |x\rangle \in \mathcal{N}_I. \tag{B8}$$

Let now $|y\rangle$ be any unit vector in $\mathbb{C}^I$ and $|x\rangle$ a point in $\mathcal{N}_I$ such that $\big\||x\rangle - |y\rangle\big\| < 1/4$. We have

$$\begin{aligned}\big|\langle y|A|y\rangle\big| &\leq \big|\left(\langle y| - \langle x|\right) A \left(|y\rangle - |x\rangle\right)\big| + \big|\langle x|A\left(|y\rangle - |x\rangle\right)\big| + \big|\left(\langle y| - \langle x|\right) A|x\rangle\big| \\ &+ \big|\langle x|A|x\rangle\big| \leq \frac{1}{16}\|A\| + \frac{1}{2}\|A\| + 3\delta.\end{aligned} \tag{B9}$$

Taking the supremum over $|y\rangle \in \mathbb{S}_{\mathbb{C}}^{N-1}$, using the fact that A is Hermitian and performing easy calculations we get

$$\|A\| \leq 7\delta, \tag{B10}$$

which implies that

$$\|U_I\|^2 \leq 1 + 7\delta \tag{B11}$$

and as a consequence

$$\|U_I\| \leq 1 + \sqrt{C_{17}\frac{k+1}{N}\ln\Big(\frac{eNL}{k+1}\Big)}. \tag{B12}$$

Now it remains to set $C_4 = \max(C_{16}, C_{17})$, take the union bound over all $k \leq NL - 1$ and note that

$$\sum_{k=1}^{NL-1} e^{-(k+1)\ln(eNL/(k+1))} \to 0, \tag{B13}$$

as $N \to \infty$ for $L \geq 2$. □

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