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Quantum dynamical entropy and decoherence rate

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Abstract

We investigate quantum dynamical systems defined on a finite-dimensional Hilbert space and subjected to an interaction with an environment. The rate of decoherence of initially pure states, measured by the increase of their von Neumann entropy, averaged over an ensemble of random pure states, is proved to be bounded from above by the partial entropy used to define the ALF-dynamical entropy. The rate of decoherence induced by the sequence of the von Neumann projectors measurements is shown to be maximal, if the measurements are performed in a randomly chosen basis. The numerically observed linear increase of entropies is attributed to free independence of the measured observable and the unitary dynamical map.

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1. Introduction

The notion of chaos in classical mechanics is well established, and any dynamical system characterized by positive Kolmogorov–Sinai (KS) entropy is called *chaotic* [1]. On the other hand, it is not at all easy to generalize the definition of chaos for quantum theory [2, 3]. There exist numerous attempts to define the quantum counterpart of the Kolmogorov–Sinai entropy both for finite and infinite quantum systems (see [4] and references therein). However, only two of them, CNT entropy [5] and ALF entropy [6], provide nonequivalent notions of quantum dynamical entropy which satisfy the following conditions:

(1) they can be formulated in an abstract algebraic framework valid for general commutative (classical) and noncommutative (quantum) dynamical systems,
(2) they coincide with the Kolmogorov–Sinai entropy when applied to classical systems, and
(3) they can be rigorously computed for several examples on noncommutative dynamical systems (different types of quantum shifts, automorphisms of quantum tori, fermionic quasi-free systems).

In contrast to the coherent-states quantum entropy defined in [4, 7], both CNT and ALF entropies are always equal to zero for quantum systems with finite-dimensional Hilbert spaces. In particular this concerns the systems being quantizations of classically chaotic dynamical systems defined on a compact phase space, we are going to analyse in this work. For the case of ALF entropy we can easily understand the mechanism leading to the apparent lack of 'correspondence principle' for the KS entropy. In fact we can see that classical–quantum correspondence exists provided the proper order of limit procedures is used. Recently this problem has been studied rigorously for both CNT and ALF entropies in the case of quantized Arnold cat maps [8].

A typical quantum state coupled with an environment suffers decoherence, i.e. a generic pure state becomes mixed as a result of the non-unitary dynamics. As shown by Zurek and Paz [9], the initial rate of decoherence is governed by the classical dynamical entropy h. Vaguely speaking, any classical density evolving in a two-dimensional phase space of a discrete invertible chaotic map T, is squeezed along the stable manifold and simultaneously stretched along the unstable manifold with the rate determined by the classical Lyapunov exponent λ. In a similar way the corresponding quantum wave packet is stretched. So in a generic case, it becomes coupled with an exponentially increasing number of states localized in the phase space. A natural assumption that these states are distinguishable by the environment implies the initially linear growth of the von Neumann entropy of a typical pure state, with the slope given by $h[T]$.

Detailed investigation of the rate of decoherence in various setups is a subject of considerable recent interest [10–18]. The main aim of this work is to establish a more precise relation between the Kolmogorov–Sinai entropy of the classical system and an increase of the average von Neumann entropy. We analyse the decoherence in finite quantum chaotic systems subjected to the sequence of periodical measurement process and establish a link between the rate of decoherence and the partial entropy used for the definition of the ALF entropy. The latter becomes equal to KS entropy in the classical limit. It is shown that such a correspondence is valid only if the measurement process possesses a well-defined classical limit, while in the case of a randomly chosen measurement the decoherence rate is governed by some global bounds essentially independent of the dynamics of the system.

The paper is organized as follows. In section 2 we recall the definition of the ALF quantum dynamical entropy. In section 3 we discuss the semiclassical limit of quantum maps and in section 4 analyse the classical limit of the ALF entropy. In section 5 we analyse the rate of decoherence and provide the upper bound (31) for the time evolution of the mean von Neumann entropy, averaged over the set of random initial pure states. These general results are used in section 6 by studying the decoherence in a model system: a periodically measured quantum baker map. Discussion of the results obtained in the context of the free-independent variables is provided in section 7.

Analysing the classical limit of the ALF entropy we found it convenient to make use of the standard C^* algebraic formalism and the notation common in papers on mathematical physics. On the other hand, we tend to believe that our work might also be useful for a reader working merely on decoherence in quantum systems and not interested in free random variables nor in subtle differences between various approaches to quantum dynamical entropy. A member of such a potential audience is kindly advised to learn about the model studied in the second part of section 3, and then proceed directly to section 6. Our numerical results presented here in figures 1 and 2 clearly illustrate the main message of the work. The initial rate of decoherence in a quantum system is governed by the degree of chaos of the corresponding classical system. This is the case if the measurement (Kraus) operators, which describe the coupling of the quantum system with an environment, possess a well-defined classical limit.

2. ALF-dynamical entropy

The original definition of the ALF-dynamical entropy [3, 6] contains an infinite time limit procedure for the entropy production which is meaningful for the infinite quantum systems described in terms of C^* algebras only. In this paper, similar to [19], we consider a finite quantum system described by d-dimensional Hilbert space $\mathbb{C}^d$ and the time dependence of the entropy production up to the saturation time of the order of $\ln d$. We use the discrete-time evolution given by the unitary $d \times d$ matrix U and the 'maximally mixed' reference state $\rho_\star = \mathbb{1}/d$. The first step in the ALF construction is the transition to the doubled quantum system described by the Hilbert space $\mathbb{C}^d \otimes \mathbb{C}^d$, and the purification of the state $\mathbb{1}/d$ given by the 'maximally entangled' state on $\mathbb{C}^d \otimes \mathbb{C}^d$ denoted by $|\Omega\rangle$. This state can be represented in the following way. Take the set $\mathcal{E}$ of pure states $\{|\alpha\rangle : \langle\alpha|\alpha\rangle = 1\}$ in $\mathbb{C}^d$ equipped with the (discrete or continuous) probability measure $\mathrm{d}\alpha$ and satisfying

$$\int_{\mathcal{E}} \mathrm{d}\alpha |\alpha\rangle\langle\alpha| = \frac{1}{d}\mathbb{1} \tag{1}$$

which will be called the *complete set of vectors*. The natural examples of $\mathcal{E}$ are orthonormal basis or coherent states generated by the irreducible representations of certain compact Lie groups on $\mathbb{C}^d$. The distinguished example is the set of all pure states $\mathcal{P}_d = \mathbb{C}P^{d-1}$ with the natural unitary invariant probability measure (Fubini–Study measure). The most general representation of maximally entangled vectors in terms of complete sets reads

$$G \leftrightarrow |\Psi_G\rangle = \sqrt{d}\int_{\mathcal{E}} \mathrm{d}\alpha |\alpha\rangle \otimes |G\alpha\rangle \tag{2}$$

where the antiunitary map G on $\mathbb{C}^d$ completely determines the vector $|\Psi_G\rangle$ irrespective of the complete set $\mathcal{E}$. It can be proved by taking two complete sets $\mathcal{E}, \mathcal{E}'$ and two antiunitary matrices G, G'. Define $|\Psi_G\rangle$ by (2), put $|\Psi_{G'}\rangle = \sqrt{d}\int_{\mathcal{E}'} \mathrm{d}\beta|\beta\rangle \otimes |G'\beta\rangle$ and compute the scalar product

$$\langle\Psi_{G'}|\Psi_G\rangle = d\int_{\mathcal{E}} \mathrm{d}\alpha \int_{\mathcal{E}'} \mathrm{d}\beta \langle\beta|\alpha\rangle\langle G(G^{-1}G')\beta|G\alpha\rangle = \int_{\mathcal{E}'} \mathrm{d}\beta\langle\beta|(G^{-1}G')\beta\rangle \tag{3}$$

equal to one if and only if $G = G'$.

The second step consists in taking a *partition of unity* $\mathbf{X} = \{X_1, X_2, \ldots, X_k\}$, where X_j, $j = 1, \ldots, k$ denote matrices of size d, which satisfy $\sum_{j=1}^k X_j^\dagger X_j = \mathbb{1}$. Such a partition describes an 'unsharp measurement' performed on the system. We shall use the same notation for the partition $\mathbf{X}$ extended trivially to the composed bi-partite system.

Partitions of unity can be composed, $\mathbf{X} \circ \mathbf{Y} = \{X_j Y_m\}$, and evolved in time $\mathcal{U}(\mathbf{X}) = \{U X_j U^\dagger\}$ to produce finer partitions, $\mathbf{X}^t = \mathcal{U}^{t-1}(\mathbf{X}) \circ \cdots \circ \mathcal{U}(\mathbf{X}) \circ \mathbf{X}$. We use the following notation for the multi-time correlation matrices given by

$$\begin{aligned}\sigma[\mathbf{X}^t]_{i_1,\ldots,i_t;j_1,\ldots,j_t} &= \big\langle\Omega\big|X_{j_1}^\dagger \mathcal{U}\big(X_{j_2}^\dagger\big)\ldots\mathcal{U}^{t-1}\big(X_{j_t}^\dagger\big)\mathcal{U}^{t-1}\big(X_{i_t}\big)\ldots\mathcal{U}\big(X_{i_2}\big)X_{i_1}\Omega\big\rangle \\ &= \big\langle\Omega\big|X_{j_1}^\dagger U^\dagger X_{j_2}^\dagger U^\dagger \ldots X_{j_t}^\dagger X_{i_t} \ldots U X_{i_2} U X_{i_1}\Omega\big\rangle.\end{aligned} \tag{4}$$

Here $\sigma[\mathbf{X}^t]$ is a positively defined, $k^t \times k^t$ complex-valued matrix with its trace equal to unity. By $S_t[\mathbf{X}, U]$ we denote the von Neumann entropy

$$S_t[\mathbf{X}, U] = -\mathrm{Tr}(\sigma[\mathbf{X}^t] \ln \sigma[\mathbf{X}^t]). \tag{5}$$

For any partition of unity $\mathbf{X}$, one can introduce the corresponding dynamical map $\Phi_{\mathbf{X}}$ in the Schrödinger picture (we use the same notation for the map acting on the system or for its trivial extension to its doubled version)

$$\rho \mapsto \Phi_{\mathbf{X}}(\rho) = \sum_{j=1}^k X_j \rho X_j^\dagger. \tag{6}$$

This map transforms an arbitrary density operator ρ into another density operator. Iterating the state $|\Omega\rangle\langle\Omega|$ t-times by the map $\Phi_{U\mathbf{X}}$ we obtain

$$\Omega[\mathbf{X}^t] = [\Phi_{U\mathbf{X}}]^t(|\Omega\rangle\langle\Omega|) \tag{7}$$

where $U\mathbf{X} = \{UX_1, UX_2, \ldots, UX_k\}$.

We have the equality

$$S_t[\mathbf{X}, U] = -\mathrm{Tr}(\Omega[\mathbf{X}^t] \ln \Omega[\mathbf{X}^t]). \tag{8}$$

The formula above gives a new interpretation of $S_t[\mathbf{X}, U]$ as the entropy of the density matrix obtained by repeated measurements performed on the evolving composed system with the initial pure entangled state $|\Omega\rangle$.

Equality (8) follows from the fact that the spectrum of the operator $\sum_{j=1}^{k} |j\rangle\langle j|$ is identical (including degeneracies) to the spectrum of the $k \times k$ matrix $[\langle i|j\rangle]$, except for the irrelevant eigenvalues equal to zero.

According to the original definition of ALF-dynamical entropy one should compute the asymptotic rate of the entropy production $\lim_{t\to\infty} \frac{1}{t} S_t[\mathbf{X}, U]$ and finally take a supremum over all physically admissible partitions of unity. Since for any finite quantum system the entropy $S_t[\mathbf{X}, U]$ is limited

$$S_t[\mathbf{X}, U] \leqslant \min\{t \ln k, d\} \tag{9}$$

the asymptotic rate of the entropy production gives zero, independently of the investigated unitary dynamics U. Hence the ALF-dynamical entropy will always be zero for finite systems. On the other hand, one may analyse the initial rate of the entropy S_t, which at small times (of order of $\ln t$) was shown [19] to be determined by the classical entropy h.

3. Classical limit of quantum maps

Consider a classical dynamical system with a compact (reduced) phase space Γ equipped with the probability measure $\mathrm{d}\gamma$. This measure is assumed to be invariant with respect to the dynamical map $\gamma \mapsto T(\gamma)$. We say that this system $(\Gamma, T, \mathrm{d}\gamma)$ is a classical limit of the sequence of quantum systems if:

(a) there exists a sequence $(\mathbb{C}^{d(n)}, U_n; n = 1, 2, \ldots)$ of $d(n)$-dimensional Hilbert spaces and $d(n) \times d(n)$-unitary matrices,
(b) there exists a *quantization procedure* which with any real function $f(\gamma)$ (usually satisfying some smoothness properties) associates the sequence of self-adjoint operators $F^{(n)}$ acting on $\mathbb{C}^d$, and
(c) for any set of observables $f_1, f_2, \ldots, f_k$ and any sequence of time steps $t_1, t_2, \ldots, t_k$ the correlation functions converge,

$$\lim_{n\to\infty} \frac{1}{d(n)} \mathrm{Tr}\big(U_n^{t_1} F_1^{(n)} U_n^{-t_1} U_n^{t_2} F_2^{(n)} U_n^{-t_2} \cdots U_n^{t_k} F_k^{(n)} U_n^{-t_k}\big) \\ = \int_\Gamma \mathrm{d}\gamma f_1(T^{t_1}(\gamma)) f_2(T^{t_2}(\gamma)) \cdots f_k(T^{t_k}(\gamma)). \tag{10}$$

Both the quantization procedure $f_j \mapsto F_j$ and the choice of U_n are not unique. The maximally mixed states $\rho_\star$ correspond to the uniform normalized measure $\mathrm{d}\gamma$. The example of such a structure has been rigorously studied for the Arnold cat maps in a recent paper [8].

To present an example of a family of quantum maps, corresponding to certain classical system, we are going to recall the construction of the quantum baker map, originally due to Balazs and Voros [20], and modified later in [21–23]. The classical baker map is defined as a transformation of a unit square, the compact phase space Γ with the coordinates q (position) and p (momentum),

$$\Gamma \ni \gamma = (q, p) \to T_B(\gamma) = (2q - [2q], (p + [2q])/2) \in \Gamma \tag{11}$$

where $[2q]$ denotes the integer part of $2q$. This map is hyperbolic and its Kolmogorov–Sinai entropy is equal to $\ln 2$. Such a transformation may be quantized in a finite Hilbert space $\mathbb{C}^d$. In an ordered orthonormal basis named position eigenbasis $\{e_j\}$ we introduce a periodic translation operator

$$Ue_j = e_{j+1} \qquad j = 1, \ldots, d-1 \qquad Ue_d = e_1. \tag{12}$$

Diagonalization of U leads to the conjugated basis—momentum eigenbasis $\{\tilde{e}_k\}$,

$$U\tilde{e}_k = \exp(2\pi \mathrm{i}k/d)\tilde{e}_k. \tag{13}$$

Analogously to (12) the translation operator in the momentum eigenbasis is introduced,

$$V\tilde{e}_k = \tilde{e}_{k+1} \qquad V\tilde{e}_d = \tilde{e}_1 \qquad k = 1, \ldots, d-1. \tag{14}$$

This operator is diagonal in the position eigenbasis,

$$Ve_j = \exp(-2\pi \mathrm{i}j/d)e_j \tag{15}$$

and the transformation between position and momentum basis is given by the discrete Fourier transform $\mathcal{F}_d$,

$$\tilde{e}_k = \sum_j [\mathcal{F}_d]_{kj} e_j = \sum_j \frac{1}{\sqrt{d}} \mathrm{e}^{-2\pi \mathrm{i}kj/d} e_j. \tag{16}$$

Having defined the group of translation operators corresponding to a classical torus it is possible [20–22] to link the unitary operator

$$U_B = (\mathcal{F}_d)^{-1} \cdot \begin{pmatrix} \mathcal{F}_{d/2} & 0 \\ 0 & \mathcal{F}_{d/2} \end{pmatrix} \tag{17}$$

acting on $\mathbb{C}^d$, where d is an even integer (e.g. $d = 2n$), to the classical transformation defined by equation (11). The translation operators also allow one to define a finite-dimensional operator corresponding to any classical observable described by a continuous function f on Γ. Let us define the Fourier expansion of f,

$$f(q, p) = \sum_{j,k} a_{jk} \, \mathrm{e}^{-2\pi \mathrm{i}jq/d} \, \mathrm{e}^{2\pi \mathrm{i}kp/d}. \tag{18}$$

Thus the operator $F^{(n)}$ corresponding to observable f may read as follows:

$$F^{(n)} = \sum_{j,k} a_{jk} V^j U^k. \tag{19}$$

As mentioned above, the quantization procedure is not unique. Another set of operators $F^{(n)}$ may be obtained if we use different ordering of translation operators, since they do not commute, $UV = VU\,\mathrm{e}^{2\pi \mathrm{i}/d}$. It is also possible to generalize the whole quantization procedure by introducing translation operators, which are not exactly periodic, but periodic up to a phase factor (e.g. $U^d = \mathrm{e}^{2\pi \mathrm{i}\chi_p/d}\mathbb{1}$, $V^d = \mathrm{e}^{2\pi \mathrm{i}\chi_q/d}\mathbb{1}$) [21, 23].

Different properties of such a quantum baker map were studied in [20–23], and the correspondence with the classical system (11) was established. Although we cannot provide a formal proof that the quantization (17) satisfies the property (10), we are going to use this model in further numerical investigations. The very same model has been recently applied by Scott and Caves [24], who interpreted the entire system as a set of subsystems and studied the increase of their von Neumann entropy as a measure of entanglement between them.

4. Correspondence principle for dynamical entropy

For any finite-dimensional quantum system inequality (9) holds, so the quantum dynamical entropy $h[U] = 0$. This fact is sometimes interpreted as the lack of correspondence principle for dynamical entropy. In section 3 we defined a family of quantum maps, parametrized by an integer index n, such that in the semiclassical limit $n \mapsto \infty$ the dimension $d(n)$ becomes infinite. Taking a sequence of quantum systems $(\mathbb{C}^d, U_n)$ with the classical limit $(\Gamma, T, \mathrm{d}\gamma)$ in the sense defined above by equation (10), we may start with a functional partition of unity $\mathbf{f} = \{f_1, f_2, \ldots, f_k; \sum_j |f_j(\gamma)|^2 = 1\}$ and construct its quantum counterparts $\mathbf{F}_n = \{F_1^{(n)}, F_2^{(n)}, \ldots, F_k^{(n)}\}$ using a suitable quantization procedure. The entropy $S_t[\mathbf{F}_n, U_n]$ is computed using equation (5),

$$\Omega[\mathbf{F}_n^t] = [\Phi_{U_n\mathbf{F}_n}]^t(|\Psi_n\rangle\langle\Psi_n|) \tag{20}$$

where

$$|\Psi_n\rangle = \frac{1}{\sqrt{d}}\sum_{m=1}^{d}|e_m\rangle\otimes|e'_m\rangle \qquad \{|e_m\rangle\}, \{|e'_m\rangle\} \text{ basis in } \mathbb{C}^d \tag{21}$$

is the purification of the tracial state $\mathbb{1}/d$ of the system given in terms of the maximally entangled vector in $\mathbb{C}^d \otimes \mathbb{C}^d$. Then according to equation (10)

$$S_t[\mathbf{f}, T] = \lim_{n\to\infty} S_t[\mathbf{F}_n, U_n] \tag{22}$$

the classical dynamical entropy of the partition can be recovered by taking first the classical limit $n \to \infty$ and then the long time limit $t \to \infty$. Usually for a given classical system with the KS entropy $h[T]$ there exist many 'optimal partitions' $\mathbf{f}$ with $\ln k \geqslant h[T]$ for which $S_t[\mathbf{f}, T] \approx t \cdot h[T]$ with a given accuracy or even exactly (generating partitions, Markovian partitions [6]). Therefore, we can expect that for large enough n and the optimal choice of the partition $\mathbf{f}$ the entropy $S_t[\mathbf{F}_n, U_n]$ displays linear growth with the rate given by the KS entropy $h[T]$ for t below $t_{\max} = 2\ln d/h[T]$, and then saturates at the maximal value $2\ln d$. For the regular dynamics T with $h[T] = 0$ we expect a slower (logarithmic) increase of $S_t[\mathbf{F}_n, U_n]$ up to the maximal value.

5. Entropy production as a measure of decoherence

For a generic quantum system S interacting with an environment (e.g. measuring apparatus) its initial pure state becomes mixed due to the increasing system–environment entanglement. Assuming that the reduced dynamics is given by a completely positive map (6), to measure the decoherence we may use the von Neumann entropy

$$E[\mathbf{X}, \alpha] = S(\Phi_{\mathbf{X}}(|\alpha\rangle\langle\alpha|)) = S(\sigma^\alpha[\mathbf{X}]) \tag{23}$$

where $|\alpha\rangle \in \mathcal{H}_S$ is an initial pure state of the system and $\sigma^\alpha[\mathbf{X}]$ is $k \times k$ correlation matrix with (ij) element $\langle\alpha|X_j^\dagger X_i|\alpha\rangle$. We extend this construction to the case of a discrete-time finite quantum dynamical system with the unitary evolution U interrupted by a measuring process (or generally interaction with an environment) described by the partition of unity $\mathbf{X}$ or equivalently by

$$E_t[\mathbf{X}, U, \alpha] = S([\Phi_{U\mathbf{X}}]^t(|\alpha\rangle\langle\alpha|)) = S(\sigma^\alpha[\mathbf{X}^t]). \tag{24}$$

The quantity above, bounded by $\ln(\dim\mathcal{H}_S)$, can be strongly dependent on the initial state of the system.

Assume now that the system S is finite, i.e. $\mathcal{H}_S = \mathbb{C}^d$. In order to obtain a more universal measure we can average the entropy over a complete set $\mathcal{E}$ of pure states $\{|\alpha\rangle\}$. The entropy averaged with respect to $\mathcal{E}$ is equal to

$$E_t[\mathbf{X}, U, \mathcal{E}] = \int_{\mathcal{E}} \mathrm{d}\alpha\, E_t[\mathbf{X}, U, \alpha] \leqslant \ln d \tag{25}$$

and its increase (entropy production) characterizes the magnitude of the decoherence process.

Since the entropy E_t is bounded from above, its asymptotic production rate, E_t/t, tends to zero for $t \to \infty$. On the other hand, we will be interested in the initial production rate. Studying a discrete dynamics we cannot define the derivative $\mathrm{d}E_t/\mathrm{d}t$, but we may, for instance, study the entropy produced after each initial time step. Analysing the trivial dynamics, $U = \mathbb{1}$, and the measurement process governed by projection orthogonal operators, $X_j = P_j = (P_j)^2$, the entropy is produced only once, and $E_t = E_1$ for all $t > 0$. Therefore, to characterize in this situation the unitary dynamics, and not the measurement process itself, we are going to use the quantity $\Delta E = E_2 - E_1$. For comparison we define the initial production of the partial ALF entropy, $\Delta S = S_2 - S_1$.

Defining the ALF entropy, which characterizes the unitary evolution U, one uses the supremum over all operational partitions of unity. Let us emphasize that there is no point in performing such a step by studying the initial decoherence rate ΔE. Since the set of transformed operators, $P_j \to P_j' = P_j V$ ($\mathbf{P} = \{P_1, \ldots, P_k\}$) with arbitrary unitary V is also a valid identity resolution, then $E_t[\mathbf{P}, U, \mathcal{E}] = E_t[\mathbf{P}', VU, \mathcal{E}]$ so the supremum over all possible measurements will be independent of the unitary dynamics U studied.

It follows from equations (5) and (7) that the time-dependent entropy $S_t[\mathbf{X}, U]$, which appears in the definition of the ALF entropy and in the semiclassical regime is related to the KS entropy, describes also the magnitude of a certain decoherence process. However, this process involves a maximally entangled state of the composed doubled system while the natural decoherence measure should be defined in terms of the system alone, such as $E_t[\mathbf{X}, U, \alpha]$ or $E_t[\mathbf{X}, U, \mathcal{E}]$.

In order to compare both entropies $E_t[\mathbf{X}, U, \mathcal{E}]$ and $S_t[\mathbf{X}, U]$ we need the following technical result.

Take the dynamical map $\Phi_{\mathbf{Y}}$ defined by the k-elements partition of unity $\mathbf{Y} = \{Y_1, Y_2, \ldots, Y_k\}$ as in (6) and an arbitrary complete set of vectors $\mathcal{E}$. We use the notation $\sigma^\alpha[\mathbf{Y}]$ and $\sigma[\mathbf{Y}]$ for the $k \times k$ correlation (density) matrices with matrix elements $\langle\alpha|Y_j^\dagger Y_i|\alpha\rangle$ and $\frac{1}{d}\operatorname{Tr}(Y_j^\dagger Y_i)$, respectively. We introduce also the tracial norm $\|A\|_1 = \operatorname{Tr}[(AA^\dagger)^{1/2}]$ and the Hilbert–Schmidt norm $\|A\|_2 = [\operatorname{Tr}(AA^\dagger)]^{1/2}$ for a matrix or an operator A and the entropy function $\eta(x) = -x\ln x$.

Theorem 1.

$$A \geqslant S(\sigma[\mathbf{Y}]) - \int_{\mathcal{E}} \mathrm{d}\alpha\, S(\sigma^\alpha[\mathbf{Y}]) \geqslant B \tag{26}$$

where

$$A = \int_{\mathcal{E}} \mathrm{d}\alpha (\|\sigma[\mathbf{Y}] - \sigma^\alpha[\mathbf{Y}]\|_1 \ln k + \eta(\|\sigma[\mathbf{Y}] - \sigma^\alpha[\mathbf{Y}]\|_1)) \tag{27}$$

$$B = \tfrac{1}{2}\max\left\{\int_{\mathcal{E}} \mathrm{d}\alpha (\|\sigma[\mathbf{Y}] - \sigma^\alpha[\mathbf{Y}]\|_1)^2, \int_{\mathcal{E}} \mathrm{d}\alpha (\|\sigma[\mathbf{Y}] - \sigma^\alpha[\mathbf{Y}]\|_2)^2\right\}. \tag{28}$$

Proof. We use the inequalities for the relative entropy [2, 3, 27]

$$S(\rho|\omega) = \operatorname{Tr}(\rho\ln\rho - \rho\ln\omega) \geqslant \tfrac{1}{2}\max\left\{\|\rho - \omega\|_1^2, \|\rho - \omega\|_2^2\right\}. \tag{29}$$

Putting $\rho = \sigma^\alpha[\mathbf{Y}]$, and $\omega = \sigma[\mathbf{Y}] = \int_{\mathcal{E}} \mathrm{d}\alpha\sigma^\alpha[\mathbf{Y}]$ and averaging over $\mathrm{d}\alpha$ we obtain the lower bound (28). The upper bound (27) follows directly from the Fannes inequality [3, 5],

$$|S(\rho) - S(\omega)| \leqslant (\|\rho - \omega\|_1 \ln(\dim \mathcal{H}) + \eta(\|\rho - \omega\|_1) \tag{30}$$

which ends the proof. □

A basic consequence of theorem 1 is the inequality

$$2 \ln d \geqslant S_t[\mathbf{X}, U] \geqslant E_t[\mathbf{X}, U, \mathcal{E}] \leqslant \ln d \tag{31}$$

which provides an upper bound for the mean von Neumann entropy, averaged over a complete set $\mathcal{E}$ of random initial pure states. If $\mathcal{E}$ is taken to be the entire set of pure states, the averaging is performed with respect to the natural, unitarily invariant (Fubini–Study) measure on space $\mathcal{P}_d = \mathbb{C}P^{d-1}$. This strict bound may be considered as one of the key results of the present work.

Furthermore, theorem 1 implies that $S_t[\mathbf{X}, U] = E_t[\mathbf{X}, U, \mathcal{E}]$ if and only if $\sigma^\alpha[\mathbf{X}^t] = \sigma[\mathbf{X}^t]$ for almost all α (except for a set of measure zero). Numerical results show that for small times both quantities are comparable $S_t[\mathbf{X}, U] \simeq E_t[\mathbf{X}, U, \mathcal{E}]$. We are going to provide some arguments in favour of this behaviour in the case $\mathcal{E} = \mathcal{P}_d$.

Consider the fluctuations of the matrix elements of the $k \times k$ matrix $\sigma^\alpha[\mathbf{Y}]$ treated as random variables with respect to the uniform measure over the set of all pure states $\mathcal{P}_d$. Deviation of a matrix element from its average is given by

$$\delta = |\langle \xi | \sigma^\alpha[\mathbf{Y}] - \sigma[\mathbf{Y}] | \xi \rangle| \tag{32}$$

where $|\xi\rangle$ is an arbitrary normalized vector from $\mathbb{C}^k$. The expectation value of the operator $\sigma^\alpha[\mathbf{Y}]$ is equal to

$$\langle \xi | \sigma^\alpha[\mathbf{Y}] | \xi \rangle = \langle \alpha | \sum_{ij} \bar{\xi}_j \xi_i Y_j^\dagger Y_i | \alpha \rangle. \tag{33}$$

The positive operator $B = \sum \bar{\xi}_i \xi_j Y_i^\dagger Y_j < \mathbb{1}$, i.e. for any normalized vector $|\phi\rangle \in \mathbb{C}^d$, $\langle \phi | B \phi \rangle \leqslant 1$. So, it can be written in the form of the sum of projectors into its eigenvectors $|\Psi\rangle$, i.e. $B = \sum_l b_l |\Psi_l\rangle\langle\Psi_l|$ $(0 \leqslant b_l \leqslant 1)$. Let α_l denote coefficients of the random normalized state $|\alpha\rangle$ with respect to the eigenvectors of the operator B, and finally

$$\delta = \left| \sum_l b_l \left(|\alpha_l|^2 - \frac{1}{d} \right) \right|. \tag{34}$$

The numbers $(|\alpha_l|^2 - d^{-1})$ take positive and negative values of the order d^{-1} but sum up to zero. However, when multiplied by another random variable $b_l \in [0, 1]$ they behave like 'steps of the random walk' yielding a sum of the order $\sqrt{d} \times d^{-1}$ and hence

$$\delta \lesssim \frac{1}{\sqrt{d}}. \tag{35}$$

Therefore, the fluctuations of the norm $\|\sigma[\mathbf{Y}] - \sigma^\alpha[\mathbf{Y}]\|_1$ behave like $k/\sqrt{d}$.

In the time-dependent case it means that for $k^t \ll d$ we have $S_t[\mathbf{X}, U] \simeq E_t[\mathbf{X}, U, \mathcal{E}]$. Moreover a random choice of $|\alpha\rangle \in \mathcal{P}_d$ gives typically $S(\sigma^\alpha[\mathbf{X}^t]) \simeq E_t[\mathbf{X}, U, \mathcal{P}_d]$.

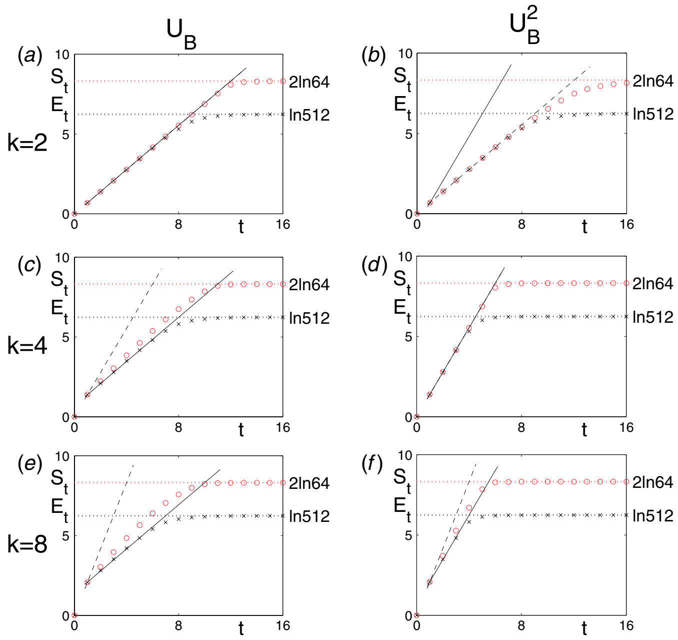

Figure 1. Initial growth of entropies $S_t[\mathbf{P^P}, U]$ (○) and $E_t[\mathbf{P^P}, U, \mathcal{P}_d]$ (×) computed for baker map $U = U_B$ ((*a*), (*c*) and (*e*)) and its square $U = U_B^2$ ((*b*), (*d*) and (*f*)) where partition $\mathbf{P^P}$ corresponds to the division of classical phase space into $k = 2$ ((*a*) and (*b*)), $k = 4$ ((*c*) and (*d*)), $k = 8$ ((*e*) and (*f*)) equal intervals in momentum coordinate.

6. Decoherence in a periodically measured baker map

To illustrate the results presented in the previous section on a concrete example we analyse the quantum baker map (17) subjected to a periodic sequence of measurements performed in the momentum basis. The entire, non-unitary dynamics of the system is described by the superoperator

$$\rho' = \sum_{j=1}^{k} P_j^P U \rho U^\dagger P_j^P. \tag{36}$$

The set $\mathbf{P^P}$ of k projection operators fulfils the identity resolution, $\sum_j P_j^P = \mathbb{1}$, since the measurement process corresponds to the partition of phase space into k equal intervals in momentum,

$$\mathbf{P} = \left\{ P_j^P : P_j^P = \sum_{i=(j-1)d/k+1}^{jd/k} |\tilde{e}_i\rangle\langle\tilde{e}_i| \right\}, \tag{37}$$

where $\tilde{e}_i$ are the momentum eigenstates defined by (13) in $\mathbb{C}^d$, and the size d of the Hilbert space is an integer multiple of k.

Iterating numerically quantum map (36) we compute how both entropies $S_t[\mathbf{X}, U]$ (7) and $E_t[\mathbf{X}, U, \mathcal{P}_d]$ (25) vary in time. Figure 1 presents the initial growth of the entropy $S_t[\mathbf{P^P}, U]$ and $E_t[\mathbf{P^P}, U, \mathcal{P}_d]$. As the evolution operator U we took the quantum baker map U_B defined by (17), panels (*a*), (*c*) and (*e*), or its square, U_B^2, panels (*b*), (*d*) and (*f*). The partition $\mathbf{P^P}$ is composed of $k = 2, 4, 8$ projection operators. The entropy E_t is averaged over a sample of

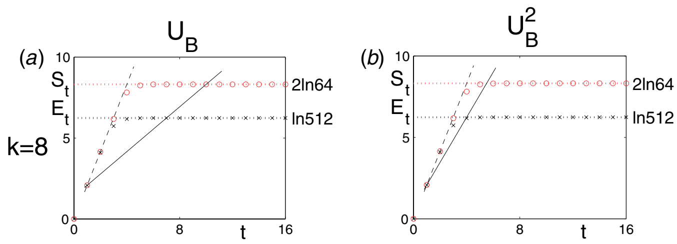

Figure 2. As in figure 1((*e*) and (*f*)) with $k = 8$ for the measurement in a random basis $\mathbf{P^R}$.

32 random initial pure states, chosen according to the Fubini–Study measure. To guide the eye we plotted solid lines corresponding to the entropy growth with the rate of the classical KS entropy, equal to $\ln 2$ in cases (*a*), (*c*) and (*e*) and $2\ln 2$ in cases (*b*), (*d*) and (*f*). The slope of the dashed lines is equal to the maximal allowed growth of entropies, equal to $\ln k$. Aiming for the semiclassical regime, we have taken the maximal dimensionality of the Hilbert space which was allowed by the computer resources at our disposal. In order to compute the entropy S_t one has to diagonalize matrices of size d^2, so we could work with matrices of size $d = 64 = 2^6$. To obtain the entropy E_t one needs to study the time evolution of density operators acting on $\mathcal{H}_d$, so we succeed in working with systems of the size $d = 512 = 2^9$. The entropy E_t obtained for $d = 64$ is smaller than S_t according to the analytical bound (31). As discussed in [26] the size d of the Hilbert space determines only the saturation level $(E_t(t \to \infty))$, but does not influence the initial entropy rate. If the measurement scheme is tuned to the classical dynamics, i.e. cases (*a*) and (*d*), the rate of the initial growth of both entropies coincides with the classical dynamical entropy $h[T]$ of the map which is equal to $\ln 2$ for the baker map, and $2\ln 2$ for its square.

We expect that the quantization property (10) holds for the classical (11) and quantum (17) baker maps, although this statement still awaits a formal proof. On the other hand, our numerical results, consistent with (22), may be treated as an argument supporting such a conjecture.

If the resolution of the measurement is not sufficient—see case (*b*), the classical chaos cannot fully manifest itself and ΔE and ΔS are smaller than $h[T]$, and equal to $\ln k$. In the opposite case, see panels (*c*), (*e*) and (*f*), a finer resolution of the measurement ($\ln k > h[T]$) allows for the decoherence with a rate faster than can be expected from the classical dynamical entropy. Hence such a measurement can be responsible for entropy production faster than may be predicted based on the degree of classical chaos. As visible in panel (*e*) this effect is larger for the entropy S_t.

To demonstrate other features of the measurement process we investigated the time dependence of both entropies $S_t[\mathbf{P^R}, U]$ and $E_t[\mathbf{P^R}, U, \mathcal{P}_d]$ for different choices of the partitions $\mathbf{P^R}$. Figure 2 shows the initial growth of both entropies calculated with the same evolution operators as in figure 1. However, the partition $\mathbf{P}^R$ was obtained by rotating the projective partition $\mathbf{P^P}$ by a random unitary matrix V, namely $P_j^R = VP_j^PV^\dagger$, for all $j = 1, \ldots, k$. The label $\mathbf{R}$ decorating the symbol $\mathbf{P^R}$ of the partition emphasizes the fact that the measurement is performed in a random basis. Such a measurement will give k different results with equal probabilities $\mathrm{Tr}\big(P_j^R\big)\big/d = 1/k$. In figure 2 both entropies initially increase with a nearly maximal slope which is equal to $\ln k = \ln 8$ (dashed line). Unitary evolution operator U does not influence the behaviour of either entropy, and the data presented in plots (*a*) and (*b*) hardly differ.

The fact that the rotated partitions lead to an almost maximal allowed growth of both entropies may be explained by the following argument. Both $S_t[\mathbf{X}, U]$ and $E_t[\mathbf{X}, U, \mathcal{P}_d]$ may be expressed as the von Neumann entropy of a mixed state obtained by the operator $[\Phi_{U\mathbf{X}}]^t$ applied to a pure state (see equations (7) and (24)). Let us denote the rotated partition by $\mathbf{Y} = V\mathbf{X}V^{\dagger}$, where V is an arbitrary unitary matrix. From the definition of the operator $\Phi_{\mathbf{X}}$ (6) we have

$$[\Phi_{U\mathbf{Y}}]^t(\rho) = [\Phi_{UV\mathbf{X}V^{\dagger}}]^t(\rho) = V[\Phi_{V^{\dagger}UV\mathbf{X}}]^t(V^{\dagger}\rho V)V^{\dagger} = V[\Phi_{U'\mathbf{X}}]^t(\rho')V^{\dagger} \tag{38}$$

where the initial state ρ becomes rotated, $\rho' = V^{\dagger}\rho V$, and the evolution U is replaced by $U' = V^{\dagger}UV$. The von Neumann entropy depends only on the spectrum of density matrix, so $S([\Phi_{U\mathbf{Y}}]^t(\rho)) = S([\Phi_{U'\mathbf{X}}]^t(\rho')$. In the case of the entropy S_t the evolution operator is a tensor product $U = U_S \otimes U_A$, so are the partition $\mathbf{X} = \mathbf{X}_S \otimes \mathbb{1}$, and the matrix $V = V_S \otimes \mathbb{1}$. Since we choose randomly only V_S, the state of the second system is not important. When ρ is a maximally entangled state so is ρ', since these states are invariant under local operations. Hence we obtain $S_t[\mathbf{Y}, U] = S_t[\mathbf{X}, U']$. The case of the entropy E_t is simpler, since here the averaging over all initial pure states $|\alpha\rangle$ automatically cancels the difference between ρ and ρ'. Therefore, the equality $E_t[\mathbf{Y}, U, \mathcal{P}_d] = E_t[\mathbf{X}, U', \mathcal{P}_d]$ holds. As we can see in both cases the randomly rotated partition $\mathbf{Y}$ is equivalent to the model with an original partition $\mathbf{X}$ and the evolution operator rotated into $U' = V^{\dagger}UV$, where V is a random unitary matrix, generated according to the Haar measure on $U(N)$. Although the spectra of U and U' are equal, the eigenvectors of U' are random, and the operator $\Phi_{U'\mathbf{X}}$ generates the maximal growth of the von Neumann entropy $\Delta E \approx \min\{\ln d, \ln k\}$. Here $\ln d$ represents the maximal von Neumann entropy of the mixed state, while the measurement $\mathbf{X}$ transforms any pure state into a mixture with entropy bounded by $\ln k$.

This argument shows that for finite systems taking the supremum over all possible partitions of unity leads to the maximal allowed growth of both entropies irrespective of the analysed unitary dynamics U. Note that the upper bound $\ln d$ is only slightly larger than that of the average quantum dynamical entropy of a random unitary matrix U distributed according to the Haar measure on $U(d)$ [7].

On the other hand, one may pose a question, how to restrict the set of possible partitions, such that the rate of growth of von Neumann entropy could correspond to the dynamical entropy of the classical system. To analyse this problem compare the properties of the momentum partition $\mathbf{P}^{\mathbf{P}}$, and the random partition $\mathbf{P}^{\mathbf{R}}$ in the phase space. To represent the partition member we make use of the Husimi-like representation,

$$x_j(q, p) \equiv \langle q, p|X_j^{\dagger}X_j|q, p\rangle. \tag{39}$$

Here $|q, p\rangle$ denotes the Gaussian states localized on the torus, the same as in [25, 26]. In figure 3 we show the phase-space representation of two partitions of unity $\mathbf{P}^{\mathbf{P}} = \{P_1, P_2\}$, and $\mathbf{P}^{\mathbf{R}} = \{VP_1V^{\dagger}, VP_2V^{\dagger}\}$, each consisting of $k = 2$ operators. $\mathbf{P}^{\mathbf{P}}$ denotes the partition into equal intervals in momentum coordinates and $\mathbf{P}^{\mathbf{R}}$ is a partition into two subspaces of equal size determined by a random unitary matrix V. As may be seen in the picture, the operators P_1^P and P_2^P are by construction localized in the lower (upper) region of the phase space, while P_1^R and P_2^R are totally delocalized.

These results suggest that in order to predict the decoherence rate ΔE (and ΔS) one should consider only these partitions $\mathbf{P}^{\mathbf{C}}$, which have a well-defined classical limit. This is the case if each of the operators X_j is localized on a subset $\epsilon_j \subset \Gamma$ of phase space, so in the classical limit $X_j(q, p) \to \chi_{\epsilon_j}(q, p)$, where χ_{ϵ_j} denotes the characteristic function of ϵ_j.

The above discussion has some important consequences for the decoherence processes in quantum systems having chaotic classical limits with the KS entropy $h[T] > 0$. Namely, if only the interaction with the environment can be described by a partition of unity having a

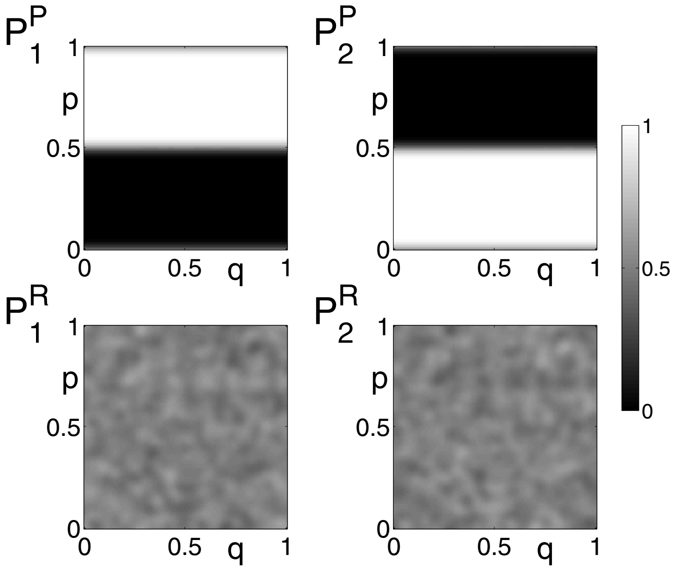

Figure 3. Phase-space representations of partitions $\mathbf{P^P} = \{P_1^P, P_2^P\}$ and $\mathbf{P^R} = \{P_1^R, P_2^R\}$ consisting of $k = 2$ operators. The real function $x_j(q, p) = \langle q, p|X_j^\dagger X_j|q, p\rangle$ is plotted in dark scale, where X_j denotes one of $P_1^P, P_2^P, P_1^R, P_2^R$. The partition $\mathbf{P^P}$ corresponds to partition on the upper and lower halves in momentum coordinates while $\mathbf{P^R}$ is partition corresponding to the projections in a a randomly selected basis. Note that the coherent-states representations of P_1^R and P_2^R are delocalized in the phase space.

well-defined classical limit (and $\ln k \geqslant h[T]$) we expect that the decoherence effects give the entropy production per single time step of the order $h[T]$ for the generic initial conditions. The linear entropy increase has to break down for times t of the order of $\ln d/h[T]$ [9].

In contrast, if the partition of unity is chosen randomly and has no well-defined classical limit we do not expect any restrictions on the entropy production rate, except the general upper bound $\Delta S \leqslant \ln k$, related to the number k of the Kraus operators.

7. Occurrence of free-independent variables

The semiclassical arguments of the previous sections do not explain the striking phenomenon observed in the numerical computations of $S_t[\mathbf{X}, U]$ and $E_t[\mathbf{X}, U, \mathcal{P}_d]$ as presented in figures 1(*a*) and (*d*) and figure 2. Namely for the two situations:

(a) the quantum system with chaotic classical limit and the semiclassical, projection valued choice of the partition $\mathbf{P} = \{P_1, P_2, \ldots, P_k\}$, $\mathrm{Tr} P_j = d/k$ satisfying $\ln k \leqslant h[T]$,

(b) nontrivial U and the random choice of the partition $\mathbf{P}$.

$S_t[\mathbf{X}, U]$ grows almost exactly linearly like $t \ln k$ and then rapidly saturates at the maximal value $2 \ln d$. The entropy $E_t[\mathbf{X}, U, \alpha]$ with a random choice of $|\alpha\rangle$ follows the same plot up to its maximal value $\ln d$. This means that in both cases (a) and (b) the correlation density matrices possess a very special structure corresponding to the maximal admissible entropy,

$$\sigma[\mathbf{P}^t]_{i_1,\ldots,i_t;j_1,\ldots,j_t} = d^{-1}\,\mathrm{Tr}\big(P_{j_1} U^\dagger P_{j_2} U^\dagger \ldots P_{j_t} P_{i_t} \ldots U P_{i_2} U P_{i_1}\big) \simeq \frac{1}{k^t}\delta_{i_1 j_1} \cdots \delta_{i_t j_t} \tag{40}$$

and similarly for a typical vector $|\alpha\rangle$,

$$\sigma^{\alpha}[\mathbf{P}^{t}]_{i_1,\ldots,i_t;j_1,\ldots,j_t} = \langle\alpha|\big(P_{j_1}U^{\dagger}P_{j_2}U^{\dagger}\ldots P_{j_t}P_{i_t}\ldots UP_{i_2}UP_{i_1}\big)|\alpha\rangle \simeq \frac{1}{k^t}\delta_{i_1j_1}\cdots\delta_{i_tj_t} \tag{41}$$

under the condition of $k^t \ll d$.

The simple form of the correlation functions (40) and (41) suggests the existence of a certain statistical law satisfied by the noncommutative variables $\{P_j, U\}$ with respect to the tracial state or a typical pure state $|\alpha\rangle$. This law should be strictly obeyed in the limit $d \to \infty$ but even for relatively low dimensions reproduces the data with very good accuracy. Such situations are common in nature. Gaussian and Poisson probability distributions are very successful in describing experimental data while their rigorous derivations involve limit theorems with strong statistical independence assumptions.

We advance the following statistical hypothesis:

For both cases (a) and (b) and large Hilbert space dimensions d the operators $\{A, U\}$ *behave asymptotically like free-independent random variables with respect to the tracial state or a typical pure state.*

Here A is an arbitrary observable with the spectral measure $(P_1, P_2, \ldots P_k)$.

We have to explain now the notion of *free independence*. In the classical probability theory (complex) random variables form a commutative $*$-algebra and the probability measure defines a positive normalized functional $f \mapsto \langle f\rangle$, the *average value*. The random variables $f_1, f_2, \ldots, f_n$ are called statistically independent if

$$\langle f_1 f_2 \ldots f_n\rangle = \langle f_1\rangle\langle f_2\rangle\cdots\langle f_n\rangle. \tag{42}$$

In *noncommutative probability* the basic object is a unital generally noncommutative $*$-algebra $\mathcal{A}$ with a state (positive normalized functional) ϕ. Due to the noncommutativity the average $\phi(x_1x_2,\ldots,x_m); x_j \in \mathcal{A}$ depends on the order of random variables. Hence, the direct extension of the definition (42) is not very interesting and essentially corresponds to product states on tensor product algebras. Instead, in noncommutative probability we have different notions of independence which take into account possible algebraic relations between random variables (e.g. CCR, CAR, etc [28]). In the last decade the so-called free families of random variables (or free independence) introduced by Voiculescu [29, 30] attracted the attention of physicists mainly due to the relations with random matrices theory.

Denote by $w(x)$ an arbitrary polynomial in noncommutative variables $X, Y \in \mathcal{A}$. The collection of noncommutative random variables $x_1, x_2, \ldots, x_k$ is called *free independent* if

$$\phi(w_1(x_{p(1)})w_2(x_{p(2)})\cdots w_m(x_{p(m)}) = 0 \tag{43}$$

whenever $\phi(w_j(x_{p(j)}) = 0$ and $p(j) \neq p(j+1)$ for all $j = 1, 2, \ldots, m$, and $p(j) \in \{1, 2, \ldots, k\}$.

It has been proved that the Wigner semicircular probability distribution is a consequence of the central limit theorem for free-independent random variables similar to the origin of the Gaussian probability distribution in the context of statistically independent commutative variables. Moreover, the free-independent variables naturally arise as limits of large random matrices [29, 30]. The consequences of free independence are illustrated by the following example.

Example. Take a family of orthogonal projections $P_1, P_2, \ldots, P_k$ and a unitary U, all from the algebra $\mathcal{A}$ with the state ϕ. Assume that for any $A = \sum a_j Pj$ the pair of random variables $\{A, U\}$ is free independent and moreover

$$\phi(P_j) = \frac{1}{k} \qquad j = 1, 2, \ldots, k \qquad \phi(U^n) = \phi\big(U^{\dagger n}\big) = 0 \qquad n = 1, 2, \ldots. \tag{44}$$

Then

$$\phi\big(P_{j_1}U^\dagger P_{j_2}U^\dagger\cdots P_{j_n}U^\dagger U P_{i_m}\cdots U P_{i_2}U P_{i_1}\big) = \delta_{nm}\frac{1}{k^n}\delta_{i_1 j_1}\cdots\delta_{i_n j_n}. \tag{45}$$

Proof. Put $Q_j = P_j - 1/k$, then

$$Q_i Q_j = \delta_{ij}Q_i - k^{-1}(Q_i + Q_j) + \delta_{ij}k^{-1} - k^{-2} \qquad \phi(Q_j) = 0. \tag{46}$$

Hence the LHS of equation (45) is a linear combination of the terms of the form

$$\phi\big(U^{\dagger n_1}Q_{k_1}U^{\dagger n_2}\cdots U^{m_1}Q_{k_1}U^{m_2}\cdots\big) \tag{47}$$

which due to the freeness and equations (44), (45), are all equal to zero except the terms which do not contain nontrivial powers of U and $U^\dagger$. This can happen, however, for $n = m$ only. In this case we can easily prove (45) by induction.

The relation (45) corresponds to the phenomenon observed in the numerical computations of $S_t[\mathbf{X}, U]$ and $E_t[\mathbf{X}, U, \mathcal{P}_d]$, and described by (40) and (41). This justifies our hypothesis formulated above. In case (b) this hypothesis is not surprising due to the random choice of the partition and the generic relations between free-random variables and random matrices. Similarly, we would expect the same phenomenon for the fixed partition and the random choice of the unitary matrix. On the other hand, for case (a) it seems to be a new characterization of chaotic quantum systems in terms of 'quantum-probabilistic' relations between the dynamics and the measurement (coarse-graining) procedure. □

8. Concluding remarks

We have analysed the decoherence in an open quantum system, the classical analogue of which is chaotic. The decoherence may be quantified by the rate of increase of the von Neumann entropy of the initially pure states. We have found an explicit upper bound for the rate of the von Neumann entropy given by the partial entropy used to define the ALF-dynamical entropy. The latter quantity is related to the Kolmogorov–Sinai entropy of the corresponding classical system. Hence our findings allow us to established a further relation between the speed of decoherence in open quantum systems and the degree of classical chaos.

Such a relation, demonstrated in several earlier works [9, 12, 15], holds if some additional assumptions concerning the coupling of the system investigated with an environment (the measurement process) are made. In particular, we proposed to study the scheme of *random measurements*, in which the usual Kraus operators X_i, which represent projectors on some well-defined fragments of classical phase space, are replaced by operators obtained by random matrices, $X_i' = V X_i V^\dagger$. In such a case the rate of von Neumann entropy becomes maximal (with probability one, with respect to the choice of random matrix V). Thus the decoherence depends only on the kind of the measurement performed (the number of Kraus operators or the dimensionality of the system), and is independent of the quantum unitary dynamics U, and of the degree of chaos (Lyapunov exponent, KS dynamical entropy) of the corresponding classical system.

From a practical point of view it is therefore natural to ask, for which class of measurement procedures the relation between classical chaos and the degree of quantum decoherence is still valid. Although we are not in a position to formulate mathematically rigorous sufficient conditions, which would imply such a relation, our numerical evidence allows us to advance the following conjecture. The interaction with an environment induces decoherence related to the degree of classical chaos, if the measurement (Kraus) operators have a well-defined classical limit. In other words, the coherent-states representation of each of the Kraus operators needs to be well localized in certain fragments of the classical phase space.

More formally, the maximal entropy growth, and its independence of the unitary dynamics, may be analytically derived from an assumption that unitary operator U and an arbitrary combination of the projector operators P_i are free independent. Obviously this statement is of a statistical nature, and does not allow one to draw rigorous conclusions for a concrete set of projection and evolution operators. The free-random variables approach concerns entire ensembles of operators and enables us to formulate exact statements concerning the decoherence rate in the limit of large Hilbert space dimension. Nevertheless, for practical purposes one may choose a set of arbitrary test states ϕ, and check whether property (45) is approximately fulfilled for the analysed unitary map U and measurement $\mathbf{P}$. It is worth emphasizing that the free-independence condition can be formulated as a condition for a pair of genuinely quantum objects, an observable and an unitary quantum map, without any reference to the classical notions. Therefore the idea of quantum chaos may be extended to systems without obvious classical counterparts or to dynamics which do not satisfy standard assumptions concerning the spectral fluctuations.

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