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Invariant sets for discontinuous parabolic area-preserving torus maps

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Abstract. We analyse a class of piecewise linear parabolic maps on the torus, namely those obtained by considering a linear map with double eigenvalue one and taking modulo one in each component. We show that within this two-parameter family of maps, the set of non-invertible maps is open and dense. For cases where the entries in the matrix are rational we show that the maximal invariant set has positive Lebesgue measure and we give bounds on the measure. For several examples we find expressions for the measure of the invariant set but we leave open the question as to whether there are parameters for which this measure is zero.

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1. Introduction

We consider a class of maps of the torus $X = [0, 1]^2$ of the form

$$\begin{aligned} x' &= ax + by \pmod 1 \\ y' &= cx + \mathrm{d}y \pmod 1 \end{aligned} \tag{1}$$

where $(x, y) \in [0, 1]^2$. This can be thought of as a map $f = g \circ M$ where

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

(we also write $M = (a, b; c, d)$ for convenience) and $g(x) = x - \lfloor x \rfloor$ is a map that takes modulo 1 in each component. Although such maps are linear except at the discontinuity induced by the map g, their dynamical behaviour can be quite complicated. Depending on the eigenvalues $\lambda_{1,2}$ of M we refer to the map as elliptic ($\lambda_1 = \overline{\lambda_2} \neq \lambda_1$), hyperbolic ($\lambda_1 > \lambda_2$) or parabolic ($\lambda_1 = \lambda_2$). In this paper we focus on the area-preserving parabolic case with determinant $ad - bc = 1$ and trace $a + d = 2$. Such maps arise naturally on examining linear maps with a periodic overflow. In particular, suppose that one would like to iterate a matrix using a digital representation with a very small discretization error but a finite range which we set to be (0, 1). If the calculation overflows such that the fractional part remains, we will obtain the map (1) (compare, for example, with [4]).

In this case f is a piecewise continuous map of $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ to itself that is area preserving and such that the linearization at almost every point in $\mathbb{T}^2$ is M. This matrix has two eigenvalues equal to one. The map is continuous everywhere on the torus except on a one-dimensional discontinuity D in $\mathbb{T}^2$. The behaviour of f on D does not affect a full measure set of X.

Parabolic area-preserving maps are not typical in the set of almost-everywhere linear maps on the torus [5]. However, their dynamical properties are of particular interest, since such maps can be considered as an interpolating case between the hyperbolic maps ($|t| > 2$) and the elliptic maps ($|t| < 2$) in the area-preserving case. They are in some sense generalizations of interval exchange maps [11, 12, 13, 16], piecewise rotations [4, 8] and interval translation maps [6, 15] to two dimensions. In fact, in section 3 and the subsequent sections, our results use the fact that for rational coefficients of M the map (1) may be decomposed into a one-parameter family of one-dimensional interval translation maps.

Hyperbolic area-preserving maps are characterized by positive Lyapunov exponents, and are often studied as model chaotic area-preserving dynamical systems (see, e.g., [5, 14] and Baker's transformation). Elliptic area-preserving maps correspond (in an appropriate eigenbasis) to rigid rotation, where the presence of a discontinuity caused by g will lead to very complicated dynamics [2–4, 9]. The elliptic–hyperbolic transition for the linear maps on the torus was studied by Amadasi and Casartelli [1], while certain properties of linear parabolic maps were analysed in [17]. In particular, a generic parabolic map displays some sort of sensitive dependence on the initial conditions. However, this property is not related to the Lyapunov exponents (which are always zero for parabolic maps), but is due to the discontinuity of the map.

1.1. Summary of main results

We parametrize the set of parabolic area-preserving maps by A and α, namely

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} (1+A)x + \dfrac{A}{\alpha}y \pmod 1 \\ -\alpha A x + (1-A)y \pmod 1 \end{pmatrix}. \tag{2}$$

Section 2 characterizes (in theorem 1) the open dense set of (A, α) such that the mapping (2) is not invertible and discusses some properties of the invertible maps. Section 3 examines properties of the *maximal invariant set* [17]

$$X^+ = \bigcap_{n=0}^{\infty} f^n(X)$$

which by the previous result is strictly smaller than X for most (A, α). Note that any $A \subset X$ with $f(A) = A$ will have $A \subset X^+$ and so in this sense the set is maximal.

We say a map (2) is *semirational* if α is rational. It is *rational* if both A and α are rational. If f is not semirational we say it is *irrational*. Note that the map is rational if and only if a, b, c and d are rational.

We investigate the two-dimensional Lebesgue measure $\ell(\cdot)$ of this set, showing in theorem 3 that for rational parabolic maps we have $\ell(X^+) > 0$. We give explicit lower and upper bounds for $\ell(X^+)$ depending on A and α.

We discuss two particular examples of rational maps where we can compute $\ell(X^+)$, namely with $A = \frac{1}{2}$ and $\alpha = 1$ or $\frac{1}{2}$. In the former case we show that $\ell(X^+) = \frac{1}{2}$ and in the latter case we show that $\ell(X^+) = \frac{1}{2} - \sqrt{3}\pi/72 - \frac{1}{8}\ln 3 \sim 0.287\,10$. In some special cases, e.g. $-1 \leqslant \alpha < 0$, $|A| \leqslant 1$ and $A = 1$, $0 < \alpha < 1$, we can compute X^+, and thus we can obtain exact values of $\ell(X^+)$. Other than these cases we have not been able to compute exact values of

$\ell(X^+)$ other than by numerical approximation, even for rational A and α. We suspect that for many $\alpha > 0$ and $A > 0$ (both irrational) then $\ell(X^+) = 0$, whereas $\ell(X^+) > 0$ for α rational. Section 4 briefly discusses examples of nonlinear parabolic maps, and we conclude with some remarks on open problems in section 5.

2. Invertibility of linear maps on the torus

Consider a matrix M corresponding to a parabolic map on the torus (1). Apart from the special *horocyclic* [7] cases

$$\begin{pmatrix} 1 & B \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} \tag{3}$$

any such matrix can be written in the form

$$M = M_{A,\alpha} = \begin{pmatrix} 1+A & A/\alpha \\ -\alpha A & 1-A \end{pmatrix} \tag{4}$$

where A and α are real parameters. Observe that the cases $\alpha, A \to 0$ (with A/α held constant) and $\alpha \to \infty$, $A \to 0$ (with αA held constant) correspond to the cases (3). Observe that

$$M_{A,\alpha}^{-1} = M_{-A,\alpha}$$

and if $S(x, y) = (y, x)$ is transposition then $S^2 = I$ and

$$SM_{A,\alpha} = M_{-A,1/\alpha} S.$$

Moreover, in the absence of the rounding g we have

$$M_{A,\alpha}^{p} = M_{pA,\alpha}$$

for any $p \in \mathbb{Z}$.

2.1. *Non-invertibility of generic parabolic maps*

The next lemma gives necessary and sufficient conditions that (1) is invertible (ignoring points that land on the discontinuity). We say the map is invertible if there is a full measure subset on which it is invertible. Since the mapping is a composition of an invertible linear map and a discontinuous map that maps open sets to open sets, both of which preserve the Lebesgue measure, a map is non-invertible if and only if there is an open set of (x, y) that has two or more pre-images. Let a, b, c, d be defined as previously.

Lemma 1. *Any map (1) with* $ad - bc = 1$ *will be non-invertible on an open set if and only if there are* $(K, L) \in \mathbb{Z}^2 \setminus (0, 0)$ *integers such that*

$$|Kc - La| < 1 \qquad \textit{and} \qquad |Kd - Lb| < 1.$$

Proof. Suppose that we have (x, y) and $(u, v) \in [0, 1)^2$ such that $(x, y) \neq (u, v)$ but $f(x, y) = f(u, v)$. Then there is $(K, L) \in \mathbb{Z}^2 \setminus (0, 0)$ such that $ax + by = au + bv + K$ and $cx + dy = cu + dv + L$. These hold if and only if

$$a(x-u) + b(y-v) = K \qquad \text{and} \qquad c(x-u) + d(y-v) = L.$$

Using the fact that a, b, c, d are non-zero and $ad - bc = 1$ this implies that this holds if and only if

$$x - u = Kc - La \qquad \text{and} \qquad y - v = Kd - Lb.$$

Thus the mapping is many-to-one if and only if there are K and L such that

$$|Kc - La| < 1 \qquad \text{and} \qquad |Kd - Lb| < 1$$

and the result follows. □

Remark 1. *It is quite possible that there are simultaneously many solutions to the inequalities of lemma 1; for example, near $A = -1$ and with α large, one can find arbitrarily large numbers of integers (K, L) satisfying both inequalities.*

Using the previous lemma we obtain the main result of this section.

Theorem 1. *The only invertible parabolic maps f have M equal to one of*

$$\begin{pmatrix} k & -(k-1)^2/l \\ l & 2-k \end{pmatrix} \qquad \begin{pmatrix} 2+l & -k \\ (l+1)^2/k & -l \end{pmatrix}$$

$$\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \qquad \text{or} \qquad \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}$$

where (k, l) are integers and b is real.

Proof. The square $V_\epsilon = (-1-\epsilon, 1+\epsilon)^2$ is a convex region symmetrical about the origin with area $4(1+\epsilon)^2$. A variant of Minkowski's theorem [10, theorem 447] says that for any $\epsilon > 0$ there will be a point on the lattice $\Lambda = \{(Kc - La, Kd - Lb) : (K, L) \in \mathbb{Z}^2\}$ other than zero inside V_ϵ, if the lattice has determinant $\Delta = ad - bc \geqslant 1$. Taking the limit $\epsilon \to 0$ and using compactness we see that there must be a point $(\kappa, \lambda) \in \Lambda$ other than zero in $\overline{V_0}$.

If there is a lattice point (κ, λ) in the interior of V_0, then the proof is complete. If not, assume without loss of generality that $(\kappa, \lambda) = (1, y)$ with $|y| \leqslant 1$. Thus exactly one point in the line segment from (but not including) $(-1, 1-y)$ to $(0, 1)$ is in the lattice (one can generate the same lattice by adding multiples of $(1, y)$ to any lattice point generating the lattice with $(1, y)$). Therefore, there will be a point in the lattice in the interior of V_0 unless this other point is $(0, 1)$. Therefore (considering the other case by interchanging x and y), there will be integers K and L such that either

$$(Kc - La = 1 \quad \text{and} \quad Kd - Lb = 0) \qquad \text{or} \qquad (Kc - La = 0 \quad \text{and} \quad Kd - Lb = 1).$$

Thus, in the invertible case the matrix M is determined by which of these cases occurs. One can easily demonstrate that either of these cases can occur. If $k = 1$ or $l = -1$ then the other (horocyclic) cases occur. □

It follows from this result that most parabolic maps are non-invertible in the following sense.

Corollary 1. *The set of non-invertible maps is open and dense within the set of parabolic maps.*

Because the (mod 1) map g commutes with integer matrix and $g^2 = g$, all conjugations of parabolic maps with integer coefficients by automorphisms of the torus $\mathbb{T}^2$ remain integer parabolic maps. For an integer parabolic map with rational $\alpha = r/s$, where $(r, s) = 1$, since there exist $m, n \in \mathbb{Z}$ such that $mr + ns = 1$, there exists an automorphism whose $GL(2, \mathbb{Z})$ matrix has the bottom row (r, s). Conjugation by this automorphism gives a parabolic map whose matrix has the bottom row $(0, *)$. Since conjugation preserves integrality, trace and determinant, the resulting matrix must be of the form $(1, B; 0, 1)$ for some B. Therefore, we conclude that all parabolic maps with integer coefficients can be reduced to one of the horocyclic cases.

3. Maximal invariant sets

By Poincaré recurrence, the maximal invariant set $X^+ = X$ (up to a set of zero measure) if and only if f is invertible at almost every point. In fact, the maximal invariant set X^+ is an upper semicontinuous function of the system parameters A, α in the Hausdorff metric. However, the measure (and dimension) of X^+ can (and does) change discontinuously with parameters; see proposition 1.

We now consider the structure of X^+ for the semirational case (i.e. α rational). Consider the family of lines L_B defined by

$$y = B - \alpha x$$

with B a fixed real number, projected onto the torus by taking modulo 1 in x and y. This can be thought of as the set

$$L_B = \{(x, y) : y = B + K - \alpha(x + L) \text{ with } (K, L) \in \mathbb{Z}^2\} \cap X.$$

Note that if

$$y = B + K - \alpha(x + L)$$

with L and K integers such that x, y in [0, 1] then $y' = -\alpha Ax + (1 - A)(B + K - \alpha(x + L) + N$ and $x' = (1 + A)x + A/\alpha(B + K - \alpha(x + L)) + M$ with M, N integers. Rearranging this we have

$$y' = B + (K + M + N) - \alpha(x + L)$$

and so the family of lines L_B is invariant under the map for any given B. We define the maximal invariant set within L_B as

$$X_B^+ = X^+ \cap L_B.$$

3.1. Semirational and rational parabolic maps

For semirational maps ($\alpha = r/s > 0$) and for any given value of B, the set L_B consists of $r + s$ (or exceptionally $r + s - 1$ if it contains the origin) intervals that are parallel to the eigenvector $v = (1, -\alpha)$ of the matrix M. We can parametrize any particular L_B by $\theta \in [0, s)$. Now we will consider $\alpha > 0$. More precisely we reparametrize X by

$$\begin{aligned} x &= \theta - [\theta] \\ y &= B - \alpha\theta - [B - \alpha\theta]. \end{aligned} \tag{5}$$

For $\theta \in [0, s)$ and $B \in [(s - 1)/s, 1)$ the mapping $(x, y) \leftrightarrow (\theta, B)$ is one-to-one and has unit Jacobian everywhere.

Since each L_B is invariant the maximal invariant set must non-trivially intersect L_B and so no single trajectory is dense in X^+. In this case we need to approximate X^+ using a distribution of trajectories on a dense set of lines.

The relation between θ and x is $x = \theta - [\theta]$. The variable y is related to θ and B by the relation $y = B - \alpha\theta - [B - \alpha\theta]$. So the point (θ, B) maps to

$$(\theta', B') = (\theta + AB/\alpha - A[\theta] - A/\alpha[B - \alpha\theta], B)$$

since

$$\begin{pmatrix}\theta'\\ v\end{pmatrix} = \begin{pmatrix}1+A & A/\alpha\\ -A\alpha & 1-A\end{pmatrix}\begin{pmatrix}\theta-[\theta]\\ B-\alpha\theta-[B-\alpha\theta]\end{pmatrix} + \begin{pmatrix}[\theta]\\ [B-\alpha\theta]\end{pmatrix}$$
$$= \begin{pmatrix}\theta + AB/\alpha - A[\theta] - A/\alpha[B-\alpha\theta]\\ -\alpha\theta + B - AB + A\alpha[\theta] + A[B-\alpha\theta]\end{pmatrix}.$$

We have added integers $[\theta]$ and $[B-\alpha\theta]$ so that $\theta' \in [0,s)$ and $v = B - \alpha\theta$ (see, e.g., [17]).

Thus the action of the map (1) on the coordinate θ is simply

$$\theta' = T_B(\theta) = \left(\theta + \frac{s}{r}AB - A[\theta] - A\frac{s}{r}\left[B - \frac{r}{s}\theta\right]\right) \pmod s. \tag{6}$$

Because the variable $B \in [(1-s)/s, 1)$ is invariant during the dynamics it is treated as a parameter. Again, the slope of these one-dimensional maps is equal to one implying that all Lyapunov exponents are zero.

In the case of a rational map, i.e. where both $A = p/q$ and $\alpha = r/s$ are rational with $(r,s) = (p,q) = 1$, we can understand a lot about the mapping T using the following factor map. Let

$$\pi(\theta) = \theta - \frac{1}{qr}[qr\theta]$$

be a projection of $I = [0,s)$ onto $J = [0, 1/qr)$: this maps qrs points onto one point.

Lemma 2. *If $A = p/q$ and $\alpha = r/s$ then the diagram*

$$\begin{array}{ccc} I & \xrightarrow{T_B} & I\\ \pi\downarrow & & \downarrow\pi\\ J & \xrightarrow{S_B} & J\end{array}$$

commutes where

$$S_B(\psi) = \psi + \frac{sp}{rq}B \quad \left(\bmod \frac{1}{qr}\right)$$

is a rotation. In other words, T_B has a factor that is a rotation.

Proof. For these assumptions we can write (6) as

$$\theta' = \left(\theta + \frac{sp}{rq}B - \frac{p}{q}[\theta] - \frac{ps}{qr}\left[B - \frac{r}{s}\theta\right]\right) \pmod s. \tag{7}$$

Defining $\psi = \pi(\theta)$ and $\psi' = \pi(\theta')$, note that

$$\psi' = \psi + C \quad \left(\bmod \frac{1}{qr}\right)$$

where $C = \pi(AB/\alpha) = \pi((sp/rq)B)$. □

Since S_B is invertible, $\cap_{n=0}^{\infty} S^n(J) = J$. The map π maps $M = qrs$ points to one point and so π^{-1} must be understood as a set-valued function. We have

$$\pi\pi^{-1}(\psi) = \psi \qquad \pi T(\theta) = S\pi(\theta)$$

for all $\psi \in J$ and $\theta \in I$, while for any set $K \subset I$

$$\pi^{-1}\pi K \supseteq K.$$

Let

$$N(\psi) = \{\theta \in X_B^+ : \pi(\theta) = \psi\} = \pi^{-1}(\psi) \cap X_B^+$$

be the set of $\theta \in X_B^+$ with $\pi(\theta) = \psi$ (we suppress the dependence of S and T on B for the next result).

Lemma 3. *For any $\psi \in J$, the set N satisfies*

$$N(S(\psi)) = T(N(\psi)).$$

Proof. To show that $T(N(\psi)) \subseteq N(S(\psi))$, note that

$$\begin{aligned} T(N(\psi)) &\subseteq \pi^{-1}\pi T(N(\psi)) = \pi^{-1}S\pi(N(\psi)) \\ &= \pi^{-1}S\pi(\pi^{-1}(\psi) \cap X_B^+) \\ &\subseteq \pi^{-1}S(\psi)) \cap X_B^+ \\ &= N(S(\psi)). \end{aligned}$$

For the other direction, suppose that $\theta \in N(S(\psi))$ and so $\pi(\theta) = S(\psi)$ and $\theta \in \cap_{n\geqslant 0}T^n(I)$. In particular, $\theta = T(\theta')$ for some θ'. Now $\pi(\theta) = \pi T(\theta') = S\pi(\theta')$ and so $S(\psi) = S\pi(\theta')$. Invertibility of S gives $\psi = \pi(\theta')$ and so $\theta' \in N(\psi)$. Hence

$$T(N(\psi)) \supseteq N(S(\psi))$$

and we have the result. □

The previous lemma relies crucially on the fact that S is invertible. The next result implies that the number of points in N is a constant almost everywhere. Let

$$\hat{N} = |N(\psi)|$$

be the cardinality of $N(\psi)$.

Lemma 4. *If B is irrational then $\hat{N}$ is constant for a set of ψ with full Lebesgue measure.*

Proof. It is a standard result that S_B is ergodic for the Lebesgue measure if and only if B is irrational. If we look at the set of ψ that gives a certain value of $N(\psi)$ this is invariant and therefore must have Lebesgue measure 0 or 1. □

One consequence of this is that the measure of X_B^+ can only take a finite number of values. We write $\hat{N}(B)$ to show the dependence on B explicitly.

Theorem 2. *Suppose that $A = p/q$ and $\alpha = r/s$. For Lebesgue almost all B we have*

$$\ell(X_B^+) = \frac{\hat{N}(B)}{qr}$$

where $\hat{N}(B)$ is an integer and $1 \leqslant \hat{N}(B) \leqslant qrs$. If there is an interval on which T has L pre-images, then

$$\hat{N}(B) \leqslant qrs + 1 - L.$$

Proof. As pre-images of $\pi^{-1}(J)$ are disjoint and all have length $1/qr$ we compute

$$\ell(X_B^+) = \sum_{k=1}^{\hat{N}(B)} \frac{1}{qr} = \frac{\hat{N}(B)}{qr}.$$

Note that $\hat{N}(B)$ must take an integer value less than or equal to qrs, which is the number of pre-images $\pi^{-1}(\psi)$.

Since T will be invertible on X_B^+, if there is an interval with more than one pre-image, only one of these will be in X_B^+ and hence $\hat{N}(B) \leqslant qrs + 1 - L$ where L is the number of pre-images. □

We can now prove the main result in this section.

Theorem 3. *Suppose that $\alpha = r/s$ and $A = p/q$ are both rational, with $(r, s) = (p, q) = 1$. Then the maximal invariant set X^+ has Hausdorff dimension two and positive Lebesgue measure, more precisely, if there is an open set on which f has K pre-images then*

$$\frac{1}{qrs} \leqslant \ell(X^+) \leqslant 1 - \frac{K-1}{qrs}.$$

Proof. Note that under the hypotheses of the theorem,

$$\ell(X^+) = \int_{B\in[1-(1/s))} \int_{s\in X_B^+} \mathrm{d}\theta \, \mathrm{d}s$$

$$= \int_{B\in[1-(1/s))} \ell(X_B^+) \, \mathrm{d}s.$$

Upon integrating the estimates in theorem 2 we obtain the result. □

We can characterize the dynamics on the lines L_B in the following way.

Corollary 2. *For any rational map, almost all B and all $x \in L_B$, $\omega(x)$ consists of a set with positive one-dimensional measure in L_B. Moreover, for a countable set of B and all $x \in L_B$, the trajectory going through x is eventually periodic.*

Proof. In the proof of theorem 3, the reduced map S_B will be an irrational rotation for all irrational B and a rational rotation for all rational B. The result follows. □

The density of irrational B simply implies the following result for the original map.

Corollary 3. *For any rational parabolic map, the pre-images of the discontinuity are dense in X^+.*

Table 1. Numerically obtained approximate values of $\ell(X^+)$ for a number of rational parabolic maps.

A	$\alpha = \frac{2}{3}$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{5}$
$\frac{1}{4}$	0.261	0.337	0.251	0.196
$\frac{1}{3}$	0.235	0.326	0.265	0.199
$\frac{2}{3}$	0.215	0.342	0.257	0.149
$\frac{5}{4}$	0.628	0.626	0.419	0.252
$\frac{3}{2}$	0.338	0.562	0.505	0.303

We do not have a precise analytical expression for $\ell(X^+)$ even for rational parabolic maps. Numerically, one can approximate the measure and obtain various values of $\ell(X^+)$; see, for example, table 1. In certain simple cases one can obtain exact values of $\ell(X^+)$ by constructing the maximal invariant set explicitly, as follows.

Proposition 1. *One can compute*

$$\ell(X^+) = \begin{cases} 1 & \text{if the map is invertible} \\ \alpha & \text{if } A = 1 \text{ and } 0 < \alpha \leqslant 1 \\ \alpha^{-1} & \text{if } A = -1 \text{ and } \alpha \geqslant 1 \\ |A\alpha| & \text{if } 0 < A \leqslant 1 \text{ and } -1 \leqslant \alpha < 0. \end{cases}$$

The invertible case is trivial, while the cases where $A = 1$ follow because the map reduces to an invertible map on a strip height α. The appendix gives a constructive proof for the case $0 < A \leqslant 1$, $-1 \leqslant \alpha < 0$. Because $f_{-A,-1}$ is conjugate to $f_{A,-1}$ by interchanging x and y we have

$$\ell(X^+) = \begin{cases} |A| & 0 < |A| \leqslant 1 \quad \alpha = -1 \\ 1 & A = 0 \quad\quad\quad\; \alpha = -1. \end{cases}$$

Note, in particular, that there is a discontinuity in $\ell(X^+)$ at $(\alpha, A) = (-1, 0)$.

The following examples (especially that in section 3.3) show that a general expression, if it can be obtained, is likely to be non-trivial.

3.2. Example I

The simplest non-trivial rational map is defined by $A = \frac{1}{2}$ and $\alpha = 1$. In this case, we can approximate the maximal invariant set numerically as in figure 1. The map in this case is

$$\begin{aligned} x' &= \tfrac{3}{2}x + \tfrac{1}{2}y \pmod 1 \\ y' &= -\tfrac{1}{2}x + \tfrac{1}{2}y \pmod 1 \end{aligned} \tag{8}$$

where $(x, y) \in X = [0, 1]^2$. In this case, as noted in [17], there appears to exist a symmetry between X^+ and its complement. In fact, we can use the results in section 3.1 to obtain the result directly.

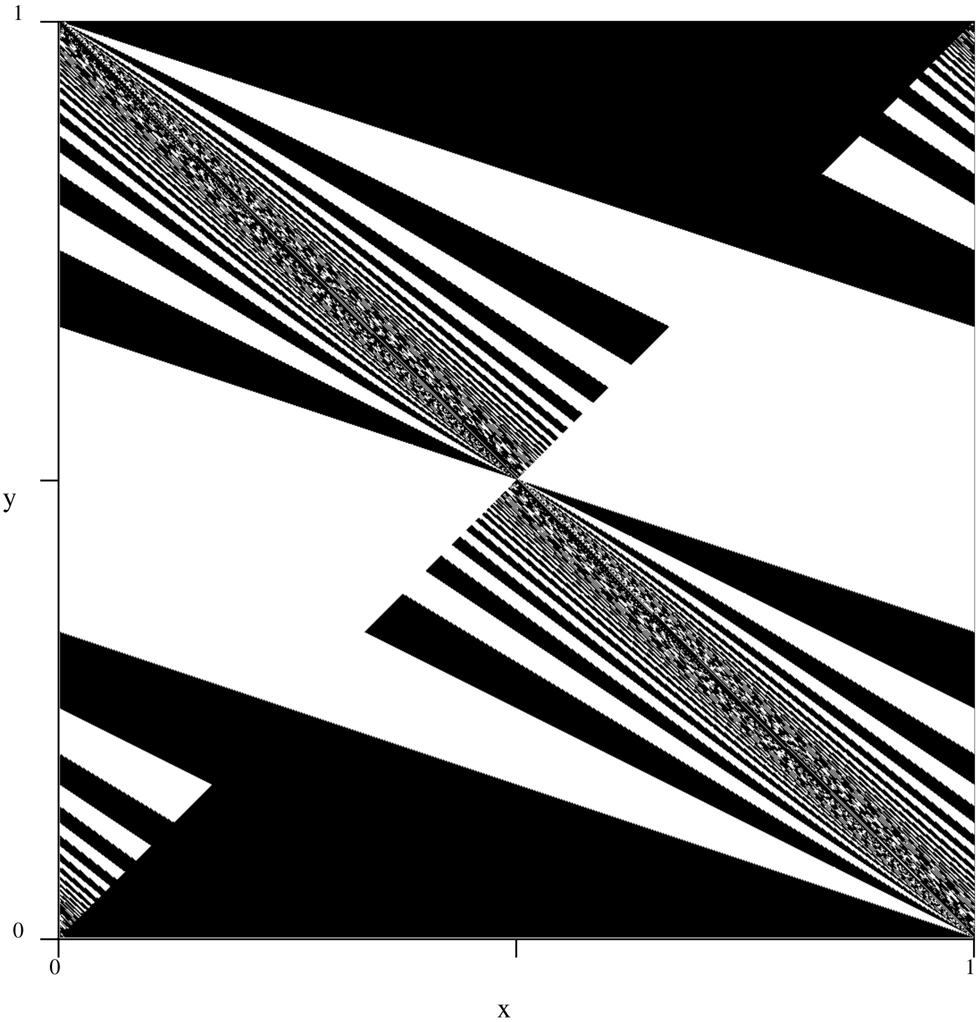

Figure 1. The black region shows the maximal invariant set X^+ for the map (8) where $A = \frac{1}{2}$ and $\alpha = 1$. Observe the symmetry of the black and white regions indicating that $\ell(X^+) = \frac{1}{2}$.

Proposition 2. *For the map (8) with* $A = \frac{1}{2}$ *and* $\alpha = 1$*, we have* $\ell(X^+) = \frac{1}{2}$.

Proof. Note that $r = s = p = 1$ and $q = 2$. Moreover, on L_B the map (6) can be written as

$$\theta' = h(\theta) = \theta + \tfrac{1}{2}B - \tfrac{1}{2}[B - \theta] \pmod 1$$

and this map is two-to-one on an open set of θ for almost all B. Therefore, for almost all B, the upper and lower bounds of theorem 2 agree and

$$\ell(X_B^+) = \tfrac{1}{2}$$

for almost all B. This implies that $\ell(X^+) = \frac{1}{2}$. □

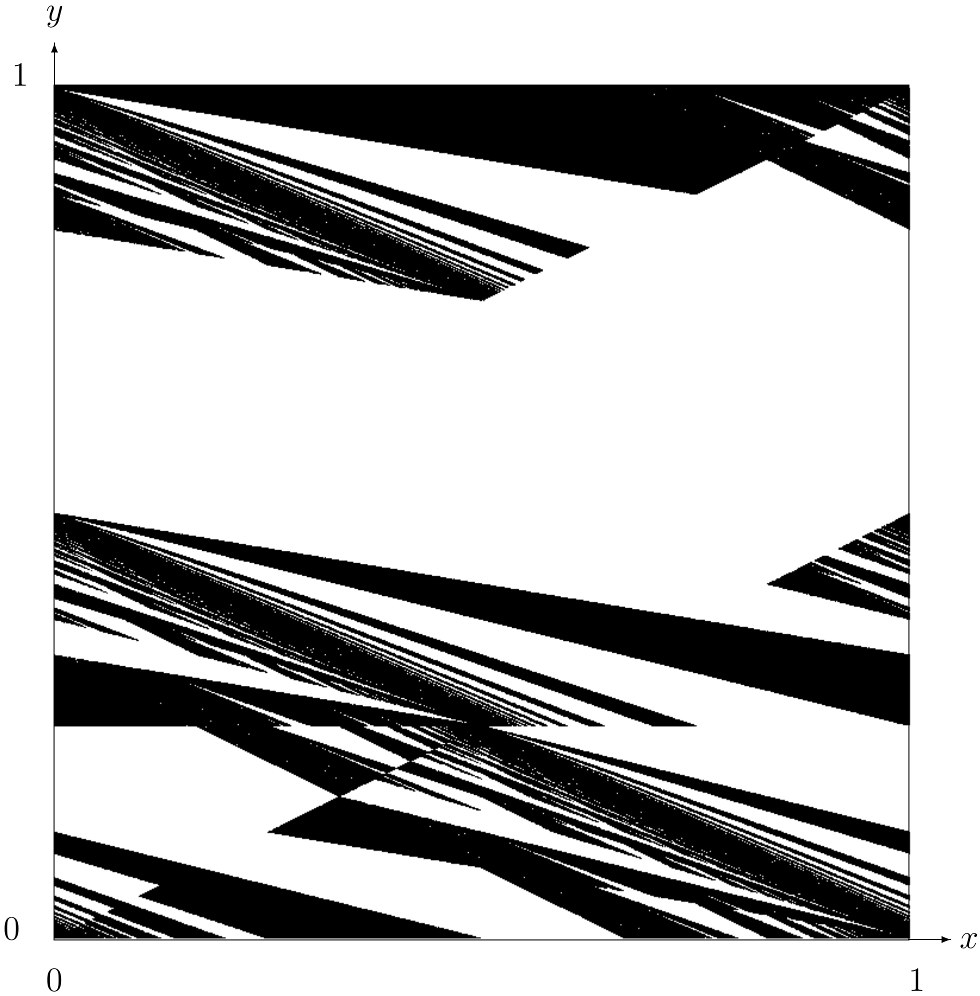

Figure 2. The black regions show the maximal invariant set X^+ of the rational map (9) where $A = \alpha = \frac{1}{2}$. This has measure corresponding to approximately 28.7% of the unit square.

3.3. Example II

We now consider the case $A = \alpha = \frac{1}{2}$; this shows that $\ell(X^+)$ may be irrational for a rational map. In this case the map is

$$\begin{aligned} x' &= \tfrac{3}{2}x + y \pmod 1 \\ y' &= -\tfrac{1}{4}x + \tfrac{1}{2}y \pmod 1. \end{aligned} \tag{9}$$

On the invariant lines L_B with $B \in [\frac{1}{2}, 1)$ equation (9) reduces to (6) with $\theta \in [0, 2)$ to

$$\theta' = T_B(\theta) = \left(\theta + B - \tfrac{1}{2}[\theta] - [B - 2\theta]\right) \pmod 2. \tag{10}$$

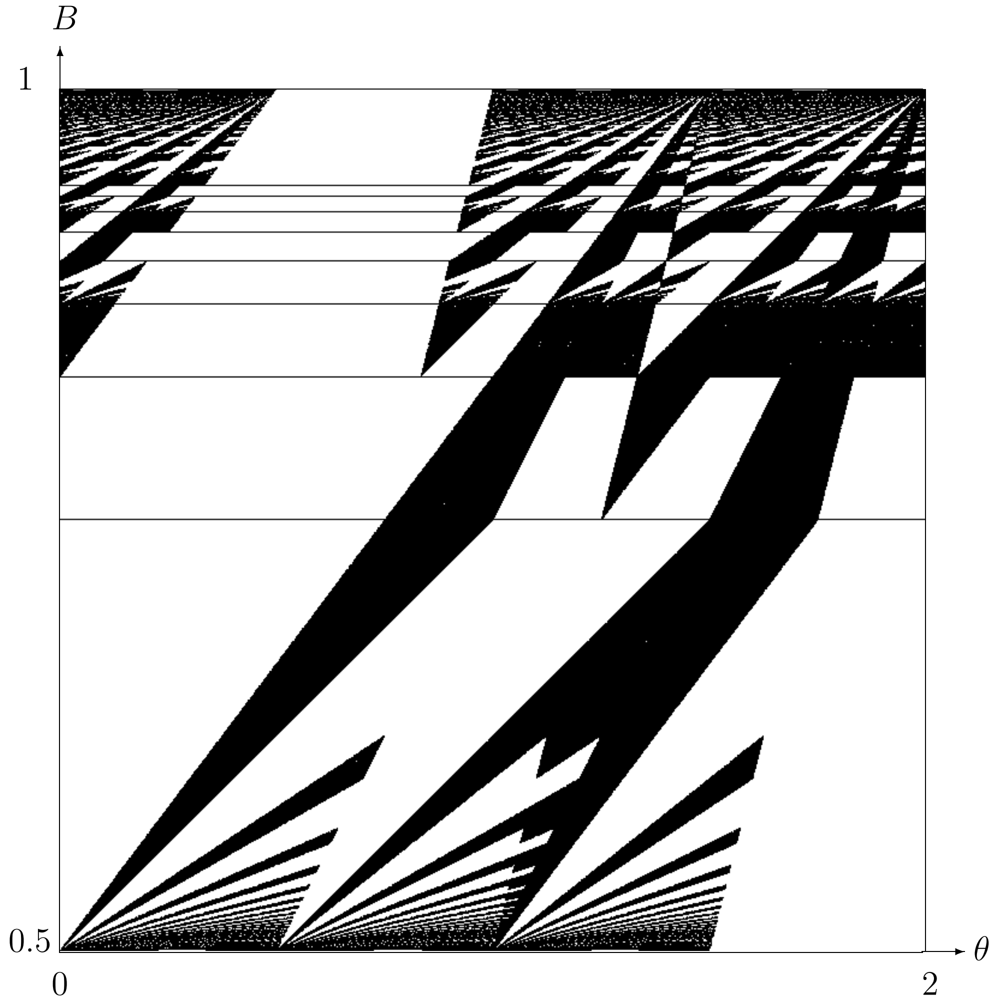

Figure 3. Maximal invariant set shown as a bifurcation diagram of the map (10). The horizontal lines $B = \frac{3}{4}, \frac{5}{6}, \frac{7}{8}, \frac{9}{10}, \frac{11}{12}, \frac{13}{14}, \frac{15}{16}$ are superimposed onto the picture to make the assertion in the text easier to see.

The maximal invariant set X^+ for the map (9) is shown in figure 2. The results in theorem 3 imply that

$$\tfrac{1}{4} \leqslant \ell(X^+) \leqslant \tfrac{3}{4}$$

but this is clearly not very precise.

We have obtained a numerical estimate by iterating a grid of initial points distributed uniformly on the square, dividing the unit square into boxes and counting the occupied cells. To avoid the transient effects, some number (say, the first hundred) of images are not marked on the graph. Applying this method we obtained $V \sim 0.287$; however, the precision of this result is limited by the highly complicated structure of some fragments of the invariant set and the very slow convergence of transients.

By using the structure noted in section 3 we show numerical approximations of X_B^+ as B changes in the map (10) as shown in figure 3, and note that the bifurcation diagram of X_B^+ has a regular structure

$$
\begin{array}{ll}
\frac{1}{2} < B < \frac{5}{6} & \ell(X_B^+) = \frac{1}{2} \\
\frac{5}{6} < B < \frac{7}{8} & \ell(X_B^+) = 1 \\
\frac{7}{8} < B < \frac{11}{12} & \ell(X_B^+) = \frac{1}{2} \\
\frac{11}{12} < B < \frac{13}{14} & \ell(X_B^+) = 1 \\
\frac{13}{14} < B < \frac{17}{18} & \ell(X_B^+) = \frac{1}{2}.
\end{array}
$$

These values agree with the predictions of theorem 3 that $\ell(X_B^+) \in \frac{1}{4}\{1, 2, 3\}$. More generally for any $n \in \mathbb{N}$ we observe that

$$
\begin{array}{ll}
\dfrac{6n-1}{6n} < B < \dfrac{6n+1}{6n+2} & \ell(X_B^+) = 1 \\
\text{otherwise} \quad \ell(X_B^+) = \frac{1}{2}. &
\end{array}
$$

Therefore, we have

$$
\begin{aligned}
\ell(X^+) &= \int_{1/2}^{1} \ell(X_B^+)\,\mathrm{d}B \\
&= \frac{1}{4} + \frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{6n+1}{6n+2} - \frac{6n-1}{6n}\right) \\
&= \frac{1}{4} + \frac{1}{12}\sum_{n=1}^{\infty}\frac{1}{3n^2+n} \\
&= \frac{1}{2} - \frac{\sqrt{3}\pi}{72} - \frac{1}{8}\ln 3 \\
&\approx 0.287\,098\,49.
\end{aligned}
$$

Thus we can obtain a series expression for the measure and in this case we can sum it explicitly. Unfortunately we have not been able to extend this method to one that applies generally.

3.4. Irrational parabolic maps

We have not yet obtained any significant results for irrational parabolic area-preserving maps and, in fact, for all the cases where we can compute it, the maximal invariant set has positive measure. Nonetheless, there is numerical evidence [17] which suggests that:

- There are (α, A) for which the Hausdorff dimension of X^+ is less than 2 (in particular, such that $\ell(X^+) = 0$). By the results shown here, this can only occur in cases where both parameters are irrational.
- This is common for $A > 0$ and $\alpha > 0$.

Note that not all irrational parabolic maps have zero measure; for example, proposition 1 shows that there are open regions in (A, α) with $\alpha < 0$ where $\ell(X^+) > 0$. For semirational cases ($\alpha = r/s$) we can reduce to maps of the form (6); however, we cannot find factors that are rotations if A is not rational. Results of [6, 15] suggest that the Hausdorff dimension of X^+ may vary between 1 and 2. The Hausdorff dimension is a rather subtle characteristic of a set. For example, two sets may be arbitrarily close in the sense of the Hausdorff metric, but their dimensions may differ by an arbitrarily large number. Therefore, small changes of the parameter α, which do not change much the structure of the maximal invariant set (in the sense of the Hausdorff metric) may induce wild fluctuations of its dimension. Note also that the cases $\alpha > 0$ and $\alpha < 0$ seem to be fundamentally different; we do not have a way of conjugating one with the other.

4. Nonlinear parabolic maps

One can also construct nonlinear parabolic maps that display similar structure in their maximal invariant sets as the piecewise linear case. Consider the two-parameter family of the maps on

Figure 4. The black region shows the maximal invariant set X^+ for the nonlinear parabolic map (11) with $\alpha = \gamma = 1$.

the torus defined by

$$\begin{aligned} x' &= x + \beta(e^{\gamma(y-x)} - 1) \pmod 1 \\ y' &= y + \beta(e^{\gamma(y-x)} - 1) \pmod 1. \end{aligned} \tag{11}$$

For any non-zero values of the real parameters β and γ the Jacobian $\partial(x', y')/\partial(x, y)$ is constant and equal to one. Moreover, for any point (x, y) the trace of the map t is equal to 2, which makes the above map similar to the linear case (1). Observe that the diagonal of the unit square $y = x$ is invariant with respect to this map. This map acting on the plane transforms the unit square into a set confined between four exponential functions. After folding it back into the unit square some fragments of it will overlap as (11) is typically non-invertible.

Figure 4 shows the invariant set of the nonlinear map in the case $\beta = \gamma = 1$. There is numerical evidence that the maximal invariant set has positive volume in this case, even though the Jacobian changes between being a rational and an irrational parabolic map at different points.

5. Discussion

In this paper we have considered some basic properties of the maps (1) in the parabolic area-preserving regime, and shown the surprising degree of sensitivity of the dynamical behaviour on the rationality of the parameters α and A. Our investigations raise some intriguing questions concerning how the Lebesgue measure of the maximal invariant set varies with parameters. For several examples we have analytical values of this measure. For general rational parabolic maps we have a result that gives upper and lower bounds for this measure. However, these bounds become weak as one examines higher denominator rational maps and do not easily give insight into the measure for irrational parabolic maps.

It would be very informative to understand the structure of invariant measures of the map (1). Note that $\ell(\cdot)$ (Lebesgue measure) restricted to X^+ is invariant under this mapping but it is not ergodic. In particular, it may be the case that this restriction is trivial in which case empirical measures can still be defined. By analogy with the interval translation maps of [6, 15] we surmise that there are cases where a Hausdorff measure restricted to X^+ is invariant, in particular, in cases where $\ell(X^+) = 0$.

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Appendix. Proof of proposition 1

We consider the case $-1 \leqslant \alpha < 0$ and $0 < A < 1$; figure A1 shows how the linear map $M_{A,\alpha}$ maps the square $[0, 1]^2$ given by $OPRQ$ into the maximal invariant set consisting of the union of the line OS and the triangles OAB and CSQ, where

$$A = (0, -\alpha) \qquad B = (A, -\alpha A) \qquad C = (1 - A, -\alpha(1 - A)) \qquad S = (1, -\alpha).$$

To see this is the maximal invariant set, note that everything in $OPRS$ decreases in its y-component unless it lands in the triangle OAB. Similarly, all points in OQS must increase in

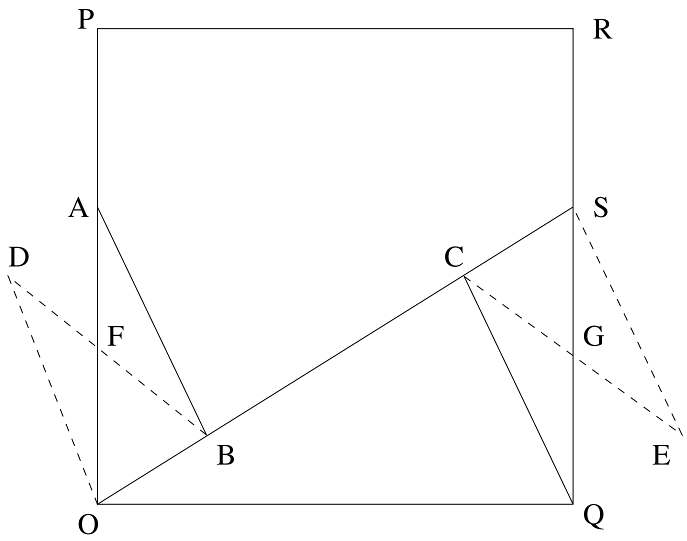

Figure A1. Construction of the maximal invariant set for parabolic maps with $-1 \leqslant \alpha < 0$ and $0 < A \leqslant 1$.

y-component unless it lands in the triangle CQS. The union of the two triangles is invariant as the dotted images ODB and CSQ show, as is the line of fixed points OS. Hence in this case we have

$$\ell(X^+) = -\alpha A = |\alpha A|.$$

A similar argument (with extra triangles) can be used to show that for $0 < A \leqslant 1$ and

$$-\frac{1}{A} < \alpha \leqslant -1$$

the maximal invariant set has the measure

$$\ell(X^+) = (1 + \alpha A)\frac{A}{1 - A}$$

but we omit this for conciseness.

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