Appendix D

Hints and answers to the exercises

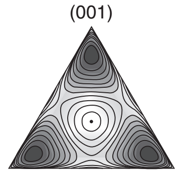
Problem 1.1. Draw a line from a pure point $\mathbf{x}_1$ through the given point $\mathbf{x}$. It ends at a point $\mathbf{y}_1$ on some face of at most $n-1$ dimensions. If this point is mixed, draw a line from a pure point $\mathbf{x}_2$ on that face through $\mathbf{y}_1$. It ends at a point $\mathbf{y}_2$ on some face of at most $n-2$ dimensions. Continue until $\mathbf{x}$ is expressed as a mixture of at most $n+1$ pure points.

Problem 1.2. One way is to construct the simplex. If we put its centre at the origin the $N = n+1$ points can be placed at

$$\begin{array}{ccccc} (-r_1, & -r_2, & \dots, & -r_{n-1}, & -r_n) \\ (R_1, & -r_2, & \dots, & -r_{n-1}, & -r_n) \\ (\ 0, & R_2, & \dots, & -r_{n-1}, & -r_n\ ). \\ \dots & \dots & \dots & \dots & \dots \\ (\ 0, & 0, & \dots, & 0, & R_n\ ) \end{array}$$

This helps.

Problem 2.1. Here is a plot of the structural entropy. The maximum $S_2 - S_1 \approx 0.22366$ is attained for $\vec{p} \approx (0.806, 0.097, 0.097)$ and its two other permutations. They are visible at the contour plot provided as three dark hills.



Problem 2.2. The $N = 3$ case shows the idea. We have $\vec{x} \cdot (\vec{y} - \vec{z}) = x_1(y_1 - z_1) + x_2(y_2 - z_2) + x_3(y_3 - z_3) = (x_1 - x_2)(y_1 - z_1) + (x_2 - x_3)(y_1 + y_2 - z_1 - z_2) + x_3(y_1 + y_2 + y_3 - z_1 - z_2 - z_3) \geq 0$ because of the conditions stated.

Problem 2.3. (a) Let $a, b \in [0,1]$ and $a' = 1 - a$, $b' = 1 - b$ and define

$$B = T_1 T_2 = \begin{bmatrix} a & a' & 0 \\ a' & a & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & b & b' \\ 0 & b' & b \end{bmatrix} = \begin{bmatrix} a & a'b & a'b' \\ a' & ab & ab' \\ 0 & b' & b \end{bmatrix}. \tag{D.1}$$

B is a bistochastic matrix. It is also orthostochastic since $B_{ij} = (O_{ij})^2$, where

$$O = \begin{bmatrix} \sqrt{a} & \sqrt{a'b} & -\sqrt{a'b'} \\ \sqrt{a'} & -\sqrt{ab} & \sqrt{ab'} \\ 0 & \sqrt{b'} & \sqrt{b} \end{bmatrix}. \tag{D.2}$$

Problem 2.4. We know that $\vec{x}$ is a (non-unique) convex combination of permutation matrices acting on $\vec{y}$; this defines a bistochastic matrix according to Birkhoff's theorem.

Problem 2.5. One obtains two important cases of the Dirichlet distribution (2.73): the round measure ($s = 1/2$) for real and flat measure ($s = 1$) for complex Gaussian random numbers [1016].

Problem 2.7. For $q \leq 2$. To see this, study the second derivative close to $p = 1$.

Problem 3.1. You will obtain

$$\frac{x^i}{X^I} = \frac{2}{1 + X^0} \quad \Rightarrow \quad X^0 = \frac{4 - r^2}{4 + r^2} \quad \Rightarrow \quad ds^2 = \left(\frac{4}{4 + r^2}\right)^2 dx^i dx^i.$$

Problem 3.2. The angles are obtained by intersecting respectively the plane and the sphere with two intersecting planes. The angles will be equal if and only if both the plane and the sphere meet the line of intersection of the two planes at the same angle. But this will happen if and only if the line of intersection forms a chord of the great circle.

Problem 3.3. You can try a calculation to see whether the natural map $(x, y) \rightarrow (x, 2y)$ between the tori is analytic (it is not). Or you can observe that the tori inherit natural flat metrics from the complex plane. On each torus there will be a pair of special closed geodesics that intersect each other, namely what used to be straight lines along the x- and y-directions on the plane. Their circumferences are equal on one of the tori, and differ by a factor of two on the other. But analytic – hence conformal – transformations do not change the ratio of two lengths, and it follows that no such analytic transformation between the tori can exist.

Problem 3.4. As an intermediate step you must prove

$$\Omega^{il}\partial_l \Omega^{jk} + \Omega^{kl}\partial_l \Omega^{ji} + \Omega^{kl}\partial_l \Omega^{ij} = 0. \tag{D.3}$$

Problem 3.5. A hyperplane through the origin in embedding space meets the 3-sphere in a 2-sphere given in stereographic coordinates by

$$a_I X^I = 0 \quad \Rightarrow \quad 2a_1 x + 2a_2 y + 2a_3 z + 1 - r^2 = 0$$

(where we assumed that the fourth component of the vector equals one). A geodesic is the intersection of two such spheres; choose them to have their centres at $(a, 0, 0)$ and $(b_1, b_2, 0)$ again without loss of generality. If you also demand $r^2 = 1$ (the equator) you get three equations with the solutions $(x, y, z) = (0, 0, \pm 1)$.

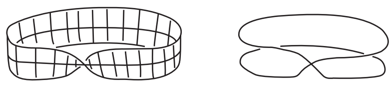
Problem 3.6. The key point is that two Hopf circles with the opposite twist meet twice. Only one-half of the circumference of a Hopf circle is needed to label the members of the family of circles that twist in the other way. (Draw the torus as a flat square to see this.)

Problem 3.7. For $\tau = -\phi$ we get

$$X + iY = \cos\frac{\theta}{2} \quad \Rightarrow \quad Y = 0 \ \& \ X > 0 \quad \Rightarrow \quad y = 0 \ \& \ x > 0. \tag{D.4}$$

With the exception of one point this maps $\mathbf{S}^2$ onto a half plane. The other two sections provide maps onto a hemisphere and a unit disc, respectively. In all three cases it is geometrically evident that we are selecting one point from each geodesics, except for a single one for which there is no prescription.

Problem 3.8. The group acting on the fibres is the discrete group with two elements, the unit element and the element that turns the fibre upside down. See the Figure. Left: the Möbius strip as a vector bundle, with a global section (i.e. an embedding of the circle in the bundle). Right: the principal bundle, with fibres equal to the group $\{\pm\}$.



Problem 4.1. Consider the case of three points and put one at the origin, one at $(1, 0)$. The location of the third point can be anywhere. That gives an $\mathbb{R}^2$. This coordinatisation of the space of triplets fails if the first two points coincide. That case evidently corresponds to one additional point 'at infinity', so we have a natural one-to-one correspondence between the space of triplets and a plane + the point at infinity, that is $\mathbb{C}\mathbf{P}^1$. Consider the case of four points: if the first two points are distinct we proceed as above; the two remaining points can be coordinatised by $\mathbb{R}^2 \times \mathbb{R}^2 = \mathbb{C}^2$. If the first two points coincide we have only a triplet of points do deal with. This is a $\mathbb{C}\mathbf{P}^1$ according to what was just shown. But $\mathbb{C}^2$ plus a $\mathbb{C}\mathbf{P}^1$ 'at

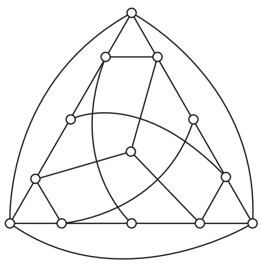
infinity' is a $\mathbb{C}\mathbf{P}^2$. And so on. This is useful in archaeology if we use the Fubini–Study metric to give a measure. Then we can answer questions like 'given $n+2$ stones, what is the probability that there are k triplets of stones lying (to a given precision) on straight lines?'.

Problem 4.2. A Klein bottle; a bottle without an inside (or outside). It takes a four dimensional being to make one that does not intersect itself.

Problem 4.3. Integrate the Fubini–Study 2-form Ω over the embedded $\mathbb{C}\mathbf{P}^1$; this gives $\int_{\mathbb{C}\mathbf{P}^1}\Omega$ = the area = π, since Ω induces the usual Fubini–Study 2-form on $\mathbb{C}\mathbf{P}^1$. But if $\mathbb{C}\mathbf{P}^1$ could be shrunk to a point then this calculation could be done within a single coordinate patch, and there could be no obstruction to the calculation $\int_{\mathbb{C}\mathbf{P}^1}\Omega = \int_{\mathbb{C}\mathbf{P}^1} d\omega = \int_{\partial(\mathbb{C}\mathbf{P}^1)}\omega = 0$, where we used Stokes' theorem and the fact that $\mathbb{C}\mathbf{P}^1$ has no boundary. This is a contradiction. Alternatively one can stare at the line at infinity in the octant picture of $\mathbb{C}\mathbf{P}^2$, and convince oneself that any attempt to move it will increase its area.

Problem 5.1. The two pure states divide a great circle into two segments. If one of the eigenstates of A lies on the shortest of these segments, the answer is $D_{\mathrm{Bhatt}} = \theta_A + \theta/2$, otherwise it is $D_{\mathrm{Bhatt}} = \theta/2$ (independent of A).

Problem 5.2. For the colouring problem, take a few photocopies of the graph. For the relation to the Escher print, begin by looking at the subgraph on the right. It is interesting in itself [990]. Its three outermost vertices correspond to rays going through the midpoints of the faces of a cube, the next layer of six vertices to rays going through the midpoints of its edges, and the inner four to rays going through the corners of the cube.



Problem 6.1. From (6.9) the Q-function of a Fock state is $Q_{|n\rangle}(z) = |z|^{2n}e^{-|z|^2}/n!$ and from (6.47) we must have $\int \mathrm{d}z^2 Q_{|n\rangle}P_{|1\rangle} = \delta_{n1}$. The solution is the moderately singular distribution $P_{|1\rangle} = e^{|z|^2}\partial_z\partial_{\bar{z}}\delta^{(2)}(z)$.

Problem 6.2. Using spherical polars in phase space (and the integral representation of the gamma function) one finds $S_W(|n\rangle) = 1 + n + \ln n! - n\Psi(n+1)$, where Ψ is the digamma function defined in Eq. (7.54).

Problem 7.1. It is a function of the maximum of the Husimi function, so it can be read off from Eq. (7.25).

Problem 7.2. Wehrl entropy and participation number for pure states of $N = 2$–5 read

N	j	m	$S_W(\|\psi\rangle)$	$R(\|\psi\rangle)$
2	1/2	±1/2	$1/2 = 0.5$	$1\frac{1}{2}$
3	1	±1	$2/3 \approx 0.667$	$1\frac{2}{3}$
3	1	0	$5/3 - \ln 2 \approx 0.974$	$2\frac{1}{2}$
4	3/2	±3/2	$3/4 = 0.75$	$1\frac{3}{4}$
4	3/2	±1/2	$9/4 - \ln 3 \approx 1.151$	$2\frac{11}{12}$
4	3/2	$\|\psi_\triangle\rangle$	$21/8 - \ln 4 \approx 1.239$	$3\frac{2}{11}$
5	2	±2	$4/5 = 0.8$	$2\frac{1}{4}$
5	2	±1	$79/30 - \ln 4 \approx 1.247$	$3\frac{3}{20}$
5	2	0	$47/15 - \ln 6 \approx 1.342$	$3\frac{1}{2}$
5	3/2	$\|\psi_{\text{tetr.}}\rangle$	$165/45 - \ln 9 \approx 1.492$	$4\frac{1}{5}$

Problem 8.2. (a) yes; (b) no. Any permutation is represented by an orthogonal matrix, and multiplication by any unitary matrix does not change the singular values

Problem 8.3. (a) absolute values of real eigenvalues; (b) equal to unity; (c) absolute values of complex eigenvalues.

Problem 8.4. In fact an even stronger property is true. Directly from the definition of the singular values it follows that $\text{sv}(A) = \text{sv}(UAV)$, for arbitrary unitary U and V. However, the special case $V = U^{-1}$ is often useful in calculations.

Problem 8.5. This is the Cauchy–Schwarz inequality for the scalar product in Hilbert–Schmidt space.

Problem 8.6. We know that $\langle\psi|P|\psi\rangle \geq 0$ for all vectors. Let $|\psi\rangle$ be a basis vector.

Problem 8.7. We can bring an arbitrary vector $\tau_i\sigma_i$ into the Cartan subalgebra, $U\tau_i\sigma_i U^\dagger = \lambda_i H_i$. Generically that is the best we can do, so the number of non-zero elements will equal the dimension of the Cartan subalgebra, i.e. $N - 1$.

Problem 8.8. Definition (8.30) applied to the Pauli matrices gives

$$O = \begin{bmatrix} c^2\vartheta\, c(2\phi) - s^2\vartheta\, c(2\psi) & c^2\vartheta\, c(2\phi) + s^2\vartheta\, c(2\psi) & -s(2\vartheta)c(\phi+\psi) \\ s^2\vartheta\, s(2\psi) - c^2\vartheta\, s(2\phi) & c^2\vartheta\, c(2\phi) + s^2\vartheta\, c(2\psi) & s(2\vartheta)s(\phi+\psi) \\ s(2\vartheta)c(\psi-\phi) & -s(2\vartheta)s(\psi-\phi) & c(2\vartheta) \end{bmatrix}. \tag{D.5}$$

where $c \equiv \cos$, $s \equiv \sin$. This is the *Cayley parametrization* of the group $SO(3)$ and describes the rotation with respect to the axis $\vec{\Omega} = (\sin\vartheta \sin\psi, \sin\vartheta\cos\psi, \cos\vartheta \sin\phi)$ by an angle t such that $\cos(t/2) = \cos\vartheta\cos\phi$.

Problem 8.9. Diagonalise! Then $\mathrm{Tr}\rho^2 = 1$ implies that all eigenvalues obey $-1 \leq \lambda_i \leq 1$, and that the eigenvalue 1 can occur at most once. Also $\mathrm{Tr}\rho^2(1-\rho) = 0$ becomes a sum of non-negative terms in which each individual term must vanish, so the eigenvalues are either 0 or 1 [507].

Problem 9.4. The spectrum consists of the MN numbers $\alpha_i\beta_j$, where $i = 1, \ldots, M$ and $j = i, \ldots, N$ (say).

Problem 9.6. Use the Schwarz inequality, $|\mathrm{Tr}(AB)|^2 \leq \mathrm{Tr}(AA^\dagger) \times \mathrm{Tr}(BB^\dagger)$, and replace A by $A\rho^{1/2}$ and B by $B\rho^{1/2}$, respectively [652].

Problem 10.1. It is enough to consider a pure POVM, so that $E_i = |\phi_i\rangle\langle\phi_i|$. The vectors have components ϕ_i^α, $\alpha = 1, \ldots, N$, $i = 1, \ldots, k$. Let these components be the elements of an $N \times k$ matrix. The completeness relation $\sum_{i=1}^k (E_i)^\alpha{}_\beta = \delta^\alpha_\beta$ implies that the rows of this matrix are orthonormal. We can always add an additional set of $k - N$ rows to the matrix, so that it becomes unitary. The columns of the new matrix form an orthonormal basis in a k dimensional Hilbert space, and there is an obvious projection of its vectors down to the original Hilbert space.

Problem 10.2. Let $a = \mathrm{diag}(A)$ and $c = \mathrm{diag}(C)$ be diagonals of complex matrices. Show that $ABC^\dagger = (ac^\dagger) \circ B$ and use it with $A = A_i$ and $C = A_i^\dagger$ for all $i = 1, ..., k$.

Problem 10.3. To prove positivity, take an arbitrary vector V^i and define $A \equiv V^i A_i$. Then $\bar{V}^i \sigma_{ij} V^j = \mathrm{Tr}\rho AA^\dagger$ is positive because the trace of two positive operators is always positive. For the final part see (10.56).

Problem 10.5. The phase flip channel.

Problem 10.6. The dynamical matrix D_Φ is represented by $D_{\substack{mn\\\mu\nu}} = \rho_{m\mu}\delta_{n\nu}$. Writing down the elements of the image $\sigma' = \Phi_\rho(\sigma) = D^R\sigma = (\rho \otimes \mathbb{1}_N)^R\sigma$ in the standard basis we obtain the desired result, $\sigma'_{m\mu} = D_{\substack{mn\\\mu\nu}}\sigma_{n\nu} = \rho_{m\mu}\mathrm{Tr}\sigma = \rho_{m\mu}$.

Problem 11.1. Write both matrices in their eigen representations, $A = \sum_i a_i|\alpha_i\rangle\langle\alpha_i|$ and $B = \sum_i b_i|\beta_i\rangle\langle\beta_i|$. Perform decompositions $A^R = \sum_i a_i\alpha^{(i)} \otimes \bar{\alpha}^{(i)}$ and $B^R = \sum_i b_i\beta^{(i)} \otimes \bar{\beta}^{(i)}$ as in (10.58), multiply them and reshuffle again to establish positivity of $(A^R B^R)^R$. For a different setting see Havel [412].

Problem 11.2. (a) The spectrum $\vec{d}$ consists of $N(N+1)/2$ elements equal to $+1$ and $N(N-1)/2$ elements equal to -1, so its sum (the trace of D) is equal to N. (b) This canonical form contains one negative term and three positive terms.

Problem 11.3. Using the non-homogeneous form of (10.36) write down the dynamical matrix D and show that some of its eigenvalues are negative.

Problem 11.4. $\Phi_T = \frac{N-1}{N}\Phi_* + \frac{1}{N}T$

Problem 11.5. The spectrum $\vec{d}$ of D_Ψ reads $(a-3, a, a, b, b, b, c, c, c)$. Hence $cp(\Psi) = \min(a-3, b, c)$ and the map is CP if $a \geq 3$ and $b, c \geq 0$. The spectrum of $D_\Psi^{T_A}$ is three-fold degenerated and contains $\{a-1, \lambda_+, \lambda_-\}$, where $\lambda_\pm = (b+c \pm \sqrt{(b-c)^2+4})/2$. Therefore $ccp(\Psi) = \min(a-1, \lambda_-)$ and the map is CcP if $a \geq 1$ and $bc \geq 1$. Note that these results are consistent with (11.9) and (11.10).

Problem 11.7. As we sum contributions from the entry-wise products of a Kraus operator A_i and its complex conjugate, the entries of T are real and non-negative. If the quantum map Φ is trace preserving, $\sum_i A_i^\dagger A_i = \mathbb{1}$, then T is stochastic. If the Choi matrix becomes diagonal, due to 'decoherence' in the space of maps, $D \to \tilde{D} = \text{diag}(D)$ then the map reduces to the classical map represented by a stochastic matrix, $\Phi = D^R \to \tilde{D}^R \to T$. If additionally, the map Φ is unital, $\sum_i A_i A_i^\dagger = \mathbb{1}$, then T is bistochastic.

Problem 12.1. For the matrix elements, deduce that

$$A = \sum_I U_I \frac{1}{N}\text{Tr}U_I^\dagger A \quad \Rightarrow \quad \sum_I (U_I)_{m\mu}(U_I^*)_{n\nu} = N\delta_{mn}\delta_{\mu\nu}. \tag{D.6}$$

Then contract with $A_{\mu\nu}$. In fact the operator basis does not even have to be unitary.

Problem 12.3.

$$H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{i\phi} & -1 & -e^{i\phi} \\ 1 & -1 & 1 & -1 \\ 1 & -e^{i\phi} & -1 & e^{i\phi} \end{bmatrix}. \tag{D.7}$$

Repeating this exercise for $N = 6$ requires much ingenuity [516].

Problem 12.4. The field with eight elements has irreducible polynomials $p_1(x) = x^3 + x + 1$ and $p_2(x) = x^3 + x^2 + 1$; α is a root of $p_1(x)$:

Element	Polynomial	tr	trx^2	order
$0 = 0$	x	0	0	–
$\alpha^7 = 1$	$x + 1$	1	1	1
$\alpha = \alpha$	$p_1(x)$	0	0	7
$\alpha^2 = \alpha^2$	$p_1(x)$	0	0	7
$\alpha^3 = \alpha + 1$	$p_2(x)$	1	1	7
$\alpha^4 = \alpha^2 + \alpha$	$p_1(x)$	0	0	7
$\alpha^5 = \alpha^2 + \alpha + 1$	$p_2(x)$	1	1	7
$\alpha^6 = \alpha^2 + 1$	$p_2(x)$	1	1	7

For the field with nine elements there are three irreducible polynomials (of degree 2) to choose from.

Problem 12.5. Applying (12.72) to the basis vectors we obtain (ignoring phase factors)

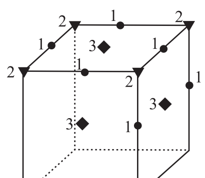
$$|z, a\rangle \to |z', a'\rangle = \left| \frac{\alpha z + \beta}{\gamma z + \delta}, \frac{a}{\gamma z + \delta} \right\rangle, \tag{D.8}$$

except that special measures must be taken if $z = \infty$ or $\gamma z + \delta = 0$.

Problem 12.6. This is a question of completing squares in exponents.

Problem 12.7. The bases from the Mermin square are real. Using the magical basis they are maximally entangled. The concurrence, when averaged over the vectors in a 2-design, is equal to its Fubini–Study average. One can then conclude that if there are two maximally entangled bases in the set, the rest must be separable [964].

Problem 12.8. A concise answer reads: $C_{ijk} = A_i \sigma_j B_k$. The indices run $i = 0, 1$; $j = 0, 1, 2, 3$ and $k = 0, 1, 2$ and the matrices are: $A_0 = \sigma_0 = B_0 = \mathbb{1}$, $A_1 = V$, $B_1 = H$ and $B_2 = HV$. These matrices (up to a phase belonging to the Clifford group) form the octahedral group and a unitary 3-design. They represent $2 \cdot 4 \cdot 3 = 24$ proper rotations of the cube: identity, $3 \cdot 3 = 9$ rotations around axes crossing the centres of opposite faces (⧫), $6 \cdot 1 = 6$ rotations around axes crossing the centres of opposite edges (•) and $4 \cdot 2 = 8$ rotations around axes passing by opposite corners (▼). The subset of 12 matrices corresponding to the tetrahedral group forms a 2-design and this number saturates the lower bound for the size $K_{\min}$ of a 2-design of unitary matrices of order two.



Octahedral group. Symbols represent axes crossing the centre of the cube labelled by the numbers of rotations around each axis.

Problem 12.9. A set of SIC vectors are

$$\begin{bmatrix} 0 & 0 & 0 & e^{i\phi} & \omega e^{i\phi} & \omega^2 e^{i\phi} & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & e^{i\phi} & \omega e^{i\phi} & \omega^2 e^{i\phi} \\ e^{i\phi} & \omega e^{i\phi} & \omega^2 e^{i\phi} & 1 & 1 & 1 & 0 & 0 & 0 \end{bmatrix}. \tag{D.9}$$

In no other (known) case is there a free parameter in a SIC. It was noted in 1844 that if we set $\phi = \pi$ there are 12 triples of linearly dependent SIC vectors, each of which is orthogonal to one of the 12 vectors in a complete set of MUB – although the language used then was that of cubic curves, not that of quantum theory [431].

Problem 13.1. Open with the observation that

$$\ln(A + xB) = \ln(A + xB + u_0) - \int_0^{u_0} \frac{\mathrm{d}u}{A + xB + u}. \tag{D.10}$$

Use the fact that $A + u$ is invertible for any positive u to rewrite this as

$$\ln(A + u_0) + \ln\left(\mathbb{1} + \frac{1}{A + u_0} xB\right) - \int_0^{u_0} \left[\frac{1}{A + u} - \frac{1}{A + u} xB \frac{1}{A + xB + u}\right] \mathrm{d}u.$$

Now do the integral over the first term, collect terms, expand the remaining logarithm, and finally let $u_0 \to \infty$.

Problem 13.2. The eigenvalues of $(\mathbb{1} - \rho/z)^{-1}$ are $z/(z - \lambda_i)$. With ρ diagonalised, and the contour chosen suitably, the first integral equals the von Neumann entropy and the second is known as subentropy [510],

$$S_Q(\rho) \equiv -\sum_{i=1}^{N} \left(\prod_{i \neq j} \frac{\lambda_i}{\lambda_i - \lambda_j}\right) \lambda_i \ln \lambda_i. \tag{D.11}$$

Problem 13.3.

$$\begin{aligned} \sum_k p_k S(\rho_k || \sigma) &= \sum_k p_k (\mathrm{Tr}\rho_k \ln \rho_k - \mathrm{Tr}\rho_k \ln \sigma) = \\ &= \sum_k p_k (\mathrm{Tr}\rho_k \ln \rho_k - \mathrm{Tr}\rho_k \ln \sigma + \mathrm{Tr}\rho_k \ln \rho - \mathrm{Tr}\rho_k \ln \rho) = \\ &= \sum_k p_k (\mathrm{Tr}\rho_k \ln \rho_k - \mathrm{Tr}\rho_k \ln \rho) + \mathrm{Tr}\rho \ln \rho - \mathrm{Tr}\rho \ln \sigma = \\ &= \sum_k p_k S(\rho_k || \rho) + S(\rho || \sigma). \end{aligned} \tag{D.12}$$

Problem 13.4. Write the Husimi function using the Schmidt decomposition $|\Psi\rangle = \sqrt{\lambda_1}|11\rangle + \sqrt{\lambda_2}|22\rangle$. Integration over the Cartesian product of two spheres gives

$$S_W(\Psi) = \frac{\lambda_1^2(1 - \ln \lambda_1)}{\lambda_1 - \lambda_2} + \frac{\lambda_2^2(1 - \ln \lambda_2)}{\lambda_2 - \lambda_1}. \tag{D.13}$$

Up to an additive constant this result is equal to the Wehrl entropy of one qubit mixed state obtained by partial trace [664] or to the subentropy (D.11) of this state.

Problem 13.5. It is enough to show that $\mathrm{Tr} d_1 d_2 \geq \mathrm{Tr} W d_1 W^\dagger d_2$, where $W = V^\dagger U$ is unitary. We can write this as an inequality for scalar products between vectors: $\vec{d}_1 \cdot \vec{d}_2 \geq (B\vec{d}_1) \cdot \vec{d}_2$ where B is unistochastic. Problem 2.2 shows that this is true.

Problem 13.6. Work in a basis where $A+B$ is diagonal. Note that $\big(\det(A+B)\big)^{1/N} = \prod_i (A_{ii} + B_{ii})^{1/N} \geq \prod_i A_{ii}^{1/N} + \prod_i B_{ii}^{1/N}$. For our purposes the second step is the more interesting: from the Schur–Horn theorem and the Schur concavity of the elementary symmetric functions it follows that $\prod_i A_{ii} \geq \det A$ (and similarly for B).

Problem 13.7. It is sufficient to compute $\mathrm{Tr} LL^\dagger$ using the representation $L = \sum_{i=1}^r A_i \otimes A_i^*$ of the superoperator.

Problem 14.1. We want to show that $||\mathbf{x}||^2 \geq ||B\mathbf{x}||^2$, so we must show that $\mathbb{1} - B^T B$ is a positive operator. This follows from the Frobenius–Perron theorem.

Problem 14.2. The probability distribution obtained from a density matrix is majorised by its eigenvalues. Helstrom's theorem implies that the trace distance between two density matrices is the maximum of the l_1-distance between the probability distributions that can be obtained from them. To maximise the trace distance the two density matrices must commute. If $\rho_2 = U\rho_1 U^\dagger$ and ρ_1 is diagonal then U must be a permutation matrix. See Figure 2.2.

Problem 14.3. We know that there is a POVM such that $\sqrt{F} = \sum_i \sqrt{p_i q_i}$, with probabilities given in Eq. (14.48). Then

$$2(1 - \sqrt{F}) = \sum_i (\sqrt{p_i} - \sqrt{q_i})^2 \leq \sum_i |\sqrt{p_i} - \sqrt{q_i}||\sqrt{p_i} + \sqrt{q_i}| = \\ = \sum_i |p_i - q_i| \leq 2D_{\mathrm{Tr}}(\rho, \sigma), \tag{D.14}$$

where Helstrom's theorem was used in the last step.

Problem 14.4. (a) This follows if we set $U = 1$ in the argument that led to the quantum Bhattacharyya coefficient. Equality holds if $[\rho, \sigma] = 0$.
(b) Equality holds if one of the states is pure. The inequality follows from this because of concavity. Incidentally, fidelity can also be bounded [668] as $E(\sigma, \rho) \leq F(\sigma, \rho) \leq G(\sigma, \rho)$, where the lower bound, called *subfidelity*, is $E(\sigma, \rho) = \mathrm{Tr}\sigma\rho + \sqrt{2}\sqrt{(\mathrm{Tr}\sigma\rho)^2 - \mathrm{Tr}\sigma\rho\sigma\rho}$, and the upper one, called *superfidelity*, is $G(\sigma, \rho) = \mathrm{Tr}\sigma\rho + \sqrt{1 - \mathrm{Tr}\rho^2}\sqrt{1 - \mathrm{Tr}\sigma^2}$. If $N = 2$ all three quantities are equal.

Problem 15.1. No, since $f_{\mathrm{WY}} = (f_{\max} + f_{\mathrm{geom}})/2$ where $f_{\mathrm{geom}} = \sqrt{t}$ is related to the geometric mean, $1/c_{\mathrm{geom}}(x, y) = \sqrt{xy}$. But it is an interesting one, because the Wigner–Yanase metric is simple, and geodesic distances can be given explicitly [332].

Problem 15.2. With $\Theta_N \equiv \prod_{k=1}^{N} \Gamma(k/2)$ we obtain:

Table D.1 *Volumes of orthogonal groups and real flag manifolds*

Manifold	Dimension	Vol[X], $a = 1/2$	Vol$'$[X], $a = 1$
$\mathbb{R}P^N$	N	$\frac{\pi^{(N+1)/2}}{\Gamma[(N+1)/2]}$	$2^{N/2}\frac{\pi^{(N+1)/2}}{\Gamma[(N+1)/2]}$
$\mathbf{F}_{\mathbb{R}}^{(N)} = \frac{O(N)}{[O(1)]^N}$	$N(N-1)/2$	$\frac{\pi^{N(N+1)/4}}{\Theta_N}$	$2^{N(N-1)/4}\frac{\pi^{N(N+1)/4}}{\Theta_N}$
$O(N)$	$N(N-1)/2$	$2^N\frac{\pi^{N(N+1)/4}}{\Theta_N}$	$2^{N(N+3)/4}\frac{\pi^{N(N+1)/4}}{\Theta_N}$
$SO(N)$	$N(N-1)/2$	$2^{N-1}\frac{\pi^{N(N+1)/4}}{\Theta_N}$	$2^{(N(N+3)/4-1}\frac{\pi^{N(N+1)/4}}{\Theta_N}$

Problem 15.4. Integrating over respective distributions we obtain $\langle S(\rho)\rangle_{\mathrm{HS}} = 1/3$, $\langle S(\rho)\rangle_{\mathrm{B}} = 2 - 7\ln 2/6$, $\langle S(\rho)\rangle_o = 2 - \ln 2$ and $\langle S(\rho)\rangle_u = \ln 2/2$.

Problem 15.5. The averages (computed by Malacarne et al. [641]) read

$$\langle \mathrm{Tr}\rho^3\rangle_{N,K} = \frac{(K+N)^2 + KN + 1}{(KN+1)(KN+2)}, \quad \langle \mathrm{Tr}\rho^4\rangle_{N,K} = \frac{(K+N)[(K+N)^2 + 3KN + 5]}{(KN+1)(KN+2)(KN+3)}. \tag{D.15}$$

Problem 15.6. Fidelity between pure states is equal to the squared component of $|\phi\rangle$ expanded in a basis containing $|\psi\rangle$. One finds $P_N(F) = (N-1)(1-F)^{N-2}$ (see Section 7.6 on random pure states).

Problem 15.7. Average fidelities, $\langle F\rangle_{\mathrm{HS}} = 1/2 + 9\pi^2/512 \approx 0.6735$ and $\langle F\rangle_{\mathrm{B}} = 1/2 + 8/(9\pi^2) \approx 0.590$, exceed the average over two random pure states, $\langle F\rangle_{\mathrm{FS}} = 1/2$ [1018].

Problem 15.9. Making use of an integral with respect to the Marchenko–Pastur distribution, $\int_0^4 [-x \ln x MP(x)]dx = -1/2$ and returning to the original variable, $\lambda = Nx$, we arrive at the asymptotic result $\langle S\rangle_{HS} \to \ln N - 1/2$. Asymptotic average purity $2/N$ is consistent with the second moment of the MP distribution, $\int_0^4 x^2 MP(x)dx = 2$. Higher traces $\langle \mathrm{Tr}\rho^k\rangle_{\mathrm{HS}}$ are given by higher moments of the MP distribution, equal to the *Catalan numbers*, $C_k = 1, 2, 5, 14, 42, \ldots$

Problem 15.10. Following [157] notice that since $\lambda_1 \geq x$ the spectrum λ is majorised by the vector $\nu = \{\lambda_1, (1-\lambda_1)/(N-1), \ldots (1-\lambda_1)/(N-1)\}$. The Schur concavity of the entropy function implies that $S(\rho) \leq S(\nu) = (1-\lambda_1)\ln(N-1) + S(\lambda_1, 1-\lambda_1) \equiv t(\lambda_1)$. The derivative $\partial t(\lambda_1)/\partial\lambda_1$ is negative for $\lambda_1 > 1/N$, so the function $t(\lambda_1)$ is monotonously decreasing for $\lambda_1 > 1/N$. Thus $t(\lambda_1) \geq t(x)$ which completes the proof.

Problem 16.1. The complete orthogonality graph must have $(2N-1)(2N-2)/2$ links. Any one of Alice's vectors can supply at most $N-1$ links, otherwise there would be linear dependencies among her vectors. The total number of links she can contribute is then the integer part of $(2N-1)(N-1)/2$, which is less than half of the total number if N is even. And Bob cannot do better.

Problem 16.3. The Schmidt vector of the state $|\phi_+\rangle^{\otimes m}$ consists of N^m components equal to N^{-m} each. The state consisting of n copies of the initial state $|\psi\rangle$ may be, for large n, approximated by $K = \exp[nE(|\psi\rangle)]$ terms in the Schmidt decomposition described by the vector $\vec{\lambda}$. Choosing $m \approx n[E(|\psi\rangle)]/\ln N$ we see that $\{N^{-m}, \ldots, N^{-m}\} \prec \{\lambda_1, \ldots, \lambda_K\}$. Thus Nielsen's majorisation theorem implies that such a conversion may be done [689]. Asymptotically the reverse transformation is also possible [115], so for any pure state the distillable entanglement is just equal to entanglement entropy, $E_D(|\psi\rangle) = E(|\psi\rangle)$.

Problem 16.4. Write a separable state in its eigenbasis, $\rho = \sum_j \lambda_j |\Psi_j\rangle\langle\Psi_j|$ and in its decomposition into pure product states, $\rho = \sum_i p_i |\phi_i^A\rangle\langle\phi_i^A| \otimes |\phi_i^B\rangle\langle\phi_i^B|$. Write the partial trace $\rho_A = \sum_i p_i |\phi_i^A\rangle\langle\phi_i^A|$ in its eigenbasis, $\rho_A = \sum_k \lambda_k^A |k\rangle\langle k|$. Apply Schrödinger's mixture theorem (8.39) twice, substituting $\sqrt{p_i}|\phi_i^A\rangle = \sum_k V_{ik}\sqrt{\lambda_k^A}|k\rangle$ into $\sqrt{\lambda_j}|\Psi_j\rangle = \sum_i U_{ji}\sqrt{p_i}|\phi_i^A\rangle|\phi_i^B\rangle$, where U and V are unitary. Multiply the result by its adjoint and obtain $\lambda_j = \sum_k B_{jk}\lambda_k^A$ making use of the orthonormality, $\langle k|k'\rangle = \delta_{k,k'}$. Show that B is bistochastic, what implies (16.67) due to HLP lemma.

Problem 16.6. Obviously not. A simple dimension counting will do. In the $N \times N$ problem the set of separable states has $N^4 - 1$ dimensions, while the set of locally diagonalizable states forms a $3(N^2-1)$ dimensional subset. (The local unitaries and the diagonalised state contribute $N^2 - 1$ parameters each.) Note that it contains the set of all product mixed states of dimension $2N^2 - 2$, equal to the dimension of the set of all pure states of the bipartite system.

Problem 16.8. This inequality follows from condition $\text{Tr}\rho^2 \leq 1$. It is not sufficient for positivity, but may be accompanied by additional inequalities involving higher traces $\text{Tr}\rho^k$, with $k = 3, 4, \ldots$ [529, 798].

Problem 16.9. Partial trace induces the HS measure (15.35) with $\lambda_1 = \cos^2\chi$. Change of variables provide the required distributions, while integrations give the expectation values $\langle \chi \rangle_{\mathbb{C}\mathbf{P}^3} = 1/3$ and $\langle C \rangle_{\mathbb{C}\mathbf{P}^3} = 3\pi/16$. The distribution $P(\chi)$ achieves maximum at $\chi_m = \arccos\left[\sqrt{1/2 + 1/\sqrt{6}}\right]$, while it is most likely to find a two-qubit random pure state with concurrence $C_m = 1/\sqrt{2}$.

Problem 16.10. The spectrum of the partial transpose ρ^{T_A} of a generic $N \times N$ state is asymptotically described by the shifted semicircular law (16.60). Hence the average negativity, $\mathcal{N}_T(\rho) = \mathrm{Tr}|\rho^{\mathrm{T}_A}| - 1$, can be expressed by an integral, $\langle \mathcal{N}_T \rangle = \int_{-1}^{0} \frac{|x|}{P}{}_{\mathrm{PT}}(x)dx = \frac{3\sqrt{3}}{4\pi} - \frac{1}{3} \simeq 0.080$. This number, compared with the negativity of the maximally entangled state, $\mathcal{N}(\rho_+) = N - 1$, shows that a generic mixed state is only weakly entangled.

Problem 17.1. For the GHZ state $\mathrm{Tr}_A|GHZ\rangle\langle GHZ| = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$, a mixture of two separable states. For the W state $\mathrm{Tr}_A|W\rangle\langle W| = \frac{1}{3}|00\rangle\langle 00| + \frac{2}{3}|\Phi^+_{BC}\rangle\langle\Phi^+_{BC}|$, an entangled mixture of a Bell state and a separable state. Its tangle equals 4/9, and the spectrum of $\mathrm{Tr}_{AB}|W\rangle\langle W|$ equals $(2/3, 1/3)$.

Problem 17.2. Expression (17.9) consists of three terms, so the rank r of $|W\rangle$ cannot be larger than three. For any three-qubit state of rank two its partial trace, e.g. $\rho_{AB} = \mathrm{Tr}_C|\psi_{ABC}\rangle\langle\psi_{ABC}|$, contains in its support at least two linearly independent product states. For $|W_3\rangle$, the reduced state ρ_{AB} analysed in Problem 17.1 contains only a single product state $|00\rangle$, so $r(|W\rangle) > 2$. The situation changes if one adds an ϵ contribution of the state $|111\rangle$. Then the reduced state ρ_{AB} supports *two* product states, $|00\rangle$ and $|11\rangle$, so the rank r *decreases* to two.

Problem 17.3. All one-party reduced density matrices are equal to

$$\rho = \begin{pmatrix} (K-k)/K & 0 \\ 0 & k/K \end{pmatrix} \tag{D.16}$$

(as can be proved by induction). For $K = 2k$ this is the maximally mixed state. The difference to the GHZ state is that entangled density matrices are encountered for bipartite reductions.

Problem 17.4. In the factors a general state is given by $|\psi_A\rangle = \cos\frac{\theta_A}{2}|0_A\rangle + \sin\frac{\theta_A}{2}e^{i\phi_A}|1_A\rangle$ etc. Write out $|\langle\psi_A|\langle\psi_B|\langle\psi_C|\psi\rangle|^2$. Take derivatives with respect to $\theta_A, \theta_B, \theta_C$. By assumption they vanish at $\theta_A = \theta_B = \theta_C = \pi$. The conclusion follows.

Problem 17.5. (a) Bernstein [126] revealed that there exist cases in which three variables X, Y, Z are pairwise independent but not independent. This is the case here. Thus the probability to register three '0' on a card is $p_{XYZ} = 0 \neq p_X p_Y p_Z$.
(b) $|\psi_\triangle\rangle$ is locally equivalent to $|GHZ\rangle$. (An easy way to check this is to use the stellar representation.) The statistics of the outcomes of measurements of the operators $\sigma_z^1, \sigma_z^2, \sigma_z^3$ coincide with that of the Bernstein cards [178].

Problem 17.6. The basis of eight orthogonal entangled states in $\mathcal{H}_{\in}^{\otimes 3}$ reads

$$|\Psi_{++}\rangle = [|000\rangle + |011\rangle + |101\rangle + |110\rangle]/2, \quad |\Psi_{+-}\rangle = [|000\rangle + |011\rangle - |101\rangle - |110\rangle]/2,$$
$$|\Psi_{-+}\rangle = [|000\rangle - |011\rangle + |101\rangle - |110\rangle]/2, \quad |\Psi_{--}\rangle = [|000\rangle - |011\rangle - |101\rangle + |110\rangle]/2,$$
$$|\Phi_{++}\rangle = [|111\rangle + |100\rangle + |010\rangle + |001\rangle]/2, \quad |\Phi_{+-}\rangle = [|111\rangle + |100\rangle - |010\rangle - |001\rangle]/2,$$
$$|\Phi_{-+}\rangle = [|111\rangle - |100\rangle + |010\rangle - |001\rangle]/2, \quad |\Phi_{--}\rangle = [|111\rangle - |100\rangle - |010\rangle + |001\rangle]/2.$$

Problem 17.7. (a) For $|GHZ_4\rangle$ all three spectra of reduced states are equal, $\Lambda_{AB} = \Lambda_{AC} = \Lambda_{AD} = \{1/2, 1/2, 0, 0\}$ so that $\bar{S}_1 = \bar{S}_2 = \bar{S}_\infty = \log 2 \approx 0.693$. (b) Two local spectra of $|B_4\rangle$ are uniform, $\{1/4, 1/4, 1/4, 1/4\}$, but the third one reads $\{1/2, 1/2, 0, 0\}$. Thus $\bar{S}_1 = \bar{S}_2 = \bar{S}_\infty = \frac{5}{3}\log 2 \approx 1.155$. (c) For the cluster state $|C_4\rangle$ the local spectra (and the mean entropies) are the same as both states are locally equivalent. (d) For the Higuchi–Sudbery state $|HS\rangle$ all three local spectra read $\{3, 1, 1, 1\}/6$, so $\bar{S}_1 = \log 2 + \frac{1}{2}\log 3 \approx 1.242$ and $\bar{S}_2 = \log 3 \approx 1.099$, while $\bar{S}_\infty = \log 2$. All the mean entropies are smaller than $\log 4 \approx 1.386$, as there are no AME states of four qubits. The notion of ‘maximally entangled’ then depends on the measure used: $|HS\rangle$ is conjectured to maximise von Neumann entropy $\bar{S}_1$ (and gives a local maximum [173]), while the mean Rényi entropy $\bar{S}_2$ (and the linear entropy $\bar{S}_2^{HC}$) is maximised by $|C_4\rangle$, as it leads to the minimal purity of the reduced state [354].

Problem 17.8. The state $|\Phi_3^6\rangle$ is a nice example of an absolutely maximally entangled state: for any partition of six subsystems into two triples the reduced density matrix on three subsystems is maximally mixed [148]. Hence the state is 3-uniform.

Problem 17.9. By inspection we see that the state is invariant under $\mathbb{1}XXX^2$, $X\mathbb{1}XX$, $X^2X\mathbb{1}X$, $XXX^2\mathbb{1}$ and under $\mathbb{1}ZZZ^2$, $Z\mathbb{1}ZZ$, $Z^2Z\mathbb{1}Z$, $ZZZ^2\mathbb{1}$, where X and Z are generators of the Heisenberg group $H(3)$. All elements cube to the unit element, all elements commute, and from these generators we can construct altogether 81 ‘words’, which is the stabilizer group for this state.

Problem 17.10. The Hamming code corrects all single errors, and no double errors. The probability that the message comes through correctly is therefore $(1-p)^7 + 7p(1-p)^6$, which is larger than $(1-p)^4$ for all $p < 1/2$. The repetition code is able to correct some double and triple errors, but gives no improvement unless $p < 0.151$.

Problem 17.11. The matrix U_9 describing (17.76) is a permutation matrix obtained by taking an arbitrary single digit (e.g. **1**) from the sudoku matrix S, while the orthogonal matrix $U_8 = \frac{1}{\sqrt{8}}H_8$ is proportional to the Hadamard matrix with entries equal ± 1. Note that S enjoys a special property: apart from the standard sudoku condition that each digit appears only once in each row, each column and each block, a stronger condition holds. Each digit appears only once in each location (position inside the block), in each broken row (e.g. three upper rows of three left blocks) and each broken column (e.g. three left columns of three lower blocks) – see below. This ensures that $U_9^{T_2}$ and U_9^R are also permutation matrices, so U_9 is multiunitary [356].

$$S = \left[\begin{array}{ccc|ccc|ccc} 8 & \mathbf{1} & 6 & 2 & 4 & 9 & 5 & 7 & 3 \\ 3 & 5 & 7 & 6 & 8 & \mathbf{1} & 9 & 2 & 4 \\ 4 & 9 & 2 & 7 & 3 & 5 & \mathbf{1} & 6 & 8 \\ \hline 7 & 3 & 5 & \mathbf{1} & 6 & 8 & 4 & 9 & 2 \\ 2 & 4 & 9 & 5 & 7 & 3 & 8 & \mathbf{1} & 6 \\ 6 & 8 & \mathbf{1} & 9 & 2 & 4 & 3 & 5 & 7 \\ \hline 9 & 2 & 4 & 3 & 5 & 7 & 6 & 8 & \mathbf{1} \\ \mathbf{1} & 6 & 8 & 4 & 9 & 2 & 7 & 3 & 5 \\ 5 & 7 & 3 & 8 & \mathbf{1} & 6 & 2 & 4 & 9 \end{array}\right], \quad H_8 = \begin{pmatrix} - & - & - & + & - & + & + & + \\ - & - & - & + & + & - & - & - \\ - & - & + & - & - & + & - & - \\ + & + & - & + & - & + & - & - \\ - & + & - & - & - & - & + & - \\ + & - & + & + & - & - & + & - \\ + & - & - & - & + & + & + & - \\ + & - & - & - & - & - & - & + \end{pmatrix}. \tag{D.17}$$

Problem 17.12. (a) Using the canonical form (17.95) one finds

$$A^{i_k} = \left\{A^0, A^1\right\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\frac{k-1}{k}} \end{pmatrix}, \begin{pmatrix} 0 & \frac{1}{\sqrt{k}} \\ 0 & 0 \end{pmatrix} \right\}. \tag{D.18}$$

and the vectors at the end are easy to find; (b) in the periodic case (17.98) one can take

$$A^{i_k} = \left\{A^0, A^1\right\} = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\} \text{ if } k < K, \tag{D.19}$$

$$A^{i_K} = \left\{A^0, A^1\right\} = \left\{ \frac{1}{\sqrt{K}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{K}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \right\}; \quad \text{c) } D = N.$$