Anderson localization of cold atoms in a disordered speckle potential

Introduction : Atomic ultra-cold matter waves may interfere, as exemplified by the development of atomic interferometry during the last 20 years. This problem studies how to create a specifically tailored disorder acting on the external motion of cold atoms and how such a disorder may influence this motion, with emphasis on Anderson localization.

In the first part, you have to calculate the statistical properties of a "speckle" pattern created by a coherent laser beam transmitted through a diffusive glass plate. In the second part, you have to use these statistical properties to study the dynamics of a cold atomic gas placed in such a speckle pattern. **The two parts are essentially independent.**

Part 1: Statistical properties of a speckle pattern

We consider the situation depicted in figure 1, where a monochromatic laser beam (approximated by a plane wave with wave-vector k_0 along the z-axis) is diffracted by a square aperture (side L) in the plane $z = 0$. We will assume that all light sources have the same polarization, so that we can forget about light polarization and describe the electromagnetic field by a single complex variable $\mathcal{E}$.

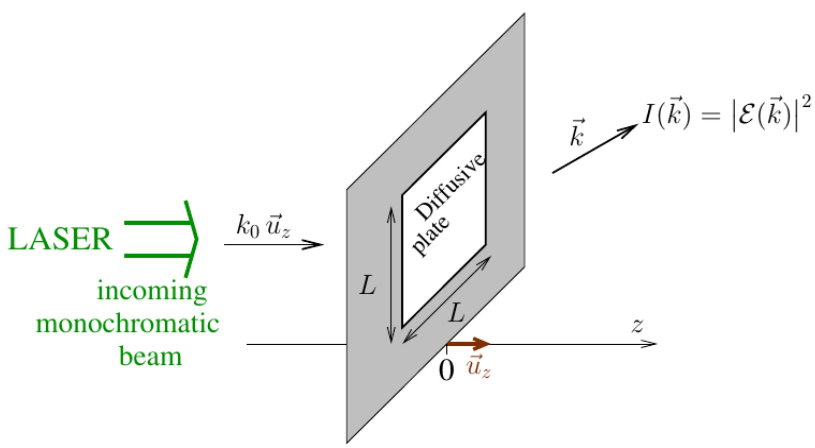


Figure 1: *A monochromatic laser beam along the z-axis is sent on a square aperture of size L in the $z = 0$ plane. We consider the diffracted field at large distance in the $\vec{k}$ direction, close to the initial direction (paraxial approximation).*

Fraunhofer regime.– In a direction close to the $z-$axis (in the paraxial approximation) one can use the Fraunhofer formula, relating the amplitude of the field at the level of the aperture ($z = 0$), denoted $A(x, y)$, to the amplitude of the field diffracted in the $\vec{k}$

direction (with $k_x,\ k_y \ll k_0$ and $k_z \approx k_0$) :

$$\mathcal{E}(\vec{k}) = \int_{-L/2}^{L/2} \mathrm{d}x \int_{-L/2}^{L/2} \mathrm{d}y\, A(x,y)\ \exp\left[\mathrm{i}(k_x x + k_y y)\right]\ . \tag{1}$$

A. Warm up : diffraction by a square aperture.– We want to study the amplitude of the wave diffracted by a square aperture, **in the absence of any diffusive plate**, i.e. $A(x,y) = \mathcal{A}$ is constant. Compute $\mathcal{E}(\vec{k})$ and the corresponding intensity $I(\vec{k}) = |\mathcal{E}(\vec{k})|^2$. It may be useful to introduce the so-called sinc function : $\mathrm{sinc}(x) \stackrel{\text{def}}{=} \sin x/x$.

B. Diffraction by a diffusive plate.– We now add on the square aperture a diffusive glass plate whose effect is to modify the *phase* of $A(x,y)$ differently at each position (x,y). We assume that the phase (mod. 2π) is a random variable, homogenously distributed in $[0, 2\pi[$, and which is moreover uncorrelated between different positions.

1/ Distribution of the electric field.– Justify, with these hypotheses, the following average values:

$$\begin{aligned} \langle A(\vec{r})\rangle &= 0 \\ \langle A(\vec{r})A(\vec{r}\,')\rangle &= 0 \\ \langle A(\vec{r})A^*(\vec{r}\,')\rangle &= \mathcal{A}^2\ \eta^2\ \delta_\eta(\vec{r}-\vec{r}\,') \end{aligned}$$

where δ_η is a (normalized, i.e. integral=1) 2D Dirac "distribution" of width η in each direction (x and y).

Explain why the wave $\mathcal{E}(\vec{k})$ diffracted in direction $\vec{k}$ is a complex random variable with a **Gaussian** distribution with zero mean, $\langle\mathcal{E}(\vec{k})\rangle = 0$.

Hint : you can discretize expression (1) as $\mathcal{E}(\vec{k}) = \eta^2 \sum_{n=1}^{N} A(\vec{r}_n)\,\mathrm{e}^{\mathrm{i}\vec{k}\cdot\vec{r}_n}$, where $\vec{r}_n$ are $N = (L/\eta)^2$ discrete and regular positions on the square (hence $\langle A(\vec{r}_n)A^*(\vec{r}_m)\rangle = \mathcal{A}^2\,\delta_{n,m}$), and use a well-known theorem of probability theory.

Justify that $\mathcal{X} = \mathrm{Re}(\mathcal{E})$ and $\mathcal{Y} = \mathrm{Im}(\mathcal{E})$ have the same variance and are uncorrelated.

2/ Rayleigh law.– Deduce that the intensity $I(\vec{k}) = |\mathcal{E}(\vec{k})|^2$ is distributed according to an exponential law :

$$P(I) = \frac{\exp\left(-I/I_0\right)}{I_0} \qquad \text{for } I \geqslant 0 \tag{2}$$

where I_0 is the average intensity. What is the most probable intensity? Compute the variance of the intensity.

3/ Field correlations.– We are interested in the correlation function of the diffracted amplitude and intensity in different directions $\vec{k}$ and $\vec{k}'$.

a) Show that $\langle\mathcal{E}(\vec{k})\mathcal{E}(\vec{k}')\rangle = 0$. Compute the correlation function $\langle\mathcal{E}(\vec{k})\mathcal{E}^*(\vec{k}')\rangle$ and express it as a function of $\Delta\vec{k} = \vec{k}-\vec{k}'$, L and I_0.

Hint : use the hint of question **1/**.

b) Using the Wick's theorem, show that:

$$\langle I(\vec{k})I(\vec{k}')\rangle = I_0^2\ +\ \left|\langle\mathcal{E}(\vec{k})\mathcal{E}^*(\vec{k}')\rangle\right|^2 \tag{3}$$

c) Show that one recovers the variance of the intensity previously computed. Extend the argument for the higher moments $\langle I(\vec{k})^n \rangle$. Compare with the Rayleigh law found above.

4/ Spatial speckle pattern.– The setting under study can be used in order to create a *spatially* (instead of angularly) correlated laser intensity.

a) Explain briefly how, using a lens of focal length f, it is possible to create such a potential in a convenient plane. If we denote by $\vec{r} = (x, y)$ the coordinate in this plane, how a variation of direction $\Delta\vec{k}$ is related to a variation in position $\Delta\vec{r}$ in this plane (we recall that $k_x,\ k_y \ll k_z \approx k_0$)?

b) Suppose that one puts a gas of non-interacting cold atoms in this plane, and that the laser frequency $\omega = c||\vec{k}||$ is quasi-resonant with an atomic resonance transition at frequency ω_0. Briefly explain the relevant mechanisms of the atom-light interaction and why, if the detuning $\delta = \omega - \omega_0$ is large compared to the width Γ of the atomic resonance, the speckle pattern acts as an effective "optical" potential V which depends on the position $\vec{r}$ of the center of mass of the atom. Show that the correlation function of the potential is given by :

$$\langle V(\vec{r})V(\vec{r}+\Delta\vec{r})\rangle = V_0^2 \left[1 + \mathrm{sinc}^2\left(\frac{\Delta x}{\sigma}\right)\mathrm{sinc}^2\left(\frac{\Delta y}{\sigma}\right)\right] \tag{4}$$

where σ is the correlation length of the speckle. Express σ as a function of the laser wavelength $\lambda = 2\pi/k_0$, L and f. Considering the approximations used in the calculation, what is the order of magnitude of the shortest correlation length achievable?

C. Diffraction by a Gaussian aperture.– We now consider the case where there is no aperture, but the incoming laser beam has a Gaussian distribution of amplitude in the $z = 0$ plane : $|A(x, y)| = \mathcal{A}\exp\left(-\frac{x^2+y^2}{2r_0^2}\right)$.

1/ By following the same lines as in question **B.3/**, deduce the field correlation $\langle \mathcal{E}(\vec{k})\mathcal{E}^*(\vec{k}')\rangle$ in this case.

2/ What would be the corresponding correlations replacing Eq. (4)?

Part 2: Cold atoms in a 1D speckle potential

We want to study the dynamics of **one atom** of mass m exposed to a spatially disordered potential $V(\vec{r})$ whose correlation function is given by eq. (4). We assume that some external constraint forces the atom to move only along the x-axis, so that the entire **dynamics is one-dimensional** with Hamiltonian :

$$H = H_0 + V(x) = \frac{p_x^2}{2m} + V(x) \tag{5}$$

with $\langle V(x)\rangle = V_0$ and

$$\langle V(x)V(x+\Delta x)\rangle = V_0^2\left[1 + \mathrm{sinc}^2\left(\frac{\Delta x}{\sigma}\right)\right] \tag{6}$$

A. Structure of the Green functions

1/ *a*) We introduce the retarded Green function $G_0(E) = 1/(E - H_0 + \mathrm{i}0^+)$ (we omit the "R" index to lighten). Express the Green function in k-space, $\widetilde{G}_0(k, k'; E) \stackrel{\text{def}}{=} \langle \phi_k | G_0(E) | \phi_{k'} \rangle$,

Hint : $\phi_k(x) = \langle x | \phi_k \rangle = (1/\sqrt{L})\mathrm{e}^{\mathrm{i}kx}$ denotes a plane wave in finite volume L, with quantized wave vectors $k = 2n\pi/L$ for $n \in \mathbb{Z}$, and orthonormalisation $\langle \phi_k | \phi_{k'} \rangle = \delta_{k,k'}$ (thus $\sum_k \leftrightarrow \frac{L}{2\pi} \int \mathrm{d}k$).

b) Recover the corresponding expression in x-space,

$$G_0(x, x'; E) = \frac{m}{\mathrm{i}\,\hbar^2 k_E} \exp(\mathrm{i}k_E|x - x'|) \qquad \text{where } k_E \stackrel{\text{def}}{=} \sqrt{2mE}/\hbar\,. \tag{7}$$

2/ Recall the relation between the Green function $G(E)$ (for Hamiltonian H), $G_0(E)$ and V. Briefly explain the notion of average Green function $\overline{G}(E)$. The self-energy $\Sigma(E)$ is the operator such that :

$$\overline{G}(E) = \frac{1}{G_0^{-1}(E) - \Sigma(E)} \tag{8}$$

Recall the Dyson equation satisfied by $\Sigma(E)$. Give a physical argument to explain that the self energy is diagonal in the momentum representation : $\langle \phi_k | \Sigma(E) | \phi_{k'} \rangle = \Sigma(k, E)\, \delta_{k,k'}$.

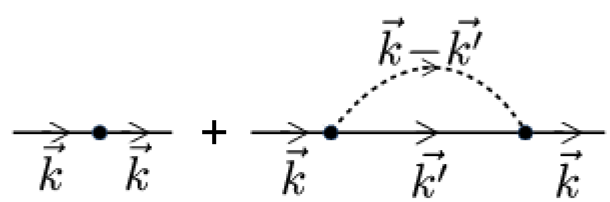
3/ We consider the weak disorder regime $|\Sigma(k, E)| \ll |E|$. We assume that the self-energy is a *smooth* function of k and E. What is the physical interpretation of $\mathrm{Re}\left[\Sigma(k, E)\right]$ and $\mathrm{Im}\left[\Sigma(k, E)\right]$? Show that the average Green function in x-space can be written as :

$$\overline{G}(x, x'; E) \approx G_0(x, x'; E) \exp\left(-\frac{|x - x'|}{2\ell(E)}\right) , \tag{9}$$

where $\ell(E)$ can be expressed in terms of $\Sigma(k_E, E)$. What is the physical meaning of $\ell(E)$?

B. Born approximation.– It is possible to perform a diagrammatic expansion of $\Sigma(k, E)$ in powers of the disorder strength V_0. Up to second order, it is :

$$\Sigma(k, E) = \tag{10}$$



1/ Explain the meaning of the various parts of these diagrams.

2/ Compute $\Sigma(k, E)$ at first order in V_0. Interpret the result.

3/ Write the integral expression for the second order term (the so-called "Born approximation"). Show that the imaginary part of the "on-shell" self-energy (that is for $k_E = \sqrt{2mE}/\hbar$) is given by :

$$\mathrm{Im}\,\Sigma(k_E, E) \simeq -\frac{mV_0^2}{2\hbar^2 k_E}[\mathcal{C}(0) + \mathcal{C}(2k_E)] \tag{11}$$

where $\mathcal{C}(k)$ is the Fourier transform of the correlation function of the potential

$$V_0^2\ \mathcal{C}(k) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} \mathrm{d}y\ \left[\langle V(x_0)V(x_0 + y)\rangle - \langle V(x_0)\rangle^2\right]\ \mathrm{e}^{-\mathrm{i}ky} \tag{12}$$

Deduce the expression for $1/k_E\ell(E)$ where $\ell(E)$ is the scattering mean free path. Which physical phenomena are respectively associated with $\mathcal{C}(0)$ and $\mathcal{C}(2k_E)$?

C. Localization length

1/ Preliminary : a Fourier transform.– By a simple argument (no need of an explicit calculation), justify that $\mathcal{C}(k)$ deduced from (6) is a triangle function $\propto (1-|k|/a)\,\theta_{\mathrm{H}}(1-|k|/a)$, where θ_{H} is the Heaviside function, and give the width a.

Hint : Fourier transform of the $\mathrm{sinc}(x)$ function is the "door" function $\pi\,\theta_{\mathrm{H}}(1-|k|)$.

2/ *a*) Explain why, when placed inside the 1D speckle potential, a cold atom is expected to be Anderson localized. When scattering by the disorder is isotropic ($\mathcal{C}(k)$ is independent on k), recall how the localization length ξ_{loc} is related to the scattering mean free path ℓ.

b) When scattering is not isotropic (i.e. forward and backward scattering have different probabilities), the simple relation discussed in **2.***a* must be understood as a relation between the localization length ξ_{loc} and the *transport* mean free path ℓ_{tr}. The transport mean free path ℓ_{tr} is obtained by changing $\mathcal{C}(0)\to\mathcal{C}(2k)$ in the expression of the mean free path ℓ (we now simplify the notation as $k_E\to k$ and omit the E dependences).

c) Show that the inverse of the localization length is given by :

$$\frac{1}{\xi_{\mathrm{loc}}} \simeq \frac{\pi\, m^2\, V_0^2\, \sigma\, (1-k\sigma)\ \theta_{\mathrm{H}}(1-k\sigma)}{\hbar^4 k^2}\,. \tag{13}$$

d) Fig. 2 shows the numerically computed inverse localization length $1/\xi_{\mathrm{loc}}^{(\mathrm{num})}$ *versus* $k\sigma$ [1]. Comment this figure. Especially, what is the problem around $k\sigma=1$? How should the above analysis be adapted in order to explain the numerics for $k\sigma>1$?

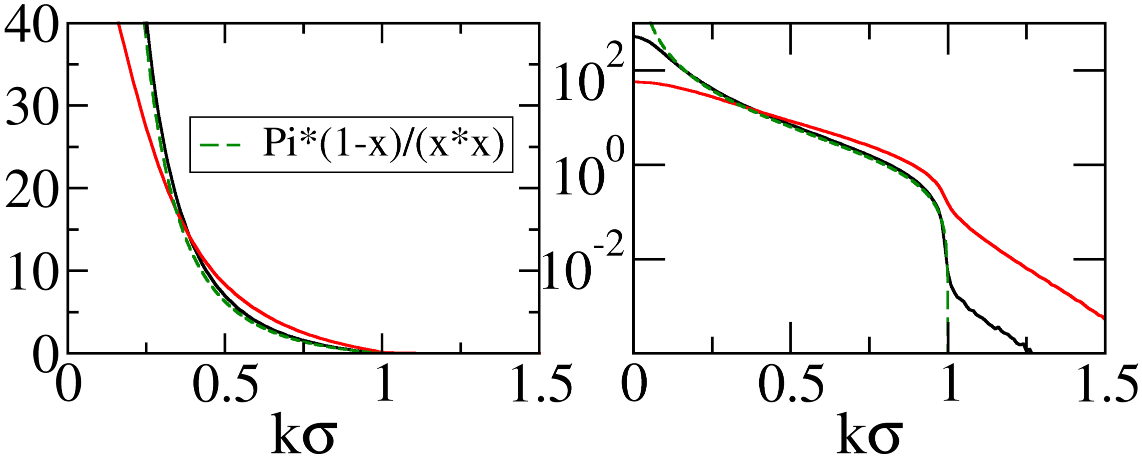


Figure 2: *Numerically computed inverse localization length for a 1D particle in a disordered "speckle" potential, versus. the parameter $k\sigma$. The dimensionless plotted quantity is $(\sigma/\xi_{\mathrm{loc}})\ (\hbar^4/m^2\sigma^4V_0^2)$ in linear scale (left plot) and log scale (right plot), for two values of the disorder strength $V_0 m\sigma^2/\hbar^2 = 0.01$ (black curve) and 0.05 (red curve). The dashed green curve is $\pi(1-x)/x^2$. Taken from [1].*

D. Experimental results Using a cold atomic cloud of Rubidium and a speckle optical potential, Billy *et al.* [2] have observed 1D Anderson localization. Figure 3 shows the experimental results, where the measured localization length is plotted versus the disorder strength. The energy of the atoms was such that $k\sigma \approx 0.65$ and $\sigma \approx 0.26\mu$m. Comment these experimental results. What is the order of magnitude of the product $k\ell$? What will happen if the same experiment is performed in 2D or 3D?

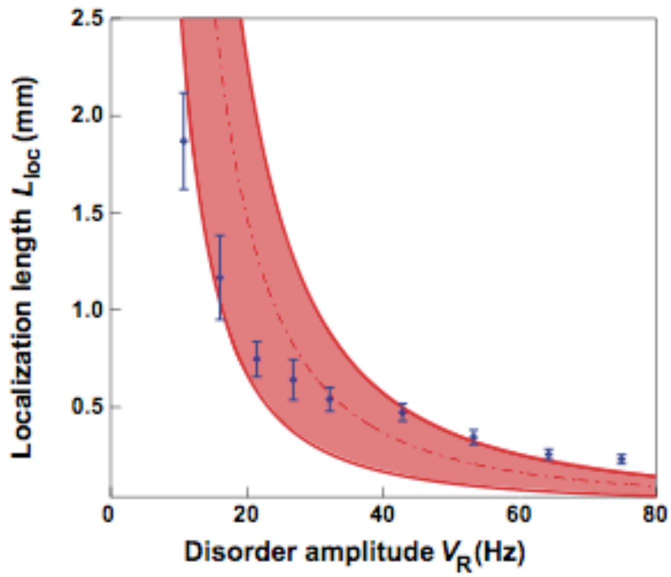


Figure 3: *Localization length for a cloud of cold Rubidium atoms in a disordered "speckle" optical potential, versus the strength V_0 (here noted V_R) of the disorder. Experimental parameters are such that $k\sigma \approx 0.65$ and $\sigma \approx 0.26\mu$m. The experimental results with error bars are the blue points, the prediction of eq.(13) shown by the red dash-dotted line. The shaded area reflects the uncertainties on the parameters. Taken from Ref. [2].*

References

[1] P. Lugan, A. Aspect, L. Sanchez-Palencia, D. Delande, B. Grémaud, C. A. Müller and C. Miniatura "*One-dimensional Anderson localization in certain correlated random potentials*", Phys. Rev. A **80**, 023605 (2009) [arxiv:0902.0107].

[2] J. Billy, V. Josse, Z. Zuo, A. Bernard, B. Hambrecht, P. Lugan, D. Clément, L. Sanchez-Palencia, P. Bouyer and A. Aspect, "*Direct observation of Anderson localization of matter-waves in a controlled disorder*", Nature **453**, 891 (2008).