Periodicity of the Aharonov-Bohm Effect in Normal-Metal Rings

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(Received 17 September 1985)

We study the Aharonov-Bohm effect in single normal-metal rings and show that averaging the conductance over many energies is equivalent to ensemble averaging. Thus raising the temperature T above a crossover T_c changes the flux periodicity of magnetoresistance oscillations from h/e to $h/2e$. T_c is determined by an energy correlation range, hD/L^2. The persistence of the h/e oscillations to high fields is explained.

PACS numbers: 72.20.My, 71.55.Jv, 72.15.Eb

The Aharonov-Bohm (AB) effect in normal-metal conductors has been the object of intense experimental and theoretical interest since the pioneering work of Al'tshuler, Aronov, and Spivak (AAS)[1,2] and Sharvin and Sharvin.[3] The effect manifests itself in metals as small magnetoresistance oscillations on top of the Ohmic residual resistance in multiply connected conductors at very low temperatures. These oscillations are periodic in the flux through the hole in the conductor and were predicted by AAS to have periodicity $h/2e$ in both rings and cylinders, unlike the AB effect for free electrons, which has periodicity h/e. The AAS effect was observed by Sharvin and Sharvin[2,3] in cylinders, and later by various groups in both cylinders[4] and arrays of rings.[5] Until recently no periodic oscillations had been clearly observed in *single* metal rings because such oscillations were hidden by aperiodic magnetoconductance fluctuations of order e^2/h which are universal in small metallic conductors.[6,7] However, Webb *et al.*[8] have now unambiguously observed oscillations with period h/e in small gold rings which persist for thousands of cycles. On the other hand, very recently Chandrasekhar *et al.*[9] and Umbach *et al.*[10] have seen $h/2e$ oscillations in single aluminum and silver rings at low magnetic fields and weaker h/e oscillations at higher fields. We propose an explanation for this diversity of experimental behavior below.

The AAS theory, which predicts $h/2e$ oscillations that decay rapidly at high fields, is based on the weak localization theory of the ensemble-averaged magnetoresistance. However, alternative approaches which do not require ensemble averaging[11,12] find that the fundamental period is h/e in rings at $T=0$. Arguments have been made by Browne *et al.*[13] and Murat *et al.*[14] that it is the process of ensemble averaging that changes the periodicity to $h/2e$. However, Ref. 13 did not examine the physical conditions necessary for the system to self-average and simply assumed that the ensemble-averaged result was representative of experiment. In contrast, recent work[6,7,12] shows that such systems will *not* self-average; and as a result it can be shown[7,15] that the h/e oscillations and the AAS effect are of comparable magnitude at $T=0$ and low fields. Here we show that raising the temperature causes a ring to self-average, making it possible to observe both h/e oscillations and the AAS effect in single rings. We also give an explanation for the insensitivity of the h/e oscillations to a field in the conductor.

We treat the conductor as a disordered region which scatters electrons incident from semi-infinite ordered regions ("leads"). At $T=0$ the dimensionless conductance $g=G/(e^2/h)$ can be expressed in terms of the elements of the $2N\times 2N$ scattering matrix of the disordered region (where N is the number of states of transverse momentum). Several formulas have been proposed for g,[12,16,17] but for the issues addressed in this paper it is sufficient to take $g=2\,\mathrm{Tr}(t^\dagger t)$, where t is the (one-spin) transmission matrix evaluated at the Fermi energy. This formula applies when $l \ll L$ (l is the elastic mean free path, and L is the sample length). To treat finite temperatures within its range of validity, we assume a thermal distribution of incident electrons and $l_{in} > L$ (l_{in} is the inelastic diffusion length), which gives

$$g(T)=2\int dE\,(-df/dE)\mathrm{Tr}[t^\dagger(E)t(E)], \qquad (1)$$

where f is the Fermi distribution function.

We consider a metal ring of inner radius r_0 in a uniform normal magnetic field *which penetrates both the hole and the annulus.* We write g for a given sample as

$$g(E,B)=g_b(E)+g_0(E,B)+\sum_{n=1}^{\infty} g_n(E,B)\cos[2\pi n\Phi_h/\Phi_0+\gamma_n(E,B)], \qquad (2)$$

where $\Phi_n = \pi B r_0^2$ is the flux through the hole, $\Phi_0 = h/e$, and g_b is the background Ohmic conductance. The AAS result implies that the odd terms in the Fourier sum ensemble average to zero, whereas the even terms do not. Equation (2) for g is entirely general; it is given content by derivation of the statistical behavior of g_0, g_n, γ_n, and their scale of variation with B and E. It has been shown[6,7,15] that in a ring with a good aspect ratio, $\pi r_0^2/S_a >> 1$ (where S_a is the area of the annulus), g_0 and g_n will vary aperiodically with B on a field scale, $B_1 \simeq (2.4)\Phi_0/S_a$, long compared to the h/e oscillations (see Fig. 2, bottom), so that we separate out the rapid variation as in Eq. (2). We now discuss the energy dependence of γ_n and g_n.

Our hypothesis is that the phases $\gamma_n(E)$ and amplitudes $g_n(E)$ depend on E in a stochastic manner and become uncorrelated when separated by more than an *energy correlation range*, E_c. Equation (1) implies that $g(T)$ is a sum over energies within kT of the Fermi level; thus, above a crossover temperature $k_B T_c \simeq E_c$, $g(T)$ is approximately the average of $k_B T/E_c$ uncorrelated $T=0$ conductance patterns. Then, if energy averaging is equivalent to an averaging over impurity configurations at $T=0$, raising the temperature in a single ring will cause a slow $T^{-1/2}$ transition from the h/e periodicity to the AAS $h/2e$ result (as long as l_{in} remains greater than $4\pi r_0$). Similar energy-averaging effects have been studied elsewhere.[6,7,14]

By analogy to optics, E_c may be defined as the energy change necessary so that two electrons which follow the same path of length S across the sample acquire a phase difference of order unity, i.e.,

$$\frac{S}{\lambda(E+E_c)} - \frac{S}{\lambda(E)} \simeq S E_c \frac{d(1/\lambda)}{dE} = \frac{S}{2\pi}\frac{dk}{dE} \simeq 1,$$

where λ is the electron wavelength. If the electrons are moving ballistically the path length S is simply the sample length L and $E_c \simeq h v_F/L$ (where v_F is the Fermi velocity). To generalize this result for metallic diffusion, which is a random walk with elastic diffusion constant D, note that the time to diffuse the sample length is L^2/D, so that the path length $S \simeq v_F L^2/D$. Thus

$$E_c \simeq h v_F/S \simeq hD/L^2, \qquad (3)$$

which agrees (up to a factor $\pi/2$) with a more rigorous derivation of E_c.[7] The quantity hD/L^2 is well known as the Thouless scaling parameter V (for a metal) which satisfies $V/\Delta E \simeq g$, where ΔE is the sample's level spacing. V is the sensitivity of the energy levels to a change in the phase of the wave functions at the boundaries[18]; it is thus very natural that it also sets the scale for phase-coherent propagation in a metallic sample connected to reservoirs. This connection gives $E_c \simeq g\,\Delta E$, which implies that in typical small devices T_c occurs in an accessible range, 0.001–10 K. Note that the condition $T_c \simeq E_c/k_B$ is equivalent to $L \simeq L_T = (hD/k_B T)^{1/2}$, where L_T is the "thermal" length, which arises in the theory of interaction effects in dirty metals.

Having determined E_c, we test our energy-averaging hypothesis by numerical simulation of $g(T)$ for a single two-dimensional ring using Eq. (1). We use a recursive Green's-function approach[6] to calculate the transmission matrix through a ring represented by a 2D nearest-neighbor tight-binding model in a normal magnetic field (the field is imposed by introducing a B-dependent phase into the hopping matrix elements[6]). To make it computationally practical to study rings with good aspect ratios by this technique (while retaining the crucial effect of the field in the annulus[6,7]), we simulate rings with only a narrow slit down the middle, but with a much larger field in the slit than in the annulus. The effective aspect ratio is then the ratio of the flux through the slit to that through the annulus. Since the AB effect depends only on the topology, this configuration should be approximately equivalent to studying a ring with a good aspect ratio in a uniform field; and in fact it does give good agreement with experiment. Figure 1(a) shows the h/e periodicity of $g(B)$ for a single ring at $T=0$ calculated at several energies separated by more than E_c, illustrating the stochastic nature of the shape and phase of the oscillation as a function of E. In Fig. 1(b) we show that energy averaging of $g(B)$ for a given sample and ensemble averaging both change the periodicity to $h/2e$, and at a rate proportional to the square root of the number of independent energies or samples [we have approximated df/dE in Eq. (1) by a square wave of width kT].

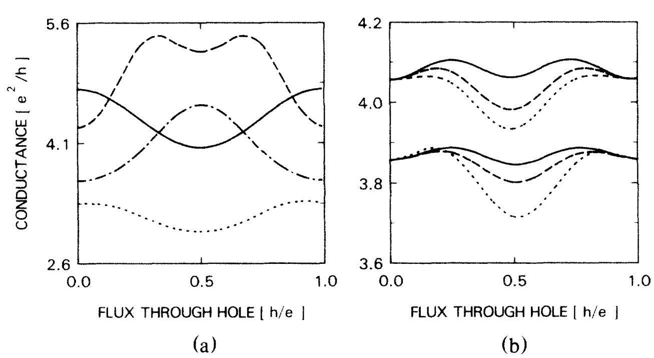

FIG. 1. (a) $g(B)$ for a single 200×20 site ring with an aspect ratio of 4.0 at $E=0.20$ (dot-dashed line), 0.21 (dashed line), 0.23 (solid line), and 0.24 (dotted line), in units of the hopping matrix element, for site disorder $W=1.0$ and $E_c \simeq 0.005$. (b) Comparison of ensemble-averaged and energy-averaged $g(B)$, for 20 (dotted line), 100 (dashed line), and 600 (solid line) samples (energies). The energy-averaged ring has the same parameters as in (a); ensemble-averaged rings are 140 by 14 sites, and otherwise the same.

The amplitude of the AAS effect, $\bar{g}_2$, is approximately $(0.044 \pm 0.004)e^2/h$. $h/2e$ oscillations of this size are expected in single rings for $T >> T_c$ but low enough that $l_{in} > 4\pi r_0$. This amplitude disagrees with AAS,[1] who predict that $\bar{g}_2$ increases as $T \to 0$, even when $l_{in} > 4\pi r_0$. It has been shown[15] that this behavior is only correct for a closed ring with no contacts, and an improved treatment gives a result which saturates at low T to a value comparable to our above result. This behavior is seen in single rings by Umbach *et al.*, who find[10] $\bar{g}_2 \simeq 0.075e^2/h$. From Fig. 1(a) we see that a typical amplitude for the h/e oscillation in our simulation is $(0.4\text{–}0.8)e^2/h$, larger than $\bar{g}_2$, a result which is expected to be independent of size,[7,15] and also agrees well with experiment.[8] Thus at $T < T_c$ the dominant periodicity will be h/e.

Now we consider the effect of the magnetic field in the annulus upon the oscillations. Experimentally it is found that the h/e oscillations persist at least up to fields of 8 T with no attenuation,[8] whereas the AAS $h/2e$ oscillations decay rapidly when a flux of order h/e penetrates the annulus.[2] This difference may be understood formally from a diagrammatic calculation,[7,15] or heuristically from the following argument. The transmission coefficients appearing in Eq. (1) for g are simply related to the probability $|C(0,L,y_1,y_2,E)|^2$ to propagate from a point $(0,y_1)$ on one side of the disordered region to a point (L,y_2) on the other side (C is the Green's function); similarly the reflection coefficients are related to $|C(0,0,y_1,y_2,E)|^2$.[6,16] The Green's function can be expressed as a path integral which (in WKB approximation) is dominated by the classical trajectories at energy E between the two points. Then

$$|C(0,L,y_1,y_2)|^2 = \sum_{p,p'} A_p A_{p'} \exp(iW_p - iW_{p'}),$$

where $W_p = S_p - Et_p$, and S_p is the action along classical trajectory p between $(0,y_1)$ and (L,y_2) traversed in a time t_p. An exactly analogous expression holds for $|C(0,0,y_1,y_2,E)|^2$. If we consider a ring with a flux only in the hole, then every pair of trajectories which go from 0 to L on opposite sides of the hole acquire a further relative phase $2\pi\Phi/\Phi_0$; also every pair of trajectories which circle the hole in opposite directions, returning to their starting point, acquire a relative phase $4\pi\Phi/\Phi_0$. Thus cross terms of this type in the above sums give rise to h/e and $h/2e$ oscillations. However, since each pair p and p' in general already have an arbitrary relative phase due to the different action along the two trajectories, upon ensemble averaging such contributions average to zero, leaving only the contribution from $p = p'$, which is insensitive to flux. The AAS effect arises because there exist a special subset of trajectories, time-reversed pairs which form a closed loop,[19] which always have a fixed relative phase ($W_p = W_{p'}$ at zero field) for any impurity configuration. Such trajectories make a contribution to the reflection coefficients which oscillates with period $h/2e$, and does not average to zero. On the other hand, there are no such time-reversed pairs of trajectories to make a configuration-independent contribution to h/e oscillations in the transmission coefficients, because such contributions come from trajectories on opposite sides of the hole. Thus the h/e contribution averages to zero. However, *for a given sample* the special AAS trajectories typically do not give a larger contribution than the "h/e" trajectories which are more numerous but have arbitrary relative phase. It has been shown[7,15] that both contributions are always of order e^2/h when $T < E_c$ and we saw above that the h/e contribution is typically larger. Now suppose that the field penetrates the annulus as well as the hole. Then two pairs of AAS trajectories typically enclose different amounts of flux, and no longer oscillate with the same phase in the sum over paths. Since it was only by having a fixed relative phase that the small subset of AAS trajectories gave a contribution of order e^2/h, the AAS effect is killed by the field in the annulus. On the other hand the "h/e" trajectories never had a fixed relative phase, even at zero field, and so their contribution is unaffected.

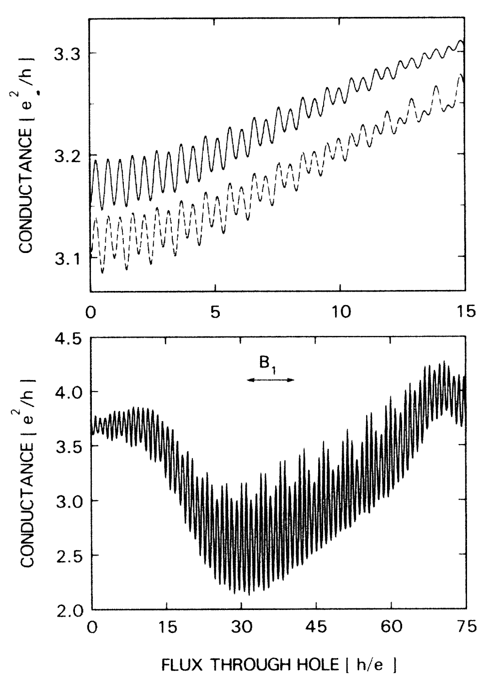

FIG. 2. (Bottom) h/e oscillations in a single 100×10 site ring of aspect ratio 4.0; B_1 is the predicted scale of the aperiodic background. (Top) Ensemble-averaged $g(B)$ for 200 (dashed line) and 950 (solid line) samples corresponding to $T \sim 200T_c$ and $T \sim 950T_c$, for the same system showing the emergence of the AAS $h/2e$ effect. Note the return to h/e periodicity at the scale B_1 for $T \sim 200T_c$.

The behavior of $g(B)$ and $\bar{g}(B)$ is shown in Fig. 2 (where we have ensemble averaged instead of energy averaged to enable us to use smaller systems). Note that $\bar{g}(B)$ shows both $h/2e$ oscillations, which die out when $B > B_1$, and weak-localization negative magnetoresistance as well, whereas $g(B)$ shows h/e oscillations persisting to high field, beats, and aperiodic background structure, just as seen experimentally.[8] This explicitly demonstrates that the ensemble-averaging process leads to the AAS result.

To summarize, because of the different sensitivities of the h/e oscillations and the AAS $h/2e$ effect to T and B, a single normal-metal ring exhibits a variety of flux-sensitive effects. At $T < T_c$ the dominant periodicity is always h/e. At $T >> T_c$, but $l_{in} > 4\pi r_0$, the AAF effect will dominate at low fields, but will give way to weaker h/e oscillations when $B >> B_1$. This picture agrees with all known experimental results. It also predicts that in the single rings where the AAS effect has been seen, lowering of the temperature further will cause a large deviation from the AAS result toward h/e periodicity, as shown in Fig. 2. There are already indications of this behavior experimentally.[9,10] Finally, *dynamic* phase incoherence is introduced by inelastic scattering, which will destroy both the h/e and AAS effects exponentially at high temperature, because they both require that some definite phase be preserved in traversal of the ring. The aperiodic fluctuations do not require phase coherence around the ring, and hence they will only disappear as a power law of T,[6,7] and will dominate the behavior in the range $T \sim 1-10$ K, which may explain the failure of some previous attempts to see flux-periodic behavior in these systems.

We thank Y. Gefen for first suggesting to us that the AAS effect came from ensemble averaging, and P. Lee for many valuable discussions; we also thank M. Büttiker, D. Prober, N. Shiren, C. Van Haesendonck, S. Washburn, and R. Webb. One of us (Y.I.) acknowledges grant support from the Fund for Basic Research of the Israeli Academy for Sciences and Humanities.

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