2-Unitary Hadamard Matrices and Multipartite Quantum Entanglement

Wojciech Bruzda

Jagiellonian University
Institute of Theoretical Physics

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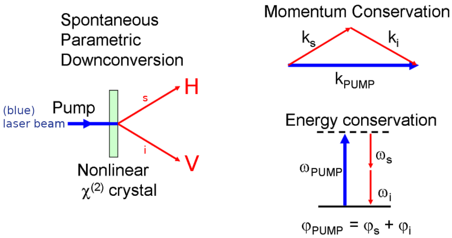
Outline

- Quantum Entanglement
- AME States (Absolutely Maximally Entangled)
- 2-Unitary Matrices
- 2-Unitary (Complex) Hadamard Matrices
- Summary and (many) Open Problems

Quantum Entanglement

quantum **entanglement** = strong correlation between physical systems



modified picture of the SPDC process producing two entangled photons

source: `https://en.wikipedia.org/wiki/Spontaneous_parametric_down-conversion`

- polarization: only H vs. V or V vs. H
- only statistical correlations

  \- no relativistic violation!

  \- no faster-than-light transfer of information

Quantum Entanglement • Formal Description

C: probability vector → Q: normalized complex **state** $|\text{vector}\rangle \in \mathcal{H}_d \cong \mathbb{C}^d$

evolution of (isolated) quantum system is governed by **unitary operators**

- (normalized) vector from the Hilbert space represents a **pure state**
- most general description: **density matrices**

Quantum Entanglement • Formal Description

- $\mathbb{S}_1$, $\mathbb{S}_2$ - two physical systems $\leftrightarrow$ $\mathcal{H}_{d_1}$, $\mathcal{H}_{d_2}$
- composite system: $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$
- wave function of non-interacting systems is factorized $|\Psi\rangle = |\Psi_1\rangle \otimes |\Psi_2\rangle$ $\rightarrow$ **separable states**
- in general $|\Psi\rangle \neq |\Psi_1\rangle \otimes |\Psi_2\rangle$

  interaction produces linear combinations of wavefunctions
- Bell state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\Big(|1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle\Big) \qquad \text{with } |0\rangle, |1\rangle \in \mathbb{C}^2$$

  cannot be written as a product: $|\Phi^-\rangle \neq (c_1|0\rangle + c_2|1\rangle) \otimes (c_3|0\rangle + c_4|1\rangle)$ with $c_j \in \mathbb{C}$

- separability vs. entanglement
- entanglement depends on the tensor structure!

$$4 \cdot 9 = 6 \cdot 6 \;\not\Longrightarrow\; \mathrm{Sep}\Big\{\mathcal{H}_4 \otimes \mathcal{H}_9\Big\} = \mathrm{Sep}\Big\{\mathcal{H}_6 \otimes \mathcal{H}_6\Big\} \;\lightning$$

Quantifying Quantum Entanglement

- measures of entanglement...
  - **separable** states
  - somehow **entangled** states
  - **maximally entangled** states

- **maximally entangled** bipartite Bell state

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}\Big(|1\rangle \otimes |0\rangle - |0\rangle \otimes |1\rangle\Big)$$

maximal correlations of measurement outputs

algebraic condition: $Tr_1|\Psi^-\rangle\langle\Psi^-| = Tr_2|\Psi^-\rangle\langle\Psi^-| \propto \mathbb{I}$

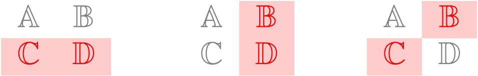
Multipartite Entanglement and AME States

- **multipartite** entanglement (for pure states)
  - here **only for** $N = 4$ (N = number of subsystems)
  - **fourpartite** physical system $\mathbb{S} = \mathbb{A}_d \otimes \mathbb{B}_d \otimes \mathbb{C}_d \otimes \mathbb{D}_d$ described by $|\psi\rangle$

    local dimension = d (for simplicity all subsystems are equalized)

> $|\psi\rangle$ is **Absolutely Maximally Entangled** state $\equiv$
> $|\psi\rangle$ **maximally entangled w.r.t. every possible even bipartition**

  - three possible (even) bipartitions of fourpartite system:



parity of splitting originates from the fact that we enforce the balance between partitions

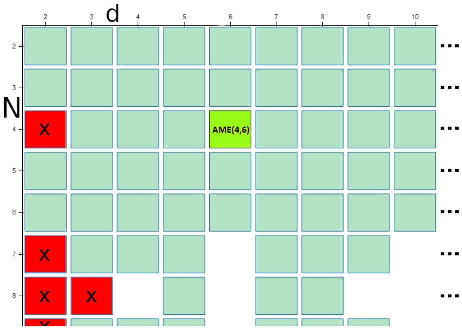
otherwise it'd be impossible to obtain maximally mixed states at the end

- notation:
  AME(N, d) → N subsystems with d degrees of freedom each

  N particles, each of d possible energy levels

AME States • Existence and Classification

- existence of AME states



modified picture of the status of AME states by N. Wyderka and F. Huber (as of 2022/03/22)

source: `https://www.tp.nt.uni-siegen.de/+fhuber/ame.html`

- existence of a particular AME state
  is related to existence of (Quantum) Orthogonal Latin Squares of particular dimension...

AME States in QIT • Applications

AME states = most "quantum" states

1. optimal bipartite unitary gate in $\mathbb{U}(d^2)$ with maximal **entangling power**
2. **perfect** tensors (tensor networks)
3. **quantum gravity**: bulk/boundary correspondence, holographic codes
4. various classes of **QECC** (quantum error corr. codes)
5. **resource** for:
   - **cryptography**
   - multiuser communication/**teleportation**
   - quantum computing (in general)
   - ...
6. **can be constructed in lab!**
7. **benchmarking**/self testing quantum computers

...

AME(4, d)

- pure state $|\psi\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d \otimes \mathbb{C}^d$

$$|\psi\rangle = \sum_{i=1}^{d} \sum_{j=1}^{d} \sum_{k=1}^{d} \sum_{l=1}^{d} \mathcal{T}_{ijkl} |i\rangle \otimes |j\rangle \otimes |k\rangle \otimes |l\rangle$$

- four-index tensor $\mathcal{T}_{ijkl}$ can be transformed into a matrix $U \in \mathbb{C}^{d^2 \times d^2}$
- for $N = 4$ there are $\binom{4}{2}$ = six such transformations
  – actually only **three** (discarding transpositions)
  they correspond to three possible splittings:

$$\begin{matrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{matrix} \leftrightarrow U_{ijkl} \qquad \begin{matrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{matrix} \overset{\mathrm{R}}{\leftrightarrow} U_{ikjl} \qquad \begin{matrix} \mathbb{A} & \mathbb{B} \\ \mathbb{C} & \mathbb{D} \end{matrix} \overset{\Gamma}{\leftrightarrow} U_{ilkj}$$

- since $|\psi\rangle$ should be a valid state, matrix U must be **unitary** for each **three** rearrangements of four-index (≡ 2-**unitary matrix**)
- tensor structure: $\mathbb{U}(d) \otimes \mathbb{U}(d)$

> Thm. $\exists$ AME(4, d) $\iff$ $\exists$ 2-unitary matrix of size d^2

for any N we have more index-rearrangements and $k = \lfloor \frac{N}{2} \rfloor$-unitarity for matrix of order d^k

Quantum Algebra

assume tensor structure: $\mathbb{U}(d_1) \otimes \mathbb{U}(d_2)$ and $U_{ijkl} = \langle ij|U|kl\rangle$

- **partial trace** over 1st or 2nd subsystem

$$\mathrm{Tr}_1(U_{d_1} \otimes U_{d_2}) = (\mathrm{Tr} \otimes \mathbb{I}_{d_2})(U_{d_1} \otimes U_{d_2}) = \left(\mathrm{Tr}\, U_{d_1}\right) \cdot U_{d_2}$$
$$\mathrm{Tr}_2(U_{d_1} \otimes U_{d_2}) = (\mathbb{I}_{d_1} \otimes \mathrm{Tr})(U_{d_1} \otimes U_{d_2}) = \left(\mathrm{Tr}\, U_{d_2}\right) \cdot U_{d_1}$$

- **partial transpose** (transpose within each block); $U^{\Gamma}_{ijkl} := U_{ilkj}$

$$U^{\Gamma} = (\mathrm{T} \otimes \mathbb{I}_{d_2})(U_{d_1} \otimes U_{d_2}) = U^{\mathrm{T}}_{d_1} \otimes U_{d_2}$$

we do not need partial transpose over the second subsystem (it is compensated by the global transpose)

- **reshuffling** (exchange rows and blocks); $U^{\mathrm{R}}_{ijkl} := U_{ikjl}$

AME(4, d) $\Longleftrightarrow$
find $\mathcal{U}$ of order d^2 such that all three matrices are unitary:

$$\mathcal{U} \in \mathbb{U}(d^2), \qquad \mathcal{U}^{\Gamma} \in \mathbb{U}(d^2), \qquad \mathcal{U}^{\mathrm{R}} \in \mathbb{U}(d^2)$$

such matrix $\mathcal{U}$ is called 2-**unitary** matrix

AME Invariants

given 2-unitary matrix $\mathcal{U} \in \mathbb{U}(d^2)$
the following operations preserve its properties:

- global transpose; $\mathcal{U}^{\mathrm{T}}$
- partial transpose; $\mathcal{U}^{\Gamma}$
- reshuffling; $\mathcal{U}^{\mathrm{R}}$
- ?
- **local unitary rotations**:

$$(U_A \otimes U_B)\mathcal{U}(U_C \otimes U_D) = \mathcal{U}' \qquad \text{for} \qquad U_X \in \mathbb{U}(d)$$

problem of local equivalence

AME and (Complex) Hadamard Matrices

H of size $d > 1$ is (complex) Hadamard iff $H \in \sqrt{d}\,\mathbb{U}(d) \cap \mathbb{T}^d$

- **usually**, a generic Hadamard matrix of order d^2 **is not** 2-unitary
- trivial constructions:
  - some AME states can be represented by permutations matrices for example $\mathrm{AME}(4, 3)$ [Phys. Rev. A **92**, 032316 (2015) by Goyeneche et al.]

$$P_9 = \left[\begin{array}{ccc|ccc|ccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]$$

then

$$H = (F_3 \otimes F_3)\, P_9 \qquad \text{or} \qquad H = (F_3 \otimes \mathbb{I}_3)\, P_9\, (F_3 \otimes \mathbb{I}_3)$$

are 2-unitary (complex) Hadamard matrices of order nine!

but e.g.

$$H = (F_3 \otimes F_3)\, P_9\, (F_3 \otimes F_3)$$ is not Hadamard!

AME and CHM

- example of permutation representation of AME$(4, 4)$

  [arXiv:2205.08842 by Suhail Ahmad Rather et al.]

$$
P_{16} = \left[\begin{array}{cccc|cccc|cccc|cccc}
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
$$

then

$$H = (F_4(a_1) \otimes F_4(a_2))P_{16}$$

is 2-unitary (complex) Hadamard matrix of order sixteen! (for any a_j)
probably there are other $\otimes$ combinations...

AME and CHM

- certain locally-inequivalent form of AME$(4, 4)$

[arXiv:2205.08842 by Suhail Ahmad Rather et al.]

$$
O_{16} = 1 \left[\begin{array}{cccc|cccc|cccc|cccc}
1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & 1 & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & -1 & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & 1 & -1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\hline
\cdot & 1 & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot \\
-1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & -1 & \cdot & \cdot & -1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 & 1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\hline
\cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot \\
\cdot & \cdot & \cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
-1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & -1 \\
\cdot & -1 & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot \\
\hline
\cdot & \cdot & \cdot & -1 & \cdot & \cdot & -1 & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\
\cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -1 & -1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot \\
\cdot & 1 & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & 1 & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & -1 & \cdot & \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot & \cdot & -1
\end{array}\right]
$$

then

$$H = O_{16} \left(F_4(a) \otimes \mathbb{I}_4 \right)$$

is 2-unitary (complex) Hadamard matrix of order sixteen! $(\forall\, a)$

but e.g.

$$H = \left(F_4 \otimes F_4 \right) O_{16}$$

is not Hadamard! (only one $F_4(a)$ is "allowed")

AME and CHM

the smallest possible 2-unitary **real** Hadamard:

$(F_2 \otimes F_2 \otimes \mathbb{I}_4)\, O_{16} =$

$$\left[\begin{array}{cccc|cccc|cccc|cccc}
1 & 1 & - & - & - & 1 & - & - & - & - & - & 1 & 1 & 1 & 1 & - \\
- & 1 & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & 1 & - & - & - \\
- & 1 & - & - & 1 & - & 1 & 1 & - & - & - & - & - & - & 1 & - \\
1 & - & - & - & - & - & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & - \\
\hline
1 & - & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - & - & 1 & - & - \\
1 & 1 & - & 1 & - & - & 1 & 1 & 1 & - & - & - & 1 & 1 & - & 1 \\
- & - & - & 1 & - & - & - & 1 & - & 1 & - & 1 & 1 & - & - & - \\
- & - & 1 & - & - & 1 & 1 & 1 & - & 1 & - & - & 1 & 1 & 1 & 1 \\
\hline
1 & 1 & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & 1 & - & - & 1 & - \\
- & 1 & - & - & - & 1 & - & 1 & 1 & 1 & 1 & - & - & 1 & - & - \\
1 & - & - & - & - & 1 & 1 & 1 & - & - & 1 & 1 & - & - & - & 1 \\
- & 1 & - & - & 1 & 1 & 1 & - & 1 & 1 & - & 1 & 1 & - & - & 1 \\
\hline
1 & - & 1 & - & 1 & 1 & - & 1 & 1 & - & - & - & 1 & - & - & - \\
1 & 1 & 1 & - & - & - & - & - & - & 1 & - & - & - & - & - & 1 \\
1 & 1 & - & 1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & - & 1 & 1 \\
1 & 1 & 1 & - & 1 & - & 1 & 1 & - & 1 & 1 & 1 & 1 & 1 & - & -
\end{array}\right] \in \mathrm{BH}(16, 2)$$

"AME" construction for <u>real</u> Hadamards works* for any dimension d of the form $d = 4k : k = 1, 2, 3, 4, \ldots$; in such cases AME is represented by permutation matrix P_{d^2} of order d^2, then taking <u>real</u> Hadamards H_d one gets 2-unitary Hadamard:

$$H = (H_d \otimes H_d)\, P_{d^2}$$

*provided that the Hadamard conjecture is true!

Cat Map by Suhail (private communication)

less trivial examples:

define a matrix U_{d^2} of order d^2

$$\langle j|U_{d^2}|k\rangle := \exp\left\{\frac{i\pi}{d^2}\left(aj^2 + bk^2 + cjk\right)\right\}$$

a, b, c - real parameters

quantization of the famous Arnold's (cat) map

observations:

- for $(a,b,c) = (1,2,-2)$ and $d \in 2\mathbb{N}+1$, U_{d^2} **is** 2-unitary **and** Hadamard!
- for <u>even</u> d, $\exists$ triplets such that U_{d^2} **is** Hadamard **but not** 2-unitary!

  no known triplets such that U_{d^2} is "close" to be 2-unitary $\wedge$ Hadamard...

- the smallest 2-unitary **complex** cat-Hadamard: $U_9 \in \mathrm{BH}(9,18)$

Open Problems

1. **any** more "really non-trivial" examples of 2-unitary CHM?!
2. there is **no** permutation representation of AME$(4, 6)$ ($\mathbb{U}(36)$)
   what about 2-unitary CHM of size 36?
3. $\exists$? 2-unitary CHM of composite orders $d = d_1 \times d_2$ for $d_1 \neq d_2$
4. general case with $N > 4$
5. given 2-unitary CHM, does its reshuffling$^{\mathrm{R}}$ or partial transpose$^{\Gamma}$ belong to the same <u>Hadamard</u> equivalence class?
6. any systematic method of constructing such classes of CHM in any dimension?

thank You!

Example of 3-Unitary Matrix

example of 3-unitary matrix of order 8 which remains unitary after any of $\binom{6}{3} = 20$ possible reorderings:

$$O_8 = \frac{1}{\sqrt{8}} \left[\begin{array}{cc|cc|cc|cc} - & - & - & 1 & - & 1 & 1 & 1 \\ - & - & - & 1 & 1 & - & - & - \\ \hline - & - & 1 & - & - & 1 & - & - \\ 1 & 1 & - & 1 & - & 1 & - & - \\ \hline - & 1 & - & - & - & - & 1 & - \\ 1 & - & 1 & 1 & - & - & 1 & - \\ \hline 1 & - & - & - & 1 & 1 & 1 & - \\ 1 & - & - & - & - & - & - & 1 \end{array} \right]$$