Measurement uncertainty and covariance

Oliver Reardon-Smith

Uniwersytet Jagielloński

oliver.reardon-smith@uj.edu.pl

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Overview

Introduction

Measurement uncertainty

Covariance

Example problem
Four? Uncertainty relations

There is no state for which the variance of $A$ and $B$ is simultaneously small.

A measurement obtaining information about $A$ disturbs the state so as to (irreversibly) change $B$.

The accuracy of an estimation of $\theta$ scales as $O(1/n)$.

A measurement that approximates $A$ well does not allow a good approximation of $B$. 

Measurement uncertainty and covariance
Four? Uncertainty relations

Preparation uncertainty

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### Four? Uncertainty relations

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**Information/disturbance**

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Four? Uncertainty relations

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“There is no state for which the variance of $A$ and $B$ is simultaneously small.”

Information/disturbance
“A measurement obtaining information about $A$ disturbs the state so as to (irreversibly) change $B$.”

Heisenberg limit
“The accuracy of an estimation of $\theta$ scales as $O(1/n)$.”
Four? Uncertainty relations

**Preparation uncertainty**

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“A measurement obtaining information about $A$ disturbs the state so as to (irreversibly) change $B$.”

**Heisenberg limit**

“The accuracy of an estimation of $\theta$ scales as $O(1/n)$.”

**Measurement uncertainty**

“A measurement that approximates $A$ well does not allow a good approximation of $B$.”
What is an observable (for the purposes of this talk)?

Given an outcome set $\Omega$, and a quantum state-space $S(\mathcal{H})$.

**An observable is**

A positive operator valued measure (POVM). Here just a function from the outcome set to the positive operators such that the sum over all outcomes is 1.
What can we do with observables (1)

Combine them with a state
If $A : \Omega \rightarrow \mathcal{L}(\mathcal{H})$ is an observable and $\rho \in S(\mathcal{H})$ is a state then

$$A^\rho : \Omega \rightarrow [0, 1]$$

$$A^\rho : \omega \mapsto \text{tr}(A(\omega)\rho)$$

is a probability distribution.
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Take convex combinations
If $A, B : \Omega \to \mathcal{L}(\mathcal{H})$ are observables, then for each $\lambda \in [0, 1]$

$$\lambda A + (1 - \lambda)B$$

represents flipping a (biased) coin and choosing to apply either $A$ or $B$ depending on the outcome.
What can we do with observables (1)

Combine them with a state
If $A : \Omega \rightarrow \mathcal{L}(\mathcal{H})$ is an observable and $\rho \in S(\mathcal{H})$ is a state then

$$A^\rho : \Omega \rightarrow [0, 1] \quad (1)$$

$$A^\rho : \omega \mapsto \text{tr}(A(\omega)\rho) \quad (2)$$

is a probability distribution.

Take convex combinations
If $A, B : \Omega \rightarrow \mathcal{L}(\mathcal{H})$ are observables, then for each $\lambda \in [0, 1]$

$$\lambda A + (1 - \lambda)B \quad (3)$$

represents flipping a (biased) coin and choosing to apply either $A$ or $B$ depending on the outcome.
What can we do with observables (2)

Compose them with quantum operations

If $A : X \rightarrow \mathcal{L}(\mathcal{H})$ is an observable, then for each TPCPM $\Phi : \mathcal{L}(\mathcal{K}) \rightarrow \mathcal{L}(\mathcal{H})$,

$$\Phi^* \circ A : X \rightarrow \mathcal{L}(\mathcal{K})$$

is an observable.

Note: If $\Phi$ is completely positive and trace-preserving, then $\Phi^*$ is completely positive and unital.
What can we do with observables (3)

Classical post-processing

If $A : X \to \mathcal{L}(\mathcal{H})$ is an observable, then for each function $f : X \to Y$ we can form

$$A_f : Y \to \mathcal{L}(\mathcal{H})$$

$$A_f : y \mapsto \sum_{x \in f^{-1}(\{y\})} A(x).$$

This amounts to combining and relabelling outcomes.
Simple example

Say $E : \{0, 1, 2\} \rightarrow \mathcal{L}(\mathcal{H})$ is an observable, but we only care if the outcome is even or odd.

Let $f(0) = f(2) = \text{even}$, and $f(1) = \text{odd}$.

If we have some apparatus $A$, that measures $E$ we can add a box that applies $f$ to the output of $A$. 
A less simple example

Compatibility

If $A : X \rightarrow \mathcal{L}(\mathcal{H})$ and $B : Y \rightarrow \mathcal{L}(\mathcal{H})$ are observables, and there is an observable $J : \Omega \rightarrow \mathcal{L}(\mathcal{H})$, and functions $f : \Omega \rightarrow X$ and $g : \Omega \rightarrow Y$ such that

\begin{align}
A &= Jf \\
B &= Jg,
\end{align}

then $A$ and $B$ are called compatible and $J$ is called a joint observable for $A$ and $B$.

Note: One can always choose $\Omega = A \times B$, to be the Cartesian product and $f, g$ to be the projection maps.
The general problem (1)

\[ \begin{align*}
A_1 & \rightarrow B_1 \\
A_2 & \rightarrow B_2 \\
\vdots & \\
A_n & \rightarrow B_n \\
\end{align*} \]
The general problem (2)

- Have some family \( \{A_i \mid i \in 1...n\} \) of incompatible observables we would like to measure \( A_i : \Omega_i \to \mathcal{L}(\mathcal{H}) \).
- Consider an arbitrary family of compatible observables, with the same outcome spaces \( B_i : \Omega_i \to \mathcal{L}(\mathcal{H}) \).
- Choose a figure of merit \( \delta \) for an approximation and explore the set of allowed vectors \( (\delta(A_1, B_1), ..., \delta(A_n, B_n)) \).
The general problem (3)

Given a fixed set of observables \( \{ A_i | i \in 1 \ldots n \} \), \( A_i : \Omega_i \rightarrow \mathcal{L}(\mathcal{H}) \) what is the set

\[
\{ (\delta(A_1, B_1), \ldots, \delta(A_n, B_n) | \text{all the } B_i \text{ are compatible} \} \quad (9)
\]
Figures of merit - probability distributions

The Minkowski $p$-distance

$$d_p(P, Q) = \left( \sum_{\omega \in \Omega} |P(\omega) - Q(\omega)|^p \right)^{\frac{1}{p}} , \quad p \in [1, \infty) \quad (10)$$

$$d_{\infty}(P, Q) = \sup_{\omega \in \Omega} |P(\omega) - Q(\omega)| \quad (11)$$
Figures of merit - observables (1)

state + observable = probability distribution

Which state to choose?

No additional context $\implies$ Take the sup over all states
A fixed set of states $\implies$ Take the sup over all states in the set
A fixed state/distribution $\implies$ Evaluate on that state
Figures of merit - observables (2)

We choose the worst case

\[ d_p(A, B) := \sup_{\rho} d_p(A^\rho, B^\rho) \]  \hspace{1cm} (12)

We can embed the set of observables in the linear space of maps from \( \Omega \) to \( \mathcal{L}(\mathcal{H}) \).
On this space \( d_p \) is a metric, and \( \| \cdot \|_p = d_p(\cdot, 0) \) is a norm.
Classical example
Classical example
Classical example
Classical example
Quantum example (1)
Quantum example(1)

\[ \sigma_z \rightarrow \text{up, down, up, up} \]

\[ z_\phi \rightarrow \text{down, up, down, down} \]
Quantum covariance (2)

Let $G$ be a group, with an action on a set $\Omega$

- $\forall g \in G$ we have $f_g : \Omega \to \Omega$
- If the identity in $G$ is $e$ then $\forall \omega \in \Omega$, $f_e(\omega) = \omega$
- $\forall g, h \in G$ we have $f_g \circ f_h = f_{gh}$
Quantum covariance (3)

Fix a Hilbert space $\mathcal{H}$, and require that $G$ has an (anti-)unitary projective representation on $\mathcal{H}$

- \( \forall g \in G \) we have an (anti-)unitary operator $U_g : \mathcal{H} \rightarrow \mathcal{H}$
- If the identity in $G$ is $e$ then $U_e = I$
- \( \forall g, h \in G \) we have $U_g U_h = e^{i a_{gh}} U_{gh}$, where $a_{gh}$ is real

The maps $R_g : \mathcal{L} (\mathcal{H}) \rightarrow \mathcal{L} (\mathcal{H})$, $R_g [A] = U_g A U_g^*$ are then a linear representation of $G$.

Each $R_g$ is positive and unital.
Quantum covariance (4)

Given this, we say an observable $A$ on $\mathcal{H}$, with outcome set $\Omega$ is \((G, R, f)\) covariant if

$$A(fg(x)) = R_g[A(x)]$$  \hspace{1cm} (13)
Why covariance?

- Many physically interesting observables are covariant
- Exploring the space of joints is hard
- \( J : \prod_i \Omega_i \rightarrow \mathcal{L}(\mathcal{H}) \) is often a POVM with very many outcomes
- Explicit parameterisations are not known
Covariantisation map

Can define this map

\[ M(A) : \omega \mapsto \frac{1}{|G|} \sum_{g \in G} R_{g^{-1}} [A(f_g(\omega))] \]  \hspace{1cm} (14)

- If \( A \) is an observable, so is \( M(A) \)
- \( M(A) \) is \((G, R, f)\) covariant
- If \( A \) is \((G, R, f)\) covariant then \( M(A) = A \)
- \( M \) is a contraction of the \( d_p \) norm
Diagram
Diagram

Covariant maps
Diagram

Covariant target

Covariant maps

Observables
Diagram

- Covariant target
- Covariant maps
- Approximator
- Observables

**Measurement uncertainty and covariance**
Diagram

Covariant target

Covariant maps

Approximator

Observables

Covariant approximator
Just a projection onto a subspace?

The invariant mean has one more useful property
Just a projection onto a subspace?

The invariant mean has one more useful property

It preserves compatibility
Just a projection onto a subspace?

If $E_1 : \Omega_1 \rightarrow L(\mathcal{H})$, $E_2 : \Omega_2 \rightarrow L(\mathcal{H})$ are compatible observables, and $G$ is a group with representation $R_g : L(\mathcal{H}) \rightarrow L(\mathcal{H})$ and actions $\alpha_g^1 : \Omega_1 \rightarrow \Omega_1$ and $\alpha_g^2 : \Omega_2 \rightarrow \Omega_2$ then

$$M_1(E_1) : \omega \mapsto \frac{1}{|G|} \sum_{g \in G} R_{g^{-1}}[E_1(\alpha_g^1(\omega))] \quad (15)$$

$$M_2(E_2) : \omega \mapsto \frac{1}{|G|} \sum_{g \in G} R_{g^{-1}}[E_2(\alpha_g^2(\omega))] \quad (16)$$

are compatible.
$M$ preserves compatibility!

$E_i$
$M$ preserves compatibility!

$\begin{align*}
J \\
E_i
\end{align*}$
$M$ preserves compatibility!

\[ J \xrightarrow{M} M[J] \]

\[ E_i \]
$M$ preserves compatibility!

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$M$ preserves compatibility!

\[ J \xrightarrow{M} M[J] \]

\[ E_i \xrightarrow{M} M[E_i] \]
A specific problem (1)

Fix a Hilbert space $\mathcal{H}$ of (finite) dimension $n$, and let $\{|c_g\rangle|g \in \mathbb{Z}_n\}$ and $\{|f_h\rangle|h \in \mathbb{Z}_n\}$ be related by the quantum Fourier transform

$$|f_h\rangle = \sqrt{\frac{1}{n}} \sum_{g \in \mathbb{Z}_n} e^{\frac{2\pi i gh}{n}} |c_g\rangle$$ \hspace{1cm} (17)

$$|c_g\rangle = \sqrt{\frac{1}{n}} \sum_{h \in \mathbb{Z}_n} e^{-\frac{2\pi i gh}{n}} |f_h\rangle$$ \hspace{1cm} (18)

Let $A: g \rightarrow |c_g\rangle\langle c_g|$ and $B: h \rightarrow |f_h\rangle\langle f_h|$ be observables on $\mathcal{H}$ with outcome set $\mathbb{Z}_n$. 
A specific problem (2)

These observables are highly symmetric!
Can define unitary “shift operators”

\[ U_k |c_g \rangle = |c_{g+k} \rangle \] \hspace{1cm} (19)

\[ V_l |f_h \rangle = |f_{h+l} \rangle \] \hspace{1cm} (20)

These obey the Weyl form of the CCR

\[ U_k V_q = e^{\frac{2\pi i}{n} k q} V_q U_k \] \hspace{1cm} (21)
A specific problem (3)

Indeed

\[ U_k V_l E V_l^* U_K^* = V_l U_k E U_K^* V_l^*, \quad \forall E \in \mathcal{L}(\mathcal{H}) \]  \hspace{1cm} (22)

If we let

\[ R_{kl} : E \mapsto U_k V_l E V_l^* U_K^* \]  \hspace{1cm} (23)

Then the \( R_{kl} \) are a representation of \( \mathbb{Z}_n \oplus \mathbb{Z}_n \)
A specific problem (4)

Let $\alpha$ and $\beta$ be group actions of $\mathbb{Z}_n \times \mathbb{Z}_n$ acting on $\mathbb{Z}_n$

\[
\alpha_{kl} : g \mapsto g + k \quad (24)
\]

\[
\beta_{kl} : h \mapsto h + l \quad (25)
\]

$A$ is $(G, R, \alpha)$ covariant and $B$ is $(G, R, \beta)$ covariant!
A specific problem (4)

Hence, for any compatible observables $B, C : \mathbb{Z}_n \rightarrow \mathcal{L}(\mathcal{H})$, $M_1(B)$ and $M_2(C)$ are better approximations to $A$ and $B$ respectively.
Solution

The following problem is *semidefinite*, for each $n \in 2, 3 \ldots$ and each $x \in [0, 1]$: 
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The following problem is *semidefinite*, for each $n \in 2, 3 \ldots$ and each $x \in [0, 1]$:

Minimise $d_\infty(B, D)$ such that $C$ and $D$ are jointly measurable, covariant, and $d_\infty(A, C) = x$
Solution

The following problem is *semidefinite*, for each $n \in 2, 3 \ldots$ and each $x \in [0, 1]$:

Minimise $d_\infty(B, D)$ such that $C$ and $D$ are jointly measurable, covariant, and $d_\infty(A, C) = x$

The semidefinite problems may be solved analytically
Solution: \( n = 2 \)
Solution: \( n = 3 \)
Solution: \( n = 10 \)
Solution: $n = 2018$
The 3 qubit Pauli observables

Figure: Various views of the uncertainty region for three mutually unbiased qubit observables $S_{\infty}(\sigma_x, \sigma_y, \sigma_z)$.
Thank you for your time
References