Generalization of Gisin's theorem to quantum fields

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Outline

Entanglement, Bell, and nonlocality

- Quantum superposition allows entangled states that cannot be factorized:
$$|\psi\rangle_{AB} \neq |\psi_A\rangle \otimes |\psi_B\rangle \,.$$
- EPR (1935): tension between quantum mechanics and *local realism*.
- Bell (1964): inequalities satisfied by all local hidden variable (LHV) models.
- Violation of a Bell inequality $\Rightarrow$ *Bell non-classicality* (nonlocality in the Bell sense).
- Experiments (Aspect, Zeilinger, loophole-free tests, . . .) confirm violation of Bell inequalities and rule out broad classes of LHV models.

Gisin's theorem (original version)

Gisin's theorem (1991)

Any pure entangled two-qubit state violates *some* Bell inequality (in particular a CHSH-type inequality).

- Consider a general pure state of two qubits

$$|\psi\rangle = \cos\theta\ |00\rangle + \sin\theta\ |11\rangle\,, \quad 0 < \theta \leq \frac{\pi}{4}.$$

- If $0 < \theta < \frac{\pi}{4}$ the state is entangled.
- Gisin proved that for each entangled $|\psi\rangle$ one can find local measurement settings such that CHSH is violated.
- Conversely, product states *never* violate CHSH.

Extensions of Gisin's theorem

- Later work extended the theorem:
  - more than two parties,
  - higher local dimension,
  - arbitrary pure states on tensor products

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \dots .$$

- Common feature: description in the *first quantization* picture.
- Fixed number of distinguishable particles, with a fixed factorization of the Hilbert space.
- Question left open: what about **quantum fields**, where:
  - particles are indistinguishable,
  - particle number may be undefined,
  - the natural language is Fock space.

From first to second quantization

First quantization

- Single particle: $|\psi\rangle \in \mathcal{H}$, usually $\mathcal{H} \simeq L^2(M)$.
- Two distinguishable particles: $\mathcal{H}_1 \otimes \mathcal{H}_2$.
- Entanglement defined with respect to this tensor product.

Second quantization

- Field modes described by a Fock space

$$\mathcal{F}(\mathcal{H})_q = \bigoplus_{n=0}^{\infty} S_q\left(\mathcal{H}^{\otimes n}\right),$$

  where $q = +$ for bosons, $q = -$ for fermions.
- Basis: occupation number states $|n_1, n_2, \ldots\rangle$ built by creation operators $a_i^\dagger$ from the vacuum $|\Omega\rangle$.
- Allows superpositions of different particle numbers.

Mode entanglement

- In a field picture, the natural subsystems are **modes**, not particles.
- Choose a single-particle space $\mathcal{H}_{AB}$ and split its orthonormal basis into two groups of modes:

$$\{a_i\}_i \quad \text{and} \quad \{b_\ell\}_\ell.$$

- A general state of the field (bosonic or fermionic) lives in $\mathcal{F}(\mathcal{H}_{AB})_q$.
- **Mode separability:**

$$|\psi_{a,b}\rangle = F(a_i^\dagger)\, G(b_\ell^\dagger)\, |\Omega\rangle\,.$$

- If the state cannot be written in this form, it is mode-entangled.

The conceptual gap

Problem

Classical Gisin-type results assume a fixed tensor product structure of *finite-dimensional or separable* Hilbert spaces with fixed particle number.

- But many physically relevant states are:
  - superpositions of different particle numbers (e.g. squeezed states),
  - states of many field modes,
  - even states in non-separable Hilbert spaces (thermodynamic limit).
- Question: **Is every pure mode-entangled state of a quantum field Bell non-classical?**
- In other words: does some Bell inequality get violated for *any* such state?

Idea: recover a tensor product structure

- Standard Schmidt decomposition works on $\mathcal{H}_A \otimes \mathcal{H}_B$.
- Fock space $\mathcal{F}(\mathcal{H}_{AB})_q$ is *not* manifestly a simple tensor product of two spaces.
- Strategy of the paper:
  1. Show an *isomorphism*

$$\mathcal{F}(\mathcal{H}_{AB})_q \simeq \mathcal{H}_A \otimes \mathcal{H}_B,$$

     where $\mathcal{H}_A$ and $\mathcal{H}_B$ are abstract separable Hilbert spaces.
  2. Perform Schmidt decomposition in $\mathcal{H}_A \otimes \mathcal{H}_B$.
  3. Map the decomposition back to Fock space.

Constructing the isomorphism

- Consider all distributions of a finite number of particles among $\{a_i\}$:

$$\vec{n}_j = (n_{j1}, n_{j2}, \dots), \quad \sum_i n_{ji} < \infty.$$

  Similarly for $\{b_\ell\}$ with vectors $\vec{m}_{j'}$.

- Any Fock basis vector can be written as

$$\left|j, j'\right\rangle = \left|\vec{n}_j; \vec{m}_{j'}\right\rangle = f(a_i^\dagger, j)\, g(b_\ell^\dagger, j')\, |\Omega\rangle\,.$$

- The set $\{|j, j'\rangle\}_{j,j'}$ is a countable orthonormal basis of $\mathcal{F}(\mathcal{H}_{AB})_q$.

Abstract Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$

- Define abstract basis vectors $\tilde{f}(j)$ and $\tilde{g}(j')$:

$$\left\langle \tilde{f}(j) \middle| \tilde{f}(k) \right\rangle = \delta_{jk}, \quad \left\langle \tilde{g}(j') \middle| \tilde{g}(k') \right\rangle = \delta_{j'k'}.$$

- $\mathcal{H}_A$ is the closure of the span of $\{\tilde{f}(j)\}_j$, $\mathcal{H}_B$ is the closure of the span of $\{\tilde{g}(j')\}_{j'}$.
- Now define a linear map

$$I : \mathcal{F}(\mathcal{H}_{AB})_q \to \mathcal{H}_A \otimes \mathcal{H}_B$$

by

$$I\left(\left|j, j'\right\rangle\right) = \tilde{f}(j) \otimes \tilde{g}(j').$$

- I is an isomorphism and preserves inner products, so it is essentially unitary (up to basis choices).

Schmidt decomposition in Fock space

- Take an arbitrary state $|\psi\rangle \in \mathcal{F}(\mathcal{H}_{AB})_q$.
- Map it to $\mathcal{H}_A \otimes \mathcal{H}_B$:

$$I(|\psi\rangle) = \sum_t \lambda_t \, \tilde{F}(t) \otimes \tilde{G}(t)$$

  where $\{\lambda_t\}$ are Schmidt coefficients.
- Map back:

$$|\psi\rangle = \sum_t \lambda_t \, F(a_i^\dagger, t) \, G(b_\ell^\dagger, t) \, |\Omega\rangle \, .$$

- This is a **Schmidt decomposition in Fock space**: each term is generated by polynomials of creation operators on each side.
- The decomposition is unique for a fixed partition of modes into $\{a_i\}$ and $\{b_\ell\}$.

Entanglement criterion in Fock space

Lemma (mode separability vs Schmidt decomposition)

A pure state $|\psi\rangle$ is separable with respect to the partition of modes $\{a_i\}$ and $\{b_\ell\}$ iff its Fock-space Schmidt decomposition has exactly one nonzero coefficient:

$$|\psi\rangle = F(a_i^\dagger, 1)\, G(b_\ell^\dagger, 1)\, |\Omega\rangle\,.$$

Otherwise the state is mode-entangled.

- Exactly analogous to the standard Schmidt rank criterion in finite dimensions.
- The Schmidt rank here is the number of nonzero λ_t in

$$|\psi\rangle = \sum_t \lambda_t\, F(a_i^\dagger, t)\, G(b_\ell^\dagger, t)\, |\Omega\rangle\,.$$

Statement of the result

Generalized Gisin's theorem (informal)

Any pure mode-entangled state of a quantum field (bosonic or fermionic) violates some Clauser–Horne (CH) Bell inequality.

- "Mode-entangled" means entangled between two groups of modes $\{a_i\}$ and $\{b_\ell\}$ in the sense of the Fock-space Schmidt decomposition.
- The theorem applies also to states with undefined particle number.
- With some extra argument, it extends to certain non-separable Hilbert spaces relevant in thermodynamic limits.

Effective two-qubit subspace

- Take the Schmidt decomposition

$$|\psi\rangle = \sum_t \lambda_t F_t G_t |\Omega\rangle, \quad F_t := F(a_i^\dagger, t),\ G_t := G(b_\ell^\dagger, t).$$

- Assume at least two nonzero coefficients, and reorder so that $|\lambda_1| \geq |\lambda_2| \geq \dots$.
- Split off the two largest terms:

$$|\psi_{\text{ent}}\rangle = \lambda_1 F_1 G_1 |\Omega\rangle + \lambda_2 F_2 G_2 |\Omega\rangle + \lambda_R |R\rangle,$$

with $|R\rangle$ orthogonal to the first two terms.

- Normalizing the first two terms defines an effective two-qubit state $|\phi_{12}\rangle$ embedded in Fock space.

Using the original Gisin theorem

- The effective state $|\phi_{12}\rangle$ is entangled in a 2×2 subspace.
- By the original Gisin theorem, there exist effective Pauli observables and settings that violate a CHSH inequality on that subspace.
- One translates Pauli operators into field operators acting on Fock space:

$$\Sigma_x(a_i) = P^A_{12} + P^A_{21},$$
$$\Sigma_y(a_i) = -i\big(P^A_{12} - P^A_{21}\big),$$
$$\Sigma_z(a_i) = P^A_{11} - P^A_{22},$$

  and similarly for the b modes.
- Here $P^A_{kk'}$ and $P^B_{kk'}$ are operators that map between the two Schmidt basis components on each side.

From CHSH to CH

- On the effective 2×2 subspace, the CHSH correlation functions
$$E(\alpha, \beta) = \langle (\alpha \cdot \Sigma(a_i))(\beta \cdot \Sigma(b_\ell)) \rangle$$
are related to joint probabilities $p(0, 0|\alpha, \beta)$.
- This gives an equivalent Clauser–Horne (CH) inequality
$$\mathrm{CH} \leq 0$$
in terms of probabilities.
- Crucial point:
$$\langle \hat{C}_{\mathrm{CH}} \rangle_\psi = (|\lambda_1|^2 + |\lambda_2|^2) \langle \hat{C}_{\mathrm{CH}} \rangle_{\phi_{12}} > 0.$$
- So the **full** state $|\psi\rangle$ violates a CH inequality, even if a CHSH violation is guaranteed only on the two-qubit subspace.

Extension to non-separable Hilbert spaces

- In some QFT contexts one considers a "big" Hilbert space

$$\mathcal{H}' = \bigoplus_{n=0}^{\infty} \mathcal{H}^{\otimes n},$$

  including actual infinite-particle states.
- This space can be non-separable.
- A generalized version of Gleason's theorem implies:
  - Any physically meaningful state on $\mathcal{H}'$ is supported on a *separable* subspace.
  - In particular, it lies in (a finite combination of) Fock spaces with well-defined mode structure.
- Since the generalized Gisin theorem holds for arbitrary pure states on Fock spaces, it effectively holds for pure states on $\mathcal{H}'$ as well.

Two-mode squeezed vacuum

- Consider the two-mode squeezed vacuum

$$|\Psi\rangle = \sum_{n=0}^{\infty} \lambda^{n+1} |n; n\rangle ,$$

  where $|n; n\rangle$ denotes n photons in mode a_1 and n in mode b_1.
- Already Schmidt form wrt. the partition $\{a_1\}$ vs $\{b_1\}$.
- It is a paradigmatic example of:
  - strong mode entanglement,
  - completely undefined particle number.
- The theorem guarantees that *some* CH inequality is violated.
- However, experimentally one would need projective measurements onto superpositions such as

$$\cos\theta\,|0\rangle + e^{i\varphi}\sin\theta\,|1\rangle ,$$

  which is very challenging with current technology.

Bright squeezed vacuum (BSV)

- Four-mode "2×2 bright squeezed vacuum" state, modes $\{a_1, a_2\}$ and $\{b_1, b_2\}$:

$$\left|\psi^{-}\right\rangle = \frac{1}{\cosh^2 \Gamma} \sum_{n=0}^{\infty} \tanh^n \Gamma \sum_{m=0}^{n} (-1)^m \left|n-m, m; m, n-m\right\rangle .$$

- For a suitable partition of modes into two subsets, the first nontrivial term is effectively a dual-rail singlet:

$$\frac{1}{\sqrt{2}}\left(|1,0;0,1\rangle - |0,1;1,0\rangle\right).$$

- In this case, the CH inequality constructed from the Schmidt decomposition can be tested with:
  - Stokes operators restricted to the single-photon subspace,
  - photon-number-resolving detectors.

BGHZ state

- BGHZ state: a GHZ-like state with undefined photon number, involving six modes $\{a_1, a_2, a_3, a_4\}$ and $\{b_1, b_2\}$.
- In Schmidt form it looks like

$$|\mathrm{BGHZ}\rangle = \sum_{k=0}^{\infty} \sum_{m=0}^{k} C_{k-m}(\Gamma) C_m(\Gamma)\ |k-m, k-m, m, m; k-m, m\rangle\,.$$

- For a suitable choice of terms we isolate an effective GHZ-like component, e.g.

$$\frac{1}{\sqrt{2}}\left(|1,1,0,0;1,0\rangle + |0,0,1,1;0,1\rangle\right).$$

- Again the theorem guarantees violation of a CH inequality, though the required projective measurements may demand nontrivial optical operations (e.g. Bell measurements with linear optics or photon subtraction).

Beamsplit single photon with ancillas

- Simple but instructive example: a single photon at the output of a balanced beamsplitter:

$$|\phi\rangle = \frac{1}{\sqrt{2}}(a_1^\dagger + b_1^\dagger)\,|\Omega\rangle = \frac{1}{\sqrt{2}}(|0;1\rangle + |1;0\rangle).$$

- This state is entangled in the *mode* picture, but all fixed total photon-number sectors have Schmidt rank 1 (there is only $n = 1$).
- One cannot demonstrate Bell non-classicality using only number-conserving projective measurements on the two modes.
- Adding coherent ancillary modes and projecting onto suitable subspaces can reveal the violation:
  - the entanglement becomes accessible in an effective dual-rail qubit encoding,
  - but at the price of ancillary resources or postselection.

When are POVMs or ancillas necessary?

Sufficient condition (qualitative)

If every projection of a pure state onto sectors with fixed total photon numbers in both mode groups has Schmidt rank 1, then no Bell violation can be revealed using only projective measurements that do not mix photon numbers.

- Intuition:
  - Number-conserving projective measurements effectively decompose the state into a classical mixture of fixed-n sectors.
  - If each such sector is separable, the overall statistics admit a local hidden-variable model for those measurements.
- To detect Bell non-classicality one then needs:
  - non-projective POVMs (e.g. homodyne detection),
  - ancilla modes (e.g. coherent states),
  - or postselection.

Physical meaning

- The generalized theorem shows:
  - Pure mode entanglement in fields is fundamentally incompatible with local realism.
  - This holds even for states with undefined or infinite particle number (within the QFT framework considered).
- But experimentally:
  - Some entangled field states are "easy" to witness via Bell tests (e.g. dual-rail encoded states).
  - Others require sophisticated measurements or remain practically inaccessible, although conceptually nonlocal.
- The paper makes this separation precise using the Fock-space Schmidt decomposition.

Summary

- Classical Gisin's theorem: any pure entangled two-qubit state violates a Bell inequality.
- The paper generalizes this to *quantum fields*:
  - Construct an isomorphism $\mathcal{F}(\mathcal{H}_{AB})_q \simeq \mathcal{H}_A \otimes \mathcal{H}_B$.
  - Build a Fock-space Schmidt decomposition.
  - Use the two largest Schmidt coefficients to define an effective two-qubit state.
  - Translate qubit observables into field observables and obtain a CH inequality violated by any pure mode-entangled state.
- Examples: two-mode squeezed vacuum, bright squeezed vacuum, BGHZ state, beamsplit single photon.
- Additional result: a criterion for when non-projective measurements or ancillas are required to see Bell non-classicality.

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