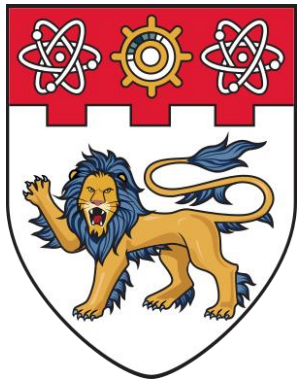

Local Thermal Operations and Classical Communication

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Based primarily on arXiv 2410.14550, developed with Rafał Bistroń

17.11.2025, Kraków


OCBC

Seminar Outline: Hold onto your armrests!

- Preliminaries:
  - Thermal Operation and their incoherent restriction
  - LOCC and Bell nonlocality
- Framework and Results:
  - LTOCC: Motivation, definition and hierarchy
  - Results on state transformability and “thermal Birkhoff polytope”
  - Correlation generation and detection

Preliminaries

Quick reminder – quantum states, operations, interpretations

- Objects of interest will be connected to finite-dimensional pure quantum states $|\psi\rangle \in \mathcal{H}_d$.
- More specifically, we will be concerned with density operators corresponding to mixed states, $\rho \in \mathcal{D}(\mathcal{H}_d) : \mathrm{Tr}\,\rho = 1, \rho \geq 0$.
- Operations on quantum states are restricted to what is known as completely positive trace-preserving maps

$$\Phi : \mathrm{Tr}[\Phi(\rho)] = \mathrm{Tr}(\rho),\ \rho \geq 0 \Rightarrow (\Phi \otimes \mathbb{I})(\rho) \geq 0$$

- Special cases are mixed unitary channels, defined by

$$\Phi(\rho) = \sum_i p_i U_i \rho U_i^\dagger,\ U_i \in \mathcal{U}(d),\ p_i \geq 0,\ \sum_i p_i = 1$$

Quick reminder – quantum states, operations, interpretations

- Probability of finding a state $|\psi\rangle$ in a state $|\phi\rangle$ is given by their squared absolute overlap $|\langle\psi|\phi\rangle|^2$ when restricted to pure states.
- This extends to measurements over entire orthonormal bases which are referred to as von Neumann measurements, with probabilities given by

$$p_i = |\langle\psi|\phi_i\rangle|^2$$

- More generally, one can consider a positive operator-valued measures, for which the probabilities are given by

$$p_i = \mathrm{Tr}(\rho P_i),\ \sum P_i = \mathbb{I},\ P_i \geq 0,\ P_i = P_i^\dagger$$

Ingredient 1: Thermodynamics in quantum regime

- Let us consider a system of dimension d with a Hamiltoniain, defining energy structure, to be given as

$$H = \sum_i E_i \, |E_i\rangle\langle E_i|$$

- In contact with a thermal bath in temperature T, such a system equilibrates to a Gibbs state given by

$$\gamma = \sum_i \frac{e^{-\beta E_i}}{\sum_j e^{-\beta E_j}} \, |E_i\rangle\langle E_i| \, , \quad \beta = \frac{1}{k_B T}$$

- Thus a simplest set of operations that respect thermal equilibrium would be so-called Gibbs-preserving (GP) operations

$$\Phi : \Phi(\gamma) = \gamma$$

Ingredient 1: Thermodynamics in quantum regime

- A refinement comes from an assumption that the evolution should be symmetric with respect to time evolution of the system. This leads to so called Gibbs-preserving covariant (GPC) operations

$$\Phi : \Phi(\gamma) = \gamma,\ e^{iHt}\Phi(\rho)e^{-iHt} = \Phi(e^{iHt}\rho e^{-iHt})$$

- Further refinement, referred to as thermal operations (TO) comes from an assumption, that the state evolves as a closed system with a thermal environment

$$\Phi : \Phi(\rho) = \mathrm{Tr}_E\left[U(\rho \otimes \gamma_E)U^\dagger\right],\ [U, H + H_E] = 0$$

Brandao, Horodecki, Oppenheim, Renes and Spekkens, Phys. Rev. Lett. 111, 250404

Energy-incoherent states – a useful restriction

- Thermal operations may seem innocent, but they are not simple to handle in general.
- A tractable closed-form formula is known for a qubit, or d = 2, with certain results on asymptotic catalytic transformation rates derived recently.
- Thus, for the most of what follows we will restrict ourselves to energy incoherent states

$$\rho = \sum_i p_i \left|E_i\right\rangle\!\left\langle E_i\right|, \; p_i \geq 0, \sum_i p_i = 1 \longleftrightarrow \mathbf{p}$$

Korzekwa, Phys. Rev. A 95, 052318 (2017)
Shiraishi and Takagi, arXiv: 2510.05642

Resource-theoretic flavour – state-to-state transformability

- Let us note that Gibbs' state is the only fixed state of the theory, and thus the only one we can refer to as free.
- Any athermality can be seen as a **resource** – thus transporting us into the realm of what is known as resource theories
- One of the usual questions is concerned with **possible transformations**.
- When considering transformation $\mathbf{p} \to \mathbf{q}$, we ask for existence of a stochastic matrix such that

$$S\mathbf{p} = \mathbf{q}, \quad S\gamma = \gamma$$

Resource-theoretic flavour – state-to-state transformability

- For infinite temperature, $\beta = 0$, we find that Gibbs state becomes a flat distribution η. Stochastic matrices preserving flat distribution are known to be bistochastic

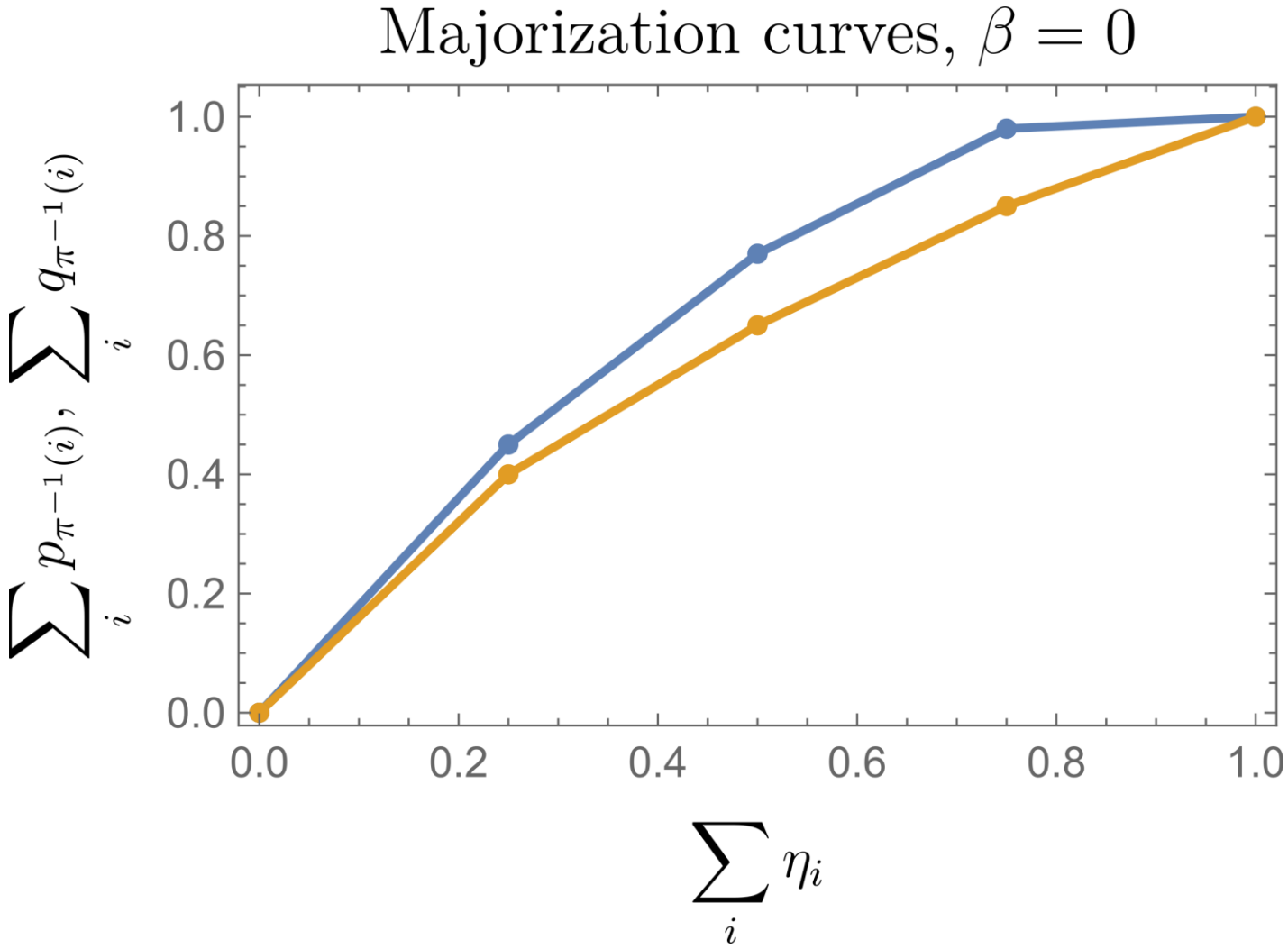
$$S\eta = \eta \iff \sum_i S_{ij} = \sum_j S_{ij} = 1$$

- It is known that transformations $\mathbf{p} \to \mathbf{q}$ are governed by majorization,

$$\exists B : B\mathbf{p} = \mathbf{q} \Leftrightarrow \mathbf{p} \prec \mathbf{q} \overset{\text{def.}}{\iff} \forall_k \sum_{i=1}^{k} (p_i^{\downarrow} - q_i^{\downarrow}) \geq 0$$

G. Hardy, J. Littlewood, and G. Polya, Inequalities, Cambridge Mathematical Library (Cambridge University Press, 1952).

Resource-theoretic flavour – state-to-state transformability



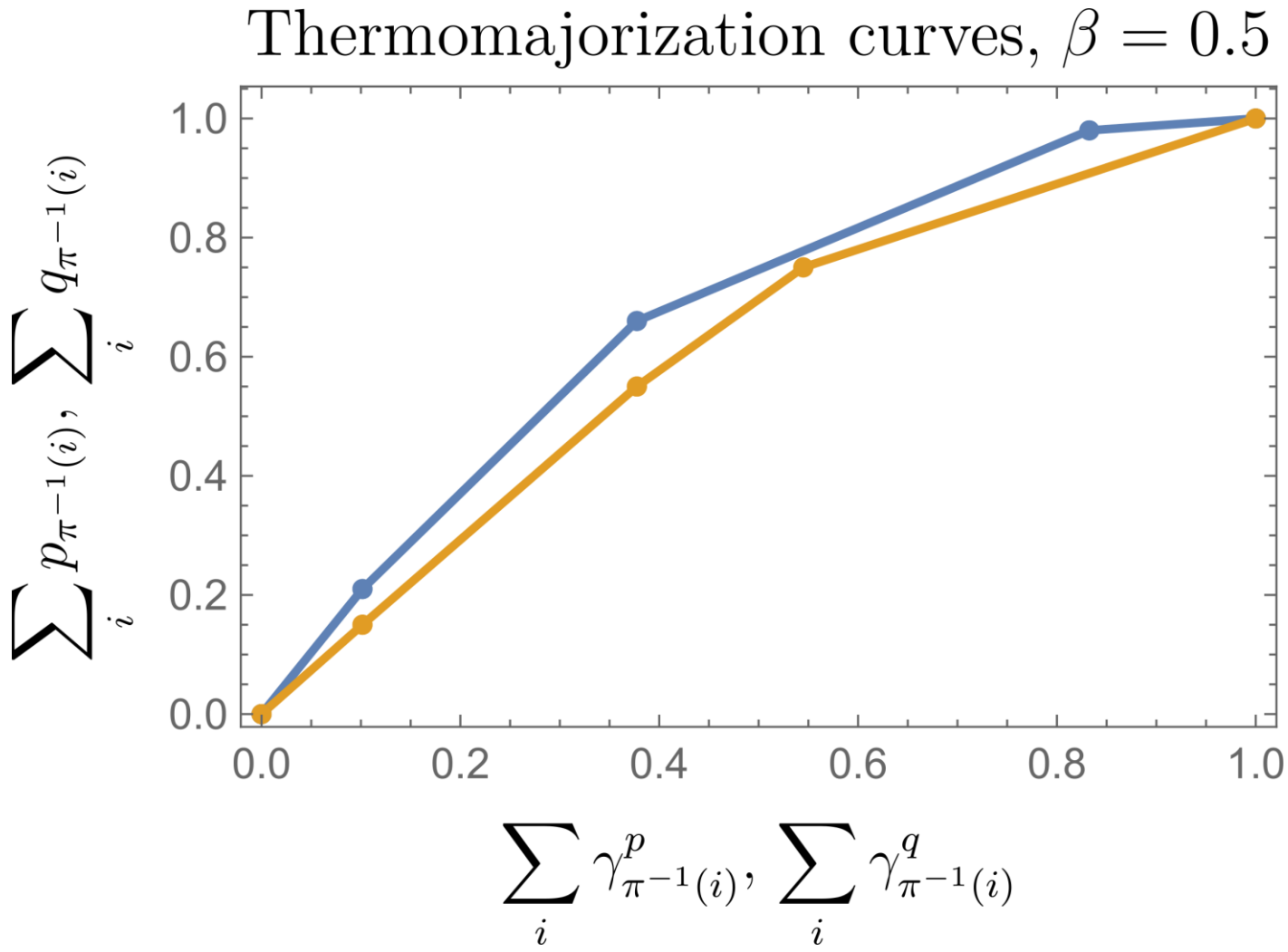
Resource-theoretic flavour – state-to-state transformability

- For finite temperatures, $\beta > 0$, it is far less obvious how to determine whether two states can be transformed into one another.
- First step is to construct thermomajorization curve $f_{\mathbf{p}}$ of **p** in the following manner:
  - Take line segments between $(0, 0)$ and (γ_i, p_i)
  - Join them end to beginning such that the resulting curve is convex.
- Do the same for **q;** if $f_{\mathbf{p}}(x) \geq f_{\mathbf{q}}(x)$ then we say that **p** thermomajorizes **q,** which we write as

$$\mathbf{p} \prec_\gamma \mathbf{q} \iff \exists S : S\mathbf{p} = \mathbf{q},\ S\gamma = \gamma$$

Horodecki and Oppenheim, Nature Communications 4, 2059 (2013)

Resource-theoretic flavour – state-to-state transformability



Two systems, two temperatures: Semilocal Thermal Operations

- Now, how do we handle two systems A and B, interacting with two disjoint baths, which can interact between themselves?
- One way to do this, referred to as *semilocal thermal operations* has been put forward in the context of nanoscale engine performance.
- In essence, it can be summed up y the following equations

$$\Phi(\rho_{AB}) = \mathrm{Tr}_{E_A,E_B}\left[U(\gamma_{E_A}\otimes\rho_{AB}\otimes\gamma_{E_B})U^\dagger\right]$$

$$[U, H_{E_A}+H_A+H_B+H_{E_B}] = 0$$

$$[U, \beta_A(H_{E_A}+H_A)+\beta_B(H_B+H_{E_B})] = 0$$

Bera, Lewenstein and Bera, npj Quantum Information 7, 31 (2021)

Ingredient 2:
Local Operations and Classical Communication

- Consider two systems – customarily referred to as Alice and Bob – which can interact only via classical communication, while operating locally on their systems; a so-called distant laboratory setting
- LOCC is notoriously difficult to handle, but can be summarised in three simple steps
  - Preprocess a system locally
  - Measure it and broadcast results
  - Process other systems conditionally based on the result obtained by the measuring party
  - Loop until the rounds are depleted
- An in-depth consideration in the framework of quantum instruments has been made by Chitambar *et al*. – we do not need it here, however.

Bell nonlocality and CHSH inequality

Bell nonlocality and CHSH inequality

- In its essence, an entangled bipartite state is a state that cannot be written as a convex mixture of local states,

$$\rho_{AB} \neq \sum_i p_i \rho_{Ai} \otimes \rho_{Bi}, \quad p_i \geq 0$$

- This notion is reflected, by construction in local hidden variable models (LHVM) proposed by John Bell – where settings of measuring devices and their outcomes are defined locally, up to a shared random variable,

$$p(ab \mid xy) = \int \mathrm{d}\lambda \, p(a \mid x, \lambda) p(b \mid y, \lambda) p(\lambda)$$

Bell nonlocality and CHSH inequality

- The famous CHSH (Clauser-Horne-Shimony-Holt) inequality is one of the simplest examples of distinguishing classical and non-classical theories by means of LHVM. It is defined by the following observable

$$\mathcal{C}_{\mathrm{CHSH}} = A_0 B_0 - A_0 B_1 + A_1 B_0 + A_1 B_1$$

  where A and B observables have eigenvalues ± 1

- Bounds on expectation values are known as

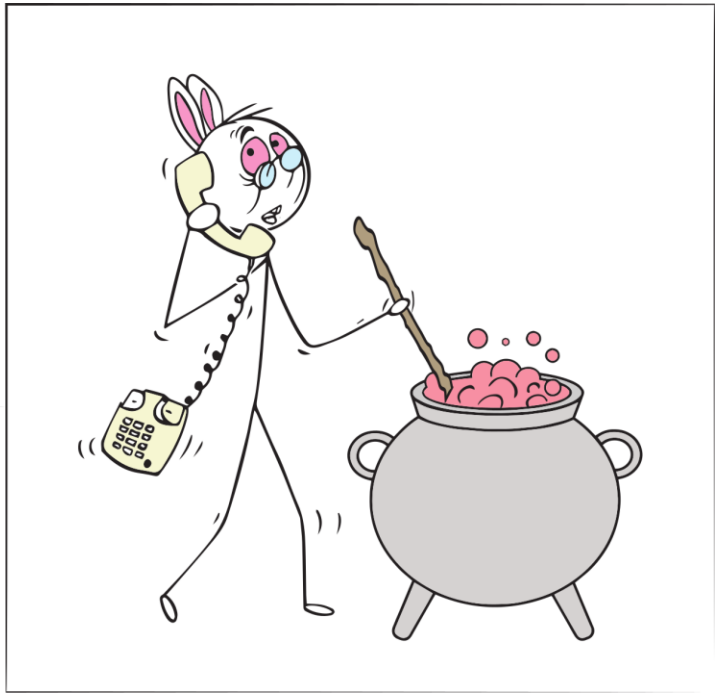
$$|\mathrm{Tr}(\rho \mathcal{C}_{\mathrm{CHSH}})| \leq \begin{cases} 2 & \text{Classical} \\ 2\sqrt{2} & \text{Quantum (Tsirelson)} \\ 4 & \text{No-signalling} \end{cases}$$

Local Thermal Operations and Classical Communication

When distant laboratories come into contact with thermal baths

What do we actually mean?

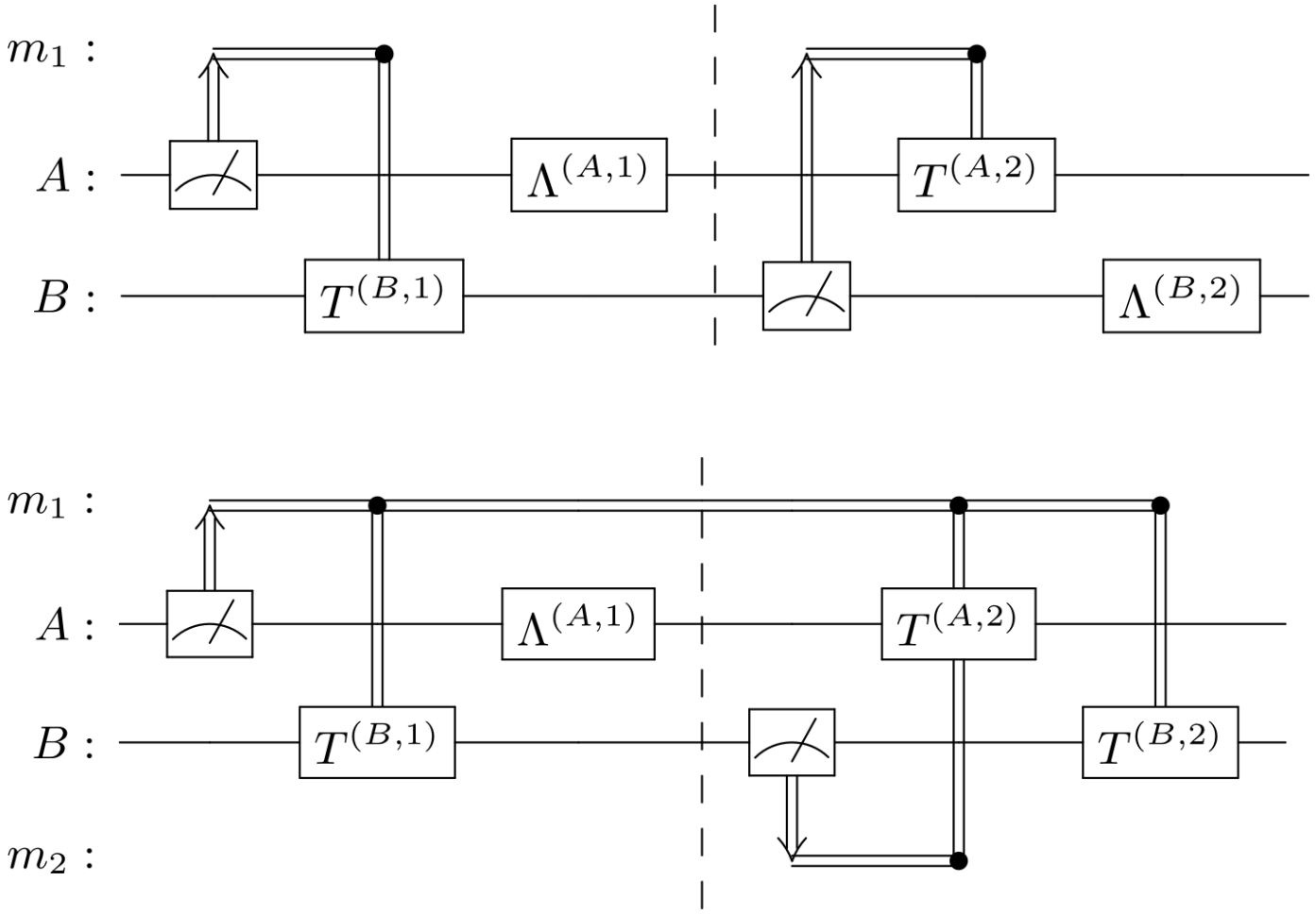
CLASSICAL COMMUNICATION

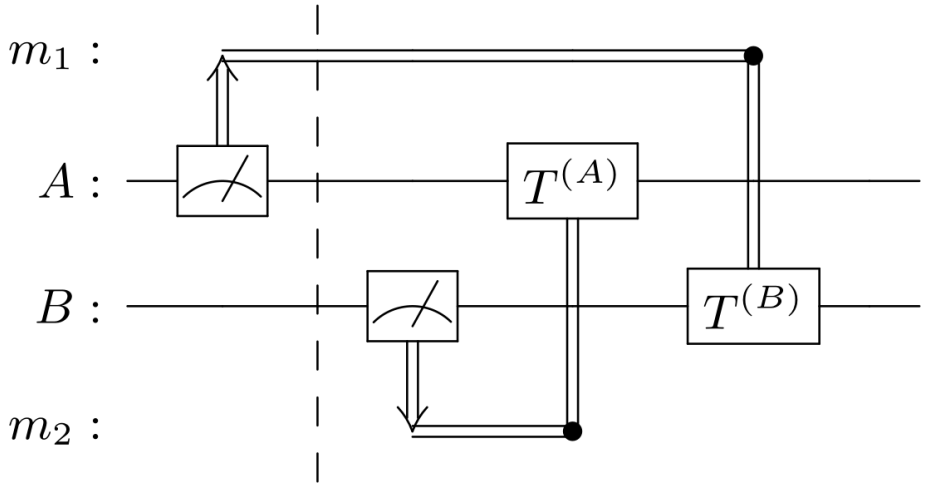


Precise set-up of the framework

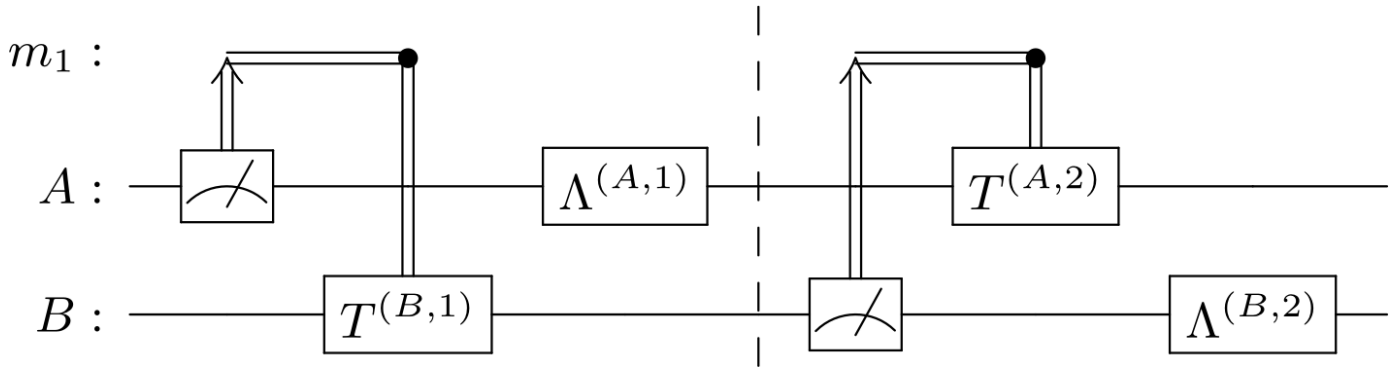
- We consider two finite-dimensional systems with their local energy structures given by Hamiltonians $H_A,\ H_B$
- Both parties are taken in contact with local thermal baths with at respective inverse temperatures $\beta_A,\ \beta_B$
- Mimicking the LOCC paradigm, the parties are restricted to:
  - Local TOs
  - Measurements in the energy eigenbasis
  - TOs conditioned on the results of measurements

Diagrammatic representation of different LTOCC variants





LTOCC (without memory)

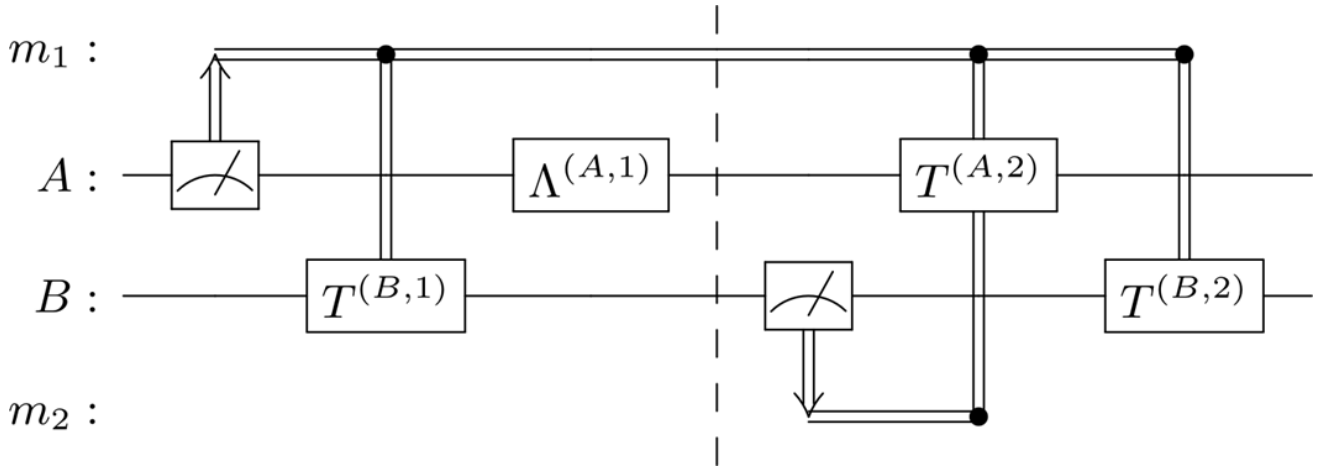


$$M = \prod_{\alpha} M^{(\alpha)} \quad \text{where} \quad M^{(\alpha)}_{ij,kl} = \Lambda^{(A,\alpha)}_{ik} T^{(B,\alpha)}_{jkl}$$

$$\text{or} \quad M^{(\alpha)}_{ij,kl} = T^{(A,\alpha)}_{ikl} \Lambda^{(B,\alpha)}_{jl} \quad ,$$

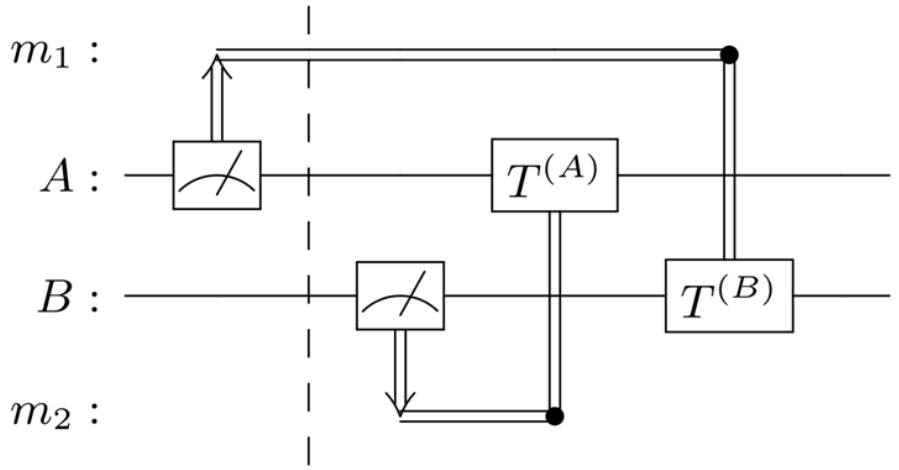
$$\forall_l \sum_k T^{(A)}_{ikl} \gamma^{(A)}_k = \gamma^{(A)}_i \quad , \quad \forall_k \sum_l T^{(B)}_{jkl} \gamma^{(B)}_l = \gamma^{(B)}_j \quad .$$

LTOCC+M (with memory)



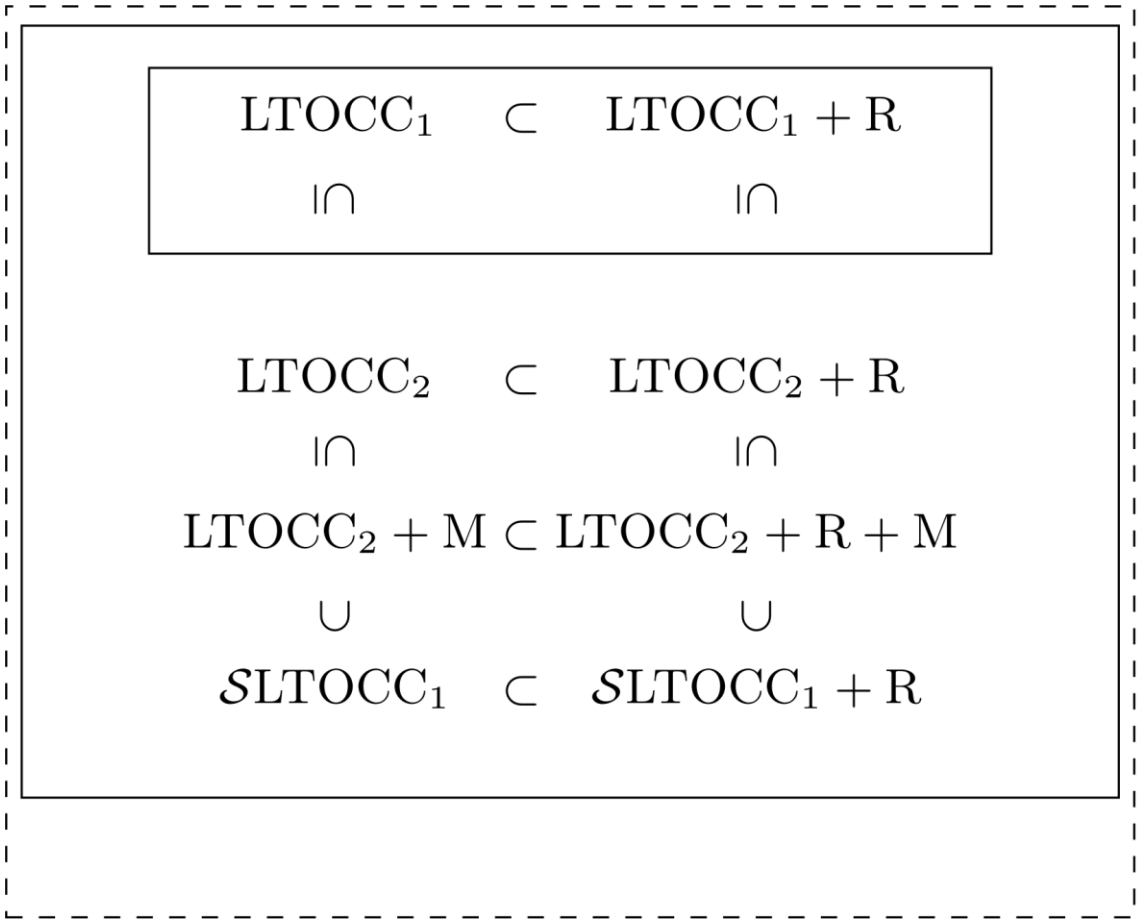
$$\mathbf{M}(\mathbf{p} \otimes \mathbf{q}) = \mathrm{Tr}_{\mathcal{M}_1,\ldots,\mathcal{M}_n} \left(\left[\bigcirc_\alpha \mathbf{M}^{(\alpha)} \right] (\mathbf{p} \otimes \mathbf{q}) \right).$$

Symmetric LTOCC



$$M_{ij,kl} = T^{(A)}_{ikl} T^{(B)}_{jkl} \quad \forall_k \sum_l T^{(A/B)}_{ikl} \gamma_l = \gamma_i \ , \quad \forall_l \sum_k T^{(A/B)}_{ikl} \gamma_k = \gamma_j$$

LTOCC Hierarchy



1 round $\subset$ SLTO

$\cap$

$\cap$

2 rounds

$\cap$

$\vdots$

$\cap$

n rounds

Marginal state transformations and relation to Birkhoff polytope

- Under LTOCC without memory, transformations of marginal states is governed by thermomajorization

$$\mathbf{p}_A \otimes \mathbf{p}_B \xrightarrow{\text{marginally}} \mathbf{q}_{A/B} \iff \mathbf{p}_{A/B} \prec_{\gamma_{A/B}} \mathbf{q}_{A/B}$$

- Symmetric restrictions complicate the situation even for infinite temperature.

Marginal state transformations and relation to Birkhoff polytope

For any distribution $\mathbf{p}$ let us define its projection $\tilde{\mathbf{p}}$ onto the boundary of the probability simplex, which we will refer to as boundary distribution of $\mathbf{p}$,

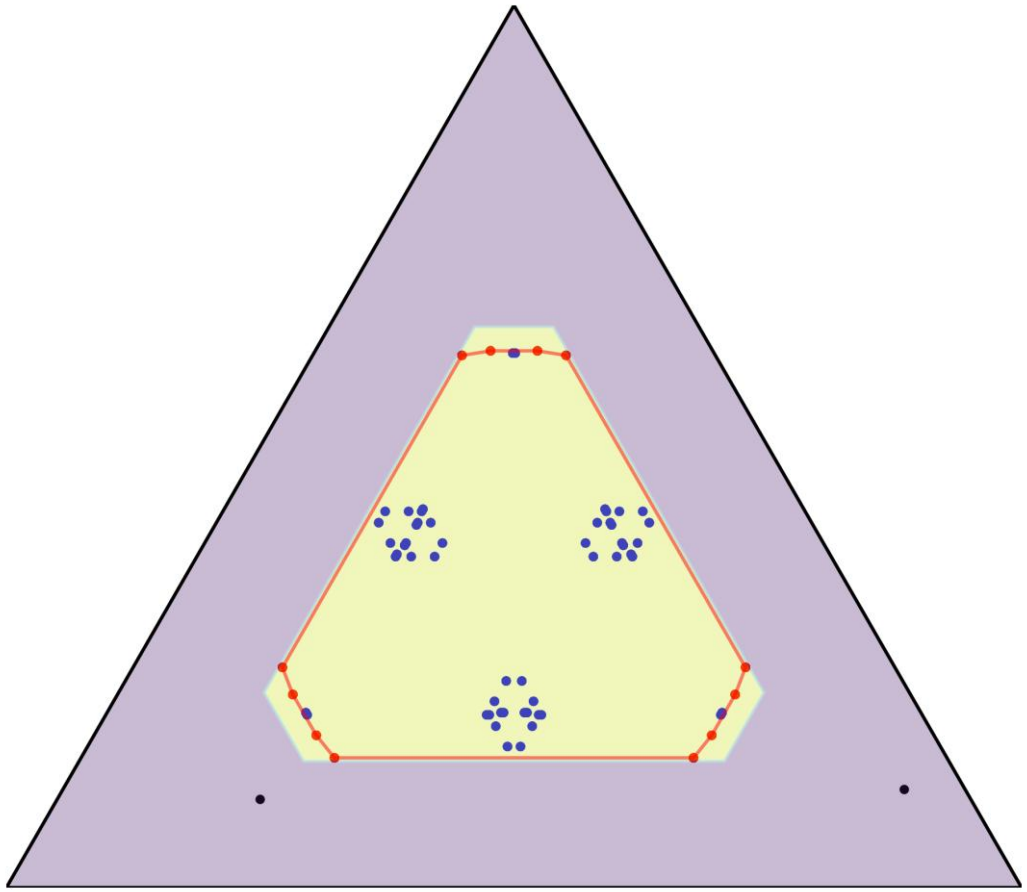
$$\tilde{\mathbf{p}} = \frac{1}{1-w_p}\mathbf{p} - \frac{w_p}{1-w_p}\gamma, \tag{40}$$

with $w_p = \min_i \frac{p_i}{\gamma_i}$; note that $\tilde{p}$ has at least one zero entry.

Theorem 12 (Enhanced thermomajorization for bithermal tensors)**.** *Let* $\mathbf{p}, \mathbf{q}, \mathbf{r}$ *be three probability distributions, and* $\tilde{\mathbf{p}}$, $\tilde{\mathbf{q}}$ *boundary distributions of* $\mathbf{p}, \mathbf{q}$*. Moreover, let us define*

$$\bar{\mathbf{r}} = \frac{1}{(1-w_p)(1-w_q)}\mathbf{r} - \frac{1-(1-w_p)(1-w_q)}{(1-w_p)(1-w_q)}\gamma \tag{41}$$

Then the necessary condition for $\mathbf{r}$ *to be achivable by a bithermal tensor from* $\mathbf{p}, \mathbf{q}$ *is not only* $\mathbf{p} \succ_\beta \mathbf{r}$ *and* $\mathbf{q} \succ_\beta \mathbf{r}$ *but also* $\tilde{\mathbf{p}} \succ_\beta \bar{\mathbf{r}}$ *and* $\tilde{\mathbf{q}} \succ_\beta \bar{\mathbf{r}}$*.*



Marginal state transformations and relation to Birkhoff polytope

- Let us take a step back and revisit bistochastic matrices

$$\sum_i S_{ij} = \sum_j S_{ij} = 1$$

- Set of all such operations, given by convex combinations of permutation matrices, is called a Birkhoff polytope
- By extension, tristochastic tensors can be defined by

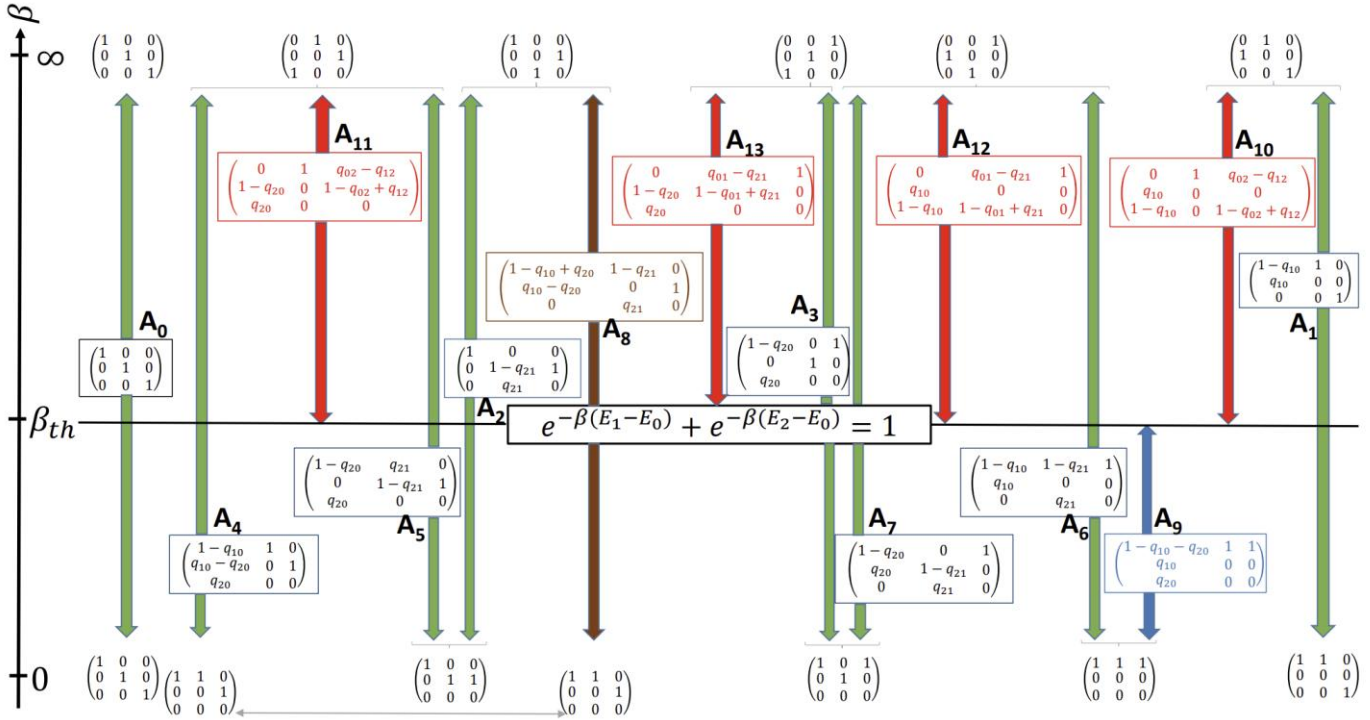
$$\sum_i T_{ijk} = \sum_j T_{ijk} = \sum_k T_{ijk} = 1$$

- Extreme points of tristochastic Birkhoff polytope, however, are not restricted to permutation tensors, composed solely of 0 and 1 entries.

Birkhoff, Garrett (1946), "Tres observaciones sobre el algebra lineal [Three observations on linear algebra]", Univ. Nac. Tucumán. Revista A., 5: 147–151

Marginal state transformations and relation to Birkhoff polytope

- Thermal counterpart of Birkhoff polytope is notoriously hard to characterize, with the only full characterization given in dimension 3 and only some extreme points known for all dimensions.



Mazurek and Horodecki, New J. Phys. 20 053040

Marginal state transformations and relation to Birkhoff polytope

- Based on the prior results on extreme points of thermal operations, we found a family of non-decomposable bi-thermal tensors that exist for all dimensions, given sufficiently low temperature

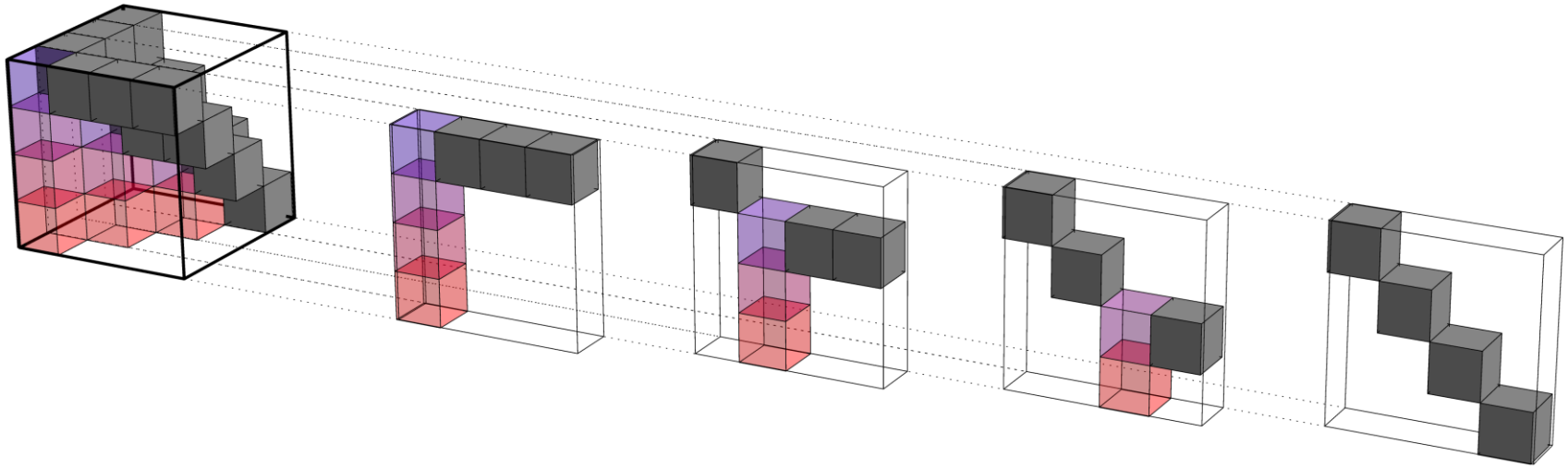
$$T_{ijk}|_{d=n} = (T_{i,0,k}|\cdots|T_{i,n,k})|_{d=n} =$$

$$= \left(\begin{array}{c|c|c|c|c} 1-\sum_{i=1}^{n} g_{i,0} & 1 \;\cdots\; 1 & 1 & 0 \;\cdots\; 0 & \\ \hline g_{1,0} & & 0 & & \\ \vdots & \mathbf{0} & \vdots & T_{i,0,k}|_{d=n-1} & \cdots \\ g_{d,0} & & 0 & & \end{array}\right.$$

$$\left.\begin{array}{c|c} 1 & 0 \;\cdots\; 0 \\ \hline 0 & \\ \vdots & T_{i,n-1,k}|_{d=n-1} \\ 0 & \end{array}\right)$$

Marginal state transformations and relation to Birkhoff polytope

- Based on the prior results on extreme points of thermal operations, we found a family of non-decomposable bi-thermal tensors that exist for all dimensions, given sufficiently low temperature

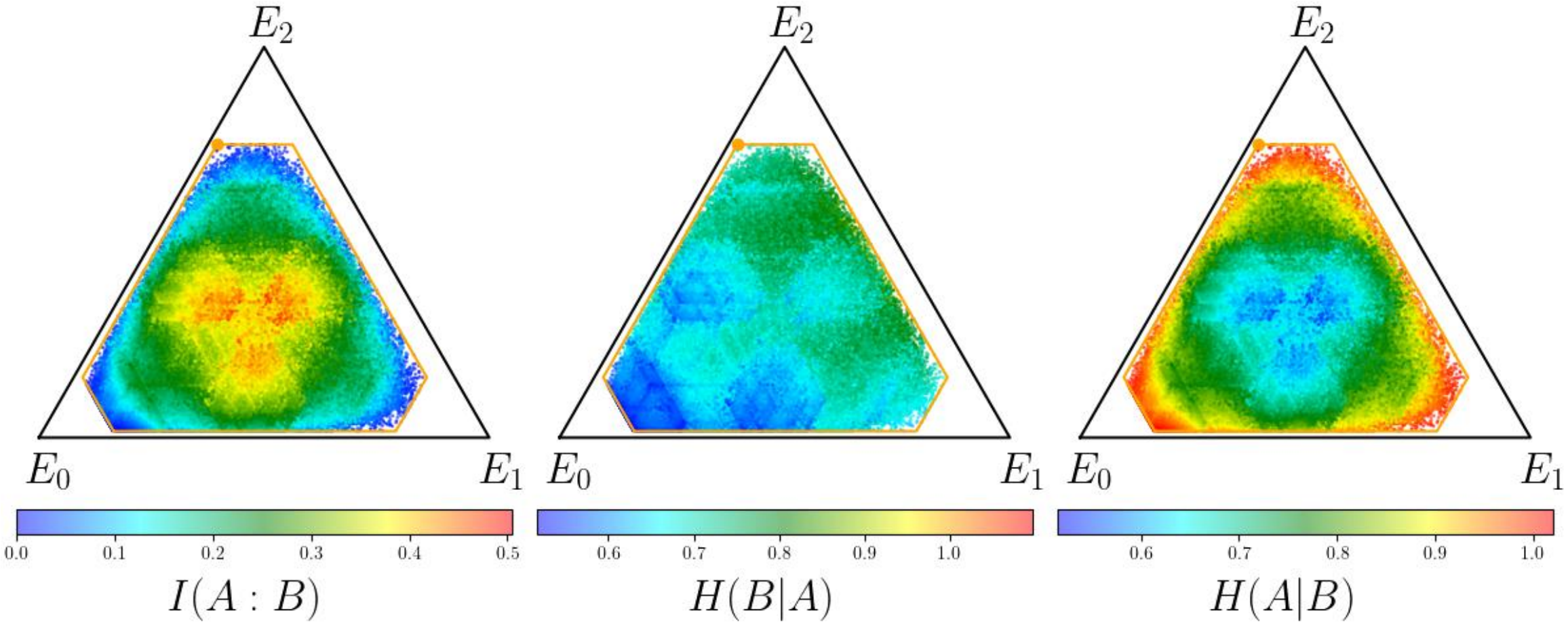


When correlations become maximal

Observation 6. *Let T_{jkl} be a thermal tensor, $\sum_l T_{jkl}\gamma_l = \gamma_j$, having an identity [or permutation matrix in infinite temperature case] as at least one layer $T_{jkl_0} = \delta_{j,k}$ [or $T_{jkl_0} = \delta_{j,\sigma(k)}$ for some permutation σ]. Then the pair of states $\mathbf{p}, \mathbf{q}$, second of which is a sharp state $q_l = \delta_{l,l_0}$ can be mapped into the maximally correlated state in one-round LTOCC.*

Observation 7. *Consider a $\mathcal{S}LTOCC$ operation consisting of two copies of the same extremal (bi)thermal tensor $M_{ij,kl} = T_{ikl}T_{jkl}$, which entries are either 0 or 1. Then for any product input state $\mathbf{p} \otimes \mathbf{q}$ the output state $\pi = M(\mathbf{p} \otimes \mathbf{q})$ is maximally correlated.*

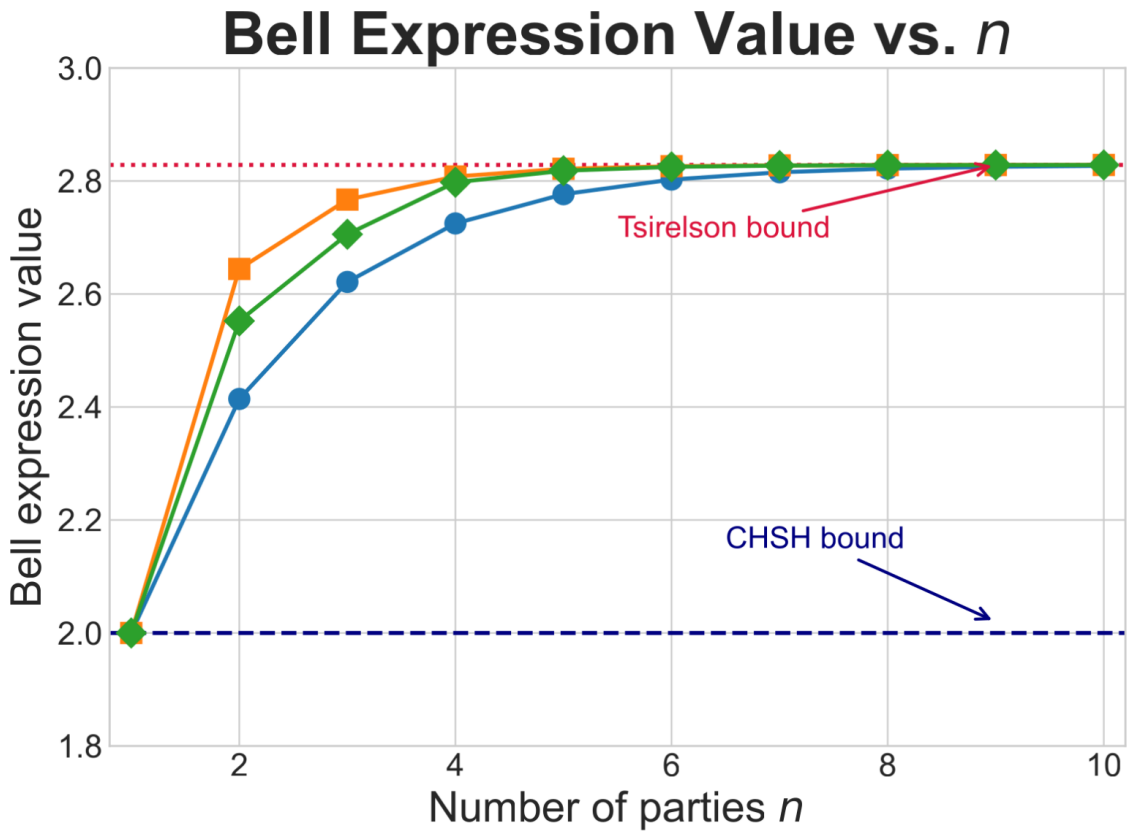
...and when they are intermediate

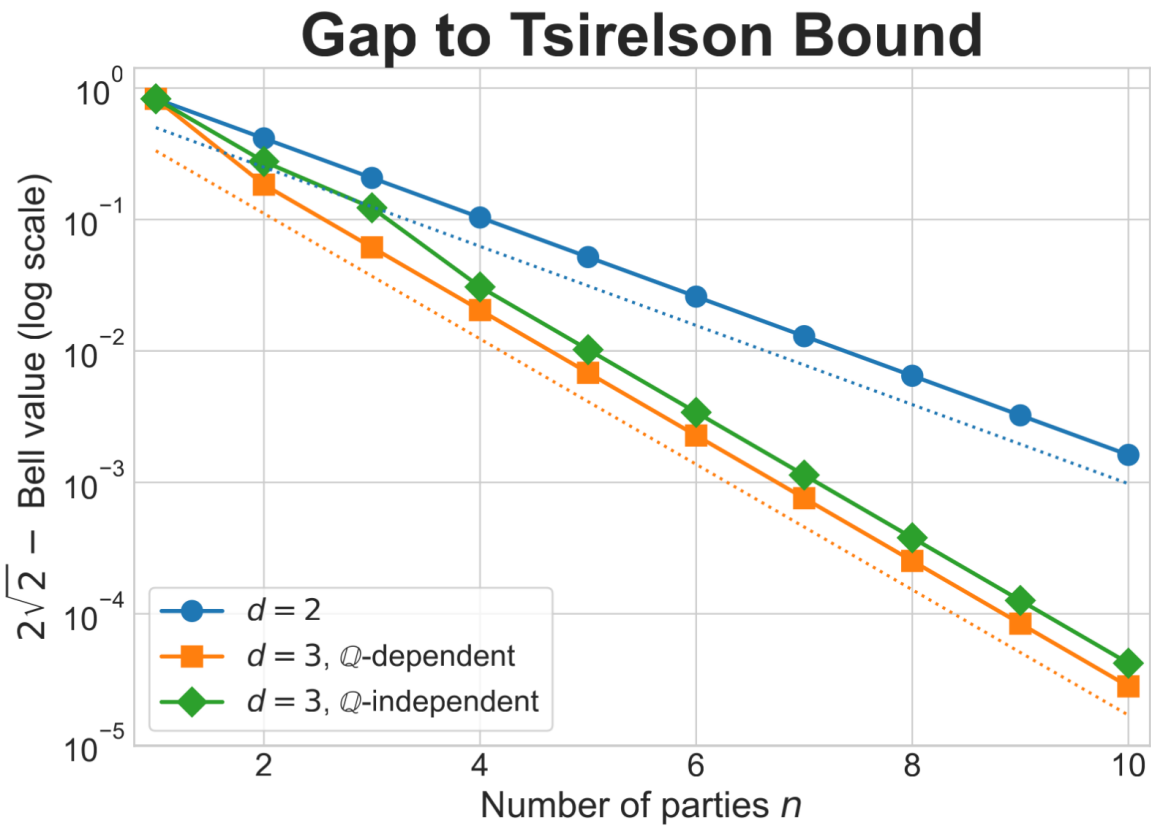


Thermal Bell nonlocality

- Perhaps the most interesting result comes from considering the possibility of detecting entanglement with measurements realisable through LTOCC protocols.
- When energy levels are non-degenerate, it is impossible to detect entanglement in single-copy regime – all accessible measurements are compatible.
- For n copies we find the following restrictions

$$|\langle\Psi|\mathcal{C}_{\mathrm{CHSH}}|\Psi\rangle| \leq \begin{cases} 2 & \text{Classical,} \\ 2\frac{D_{\mathrm{ndeg}}+\sqrt{2}D_{\mathrm{deg}}}{d^n} & \text{Thermal (LTOCC+M),} \\ 2\sqrt{2} & \text{Quantum.} \end{cases}$$


Bell Expression Value vs. n
Bell expression value
Number of parties n
Tsirelson bound
CHSH bound
3.0
2.8
2.6
2.4
2.2
2.0
1.8
2
4
6
8
10


Gap to Tsirelson Bound
$2\sqrt{2}$ − Bell value (log scale)
Number of parties n
$d = 2$
$d = 3$, $\mathbb{Q}$-dependent
$d = 3$, $\mathbb{Q}$-independent
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
2
4
6
8
10

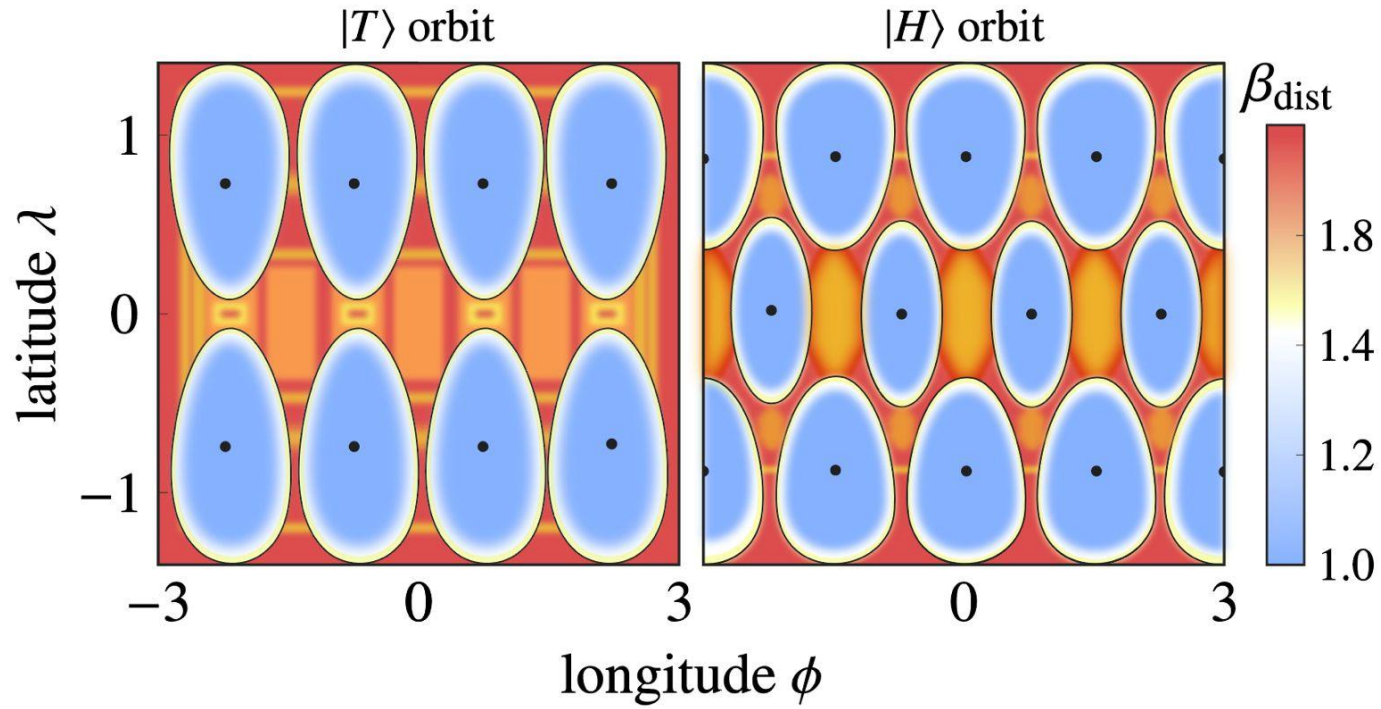
Summary

- We studied interaction of two rarely interacting frameworks, studied in the context of quantum resources:
  - Local Operations and Classical Communications (LOCC), used in the context of entanglement manipulation
  - Thermal Operations (TO), used for manipulation of athermality
- We introduced general framework for studying such operations and demonstrated hierarchy and relation to earlier work on semilocal TO.
- We introduced necessary conditions on marginal transformability of energy-incoherent states and demonstrated nondecomposable extreme operation, which exists for all dimensions given temperature low enough
- We investigated ability to both generate and detect correlations using LTOCC measurements and demonstrated thermal restrictions on detectable Bell nonlocality

Closing remarks – what came after, where it might go

- The study feeds into a larger question of interactions between different quantum resources and potential limitations and advantages stemming from such combinations
- Other directions that have been already explored are:
  - Resource engines [Wojewódka-Ściążko *et al.*]
  - Quantum resource allocation [Salazar, Biswas, J. Cz. *et al.*]
  - Generation of entanglement from athermality [de Oliveira, Son, J. Cz. and Ng] and heat engines [J. Cz. and Bistroń]
  - Detection of entanglement by anomalous heatflows [de Oliveira, Bohr-Brask and Lipka-Bartosik]
- Directions that are still work in progress include
  - Limitations on resource compositions [Ganardi, Son, J. Cz. *et al.]*
  - Generation of magic from athermality [de Oliveira, Macedo, J. Cz. *et al.*]
  - …and more?


q_1
H
S
X
q_2
Z
q_3
q_1
H
T
q_2
T
q_3
T


$|T\rangle$ orbit
$|H\rangle$ orbit
β_{dist}
1.8
1.4
1.2
1.0
latitude λ
1
0
−1
−3
0
3
0
3
longitude ϕ


Thank you for your attention!
Grazie per l’attenzione!