Typicality of contextuality certification in prepare and measure scenarios

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Outline

1. Contextuality

❖ Intuitive explanation

❖ Generalized contextuality in prepare-and-measure scenarios

❖ Contextual correlations

❖ Contextuality as a notion of nonclassicality

2. Contextuality certification

❖ Classical GPTs, simplex embeddings

3. Typicality

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Illustrated by Vinicius P. Rossi

(i) Simple explanation: a raccoon

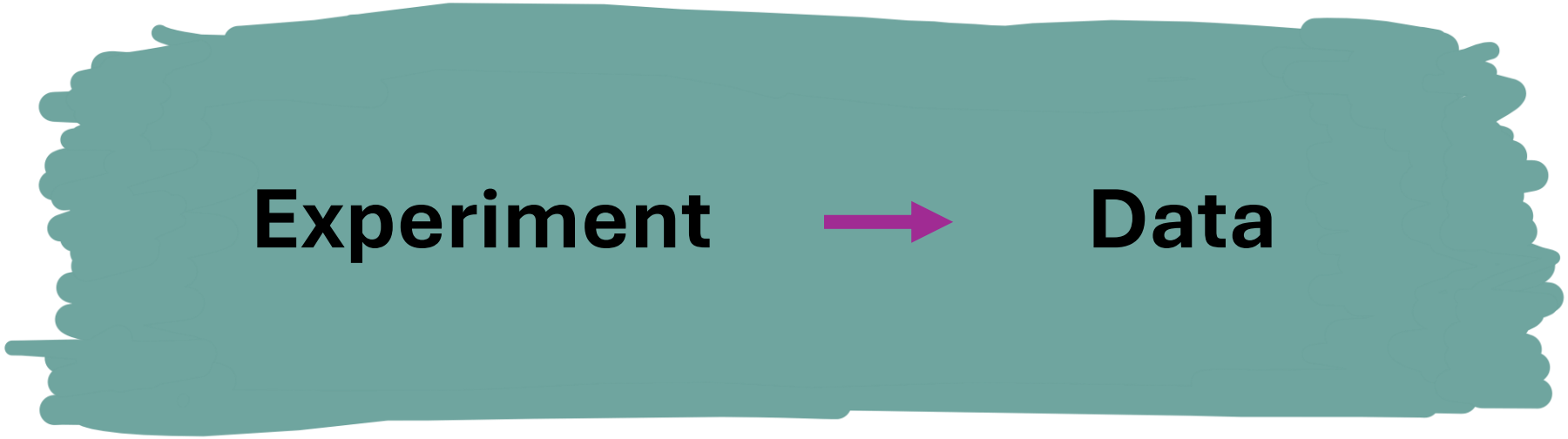
(ii) Second explanation: Animal conspiracy

(ii) Second explanation: Animal conspiracy

Contextuality is when you need (ii)

(i) (ii)

Quantum kitchen is incompatible with explanation (i)



1) Look for an intuitive explanation
2) If it doesn’t work, maybe some of the basic intuitions don’t work

Nomenclature

Preparation

Context

(impossible to distinguish by any measurement)

Measurement

Nomenclature

Explanation

Noncontextual

Contextual

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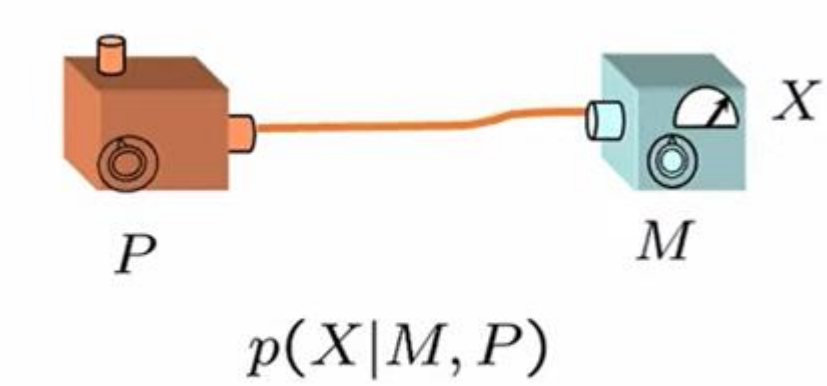
Prepare-and-measure scenario

Operational procedure:

List of laboratory instructions



Observed probabilities

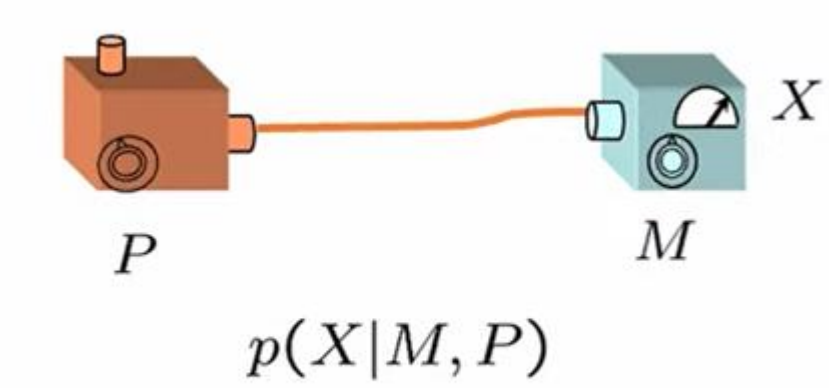
Prepare-and-measure scenario

(Hypothesized) **Explenation for the observed probabilities:**
Ontological model of an operational theory

Ontic state

Possible properties

of the system



Outcomes

Reveal something about
the ontic state

Prepare-and-measure scenario

Ontological model of an operational theory

Probability distribution over properties of the system

(maybe you can't prepare it exactly)

$\lambda \in \Lambda$ Ontic state space

$$\mathsf{P} \leftrightarrow \mu(\lambda|\mathsf{P})$$

Determine what properties you learn about

$$\mathsf{M} \leftrightarrow \xi(X|\mathsf{M},\lambda)$$

$$p(X|M,P) = \sum_\lambda \xi(X|M,\lambda)\,\mu(\lambda|P)$$

Generalized contextuality

Motivated by Leibniz's principle of the identity of indiscernibles:

> *If a difference in set-up is not distinguished in the observable phenomena, then it should not be distinguished in the ontological picture either.*

An ontological model of an operational theory is *noncontextual* if

experimental procedures which always lead to the same observational data → **identical representations in the ontological model**

R. W. Spekkens, Contextuality for preparations, transformations, and unsharp measurements, https://arxiv.org/abs/quant-ph/0406166

Generalized contextuality

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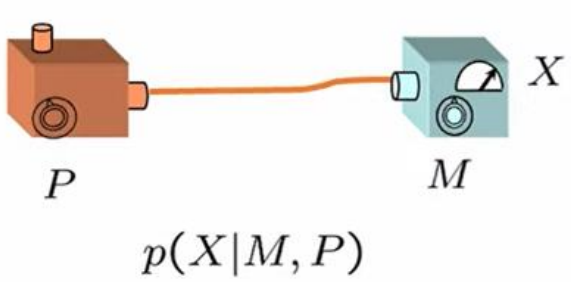

→

identical representations in the ontological model

$$p(X|M,P) = \sum_\lambda \xi(X|M,\lambda)\,\mu(\lambda|P)$$

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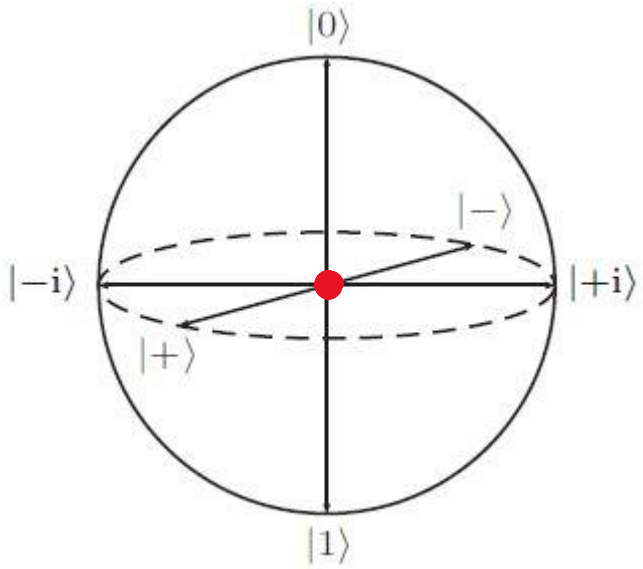
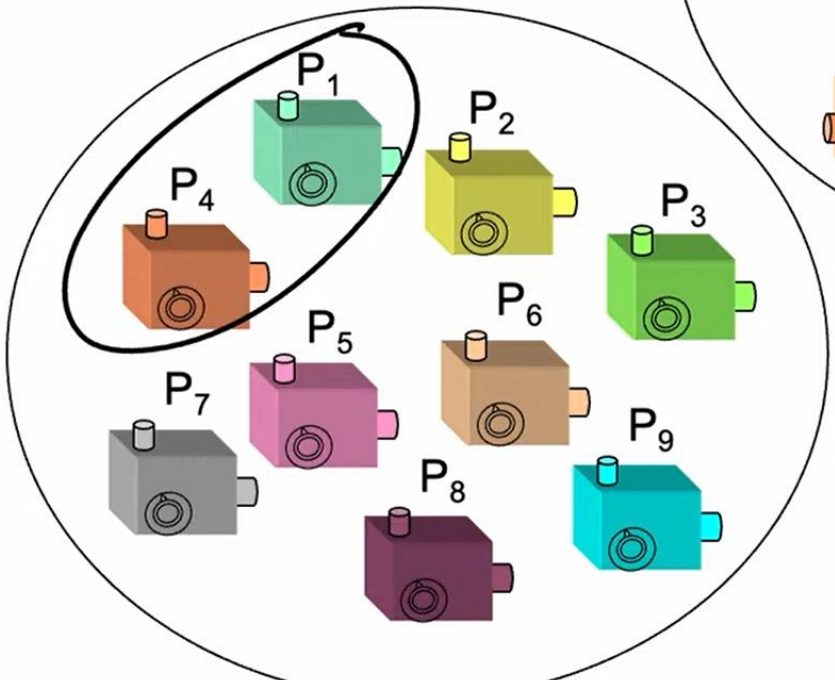
Operational equivalences

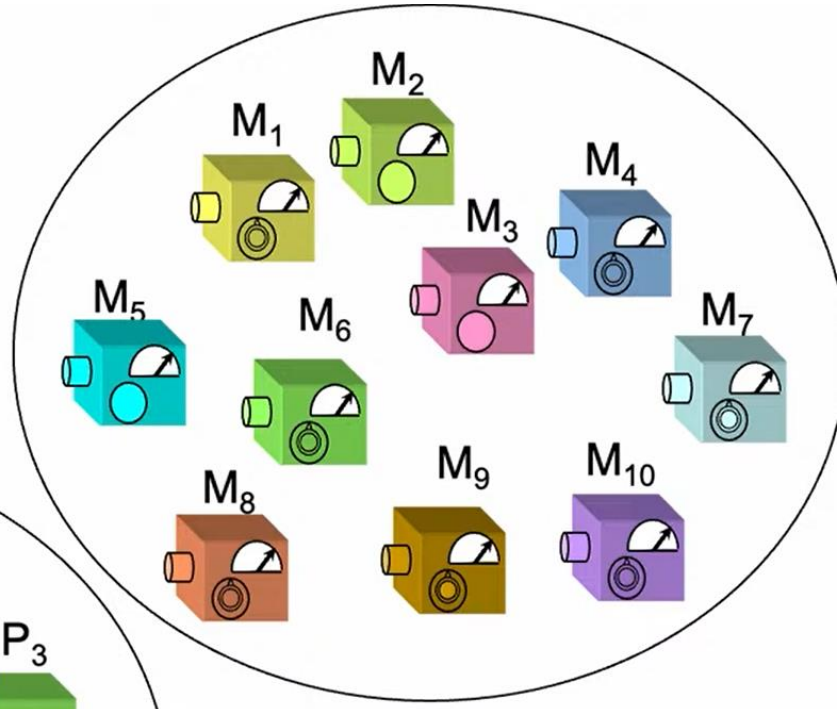


Context distinguishes operationally equivalent objects

$$P_1 \simeq P_4$$

$$\forall M : p(X|P_1, M) = p(X|P_4, M)$$





Noncontextuality inequalities

Operational equivalences

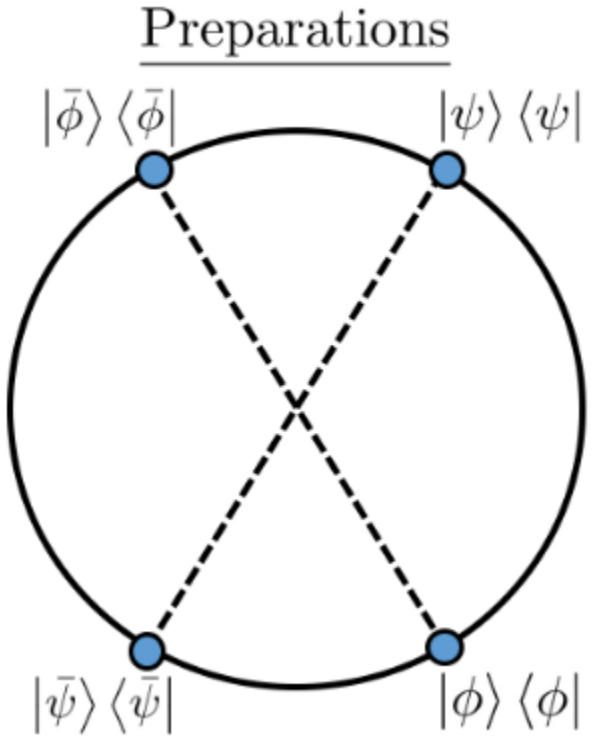
$$\frac{1}{2}\left|\phi\right\rangle\left\langle\phi\right| + \frac{1}{2}\left|\bar{\phi}\right\rangle\left\langle\bar{\phi}\right| = \frac{1}{2}\left|\psi\right\rangle\left\langle\psi\right| + \frac{1}{2}\left|\bar{\psi}\right\rangle\left\langle\bar{\psi}\right| = \frac{\mathbb{1}}{2}$$

imply constraints

$$\frac{1}{2}\mu_{\phi}(\lambda) + \frac{1}{2}\mu_{\bar{\phi}}(\lambda) = \frac{1}{2}\mu_{\psi}(\lambda) + \frac{1}{2}\mu_{\bar{\psi}}(\lambda)$$

which in turn imply noncontextuality inequalities

$$\sum_{P,k,M} \alpha_{P,k,M}\ \mathrm{pr}(k|P,M) \leq r$$



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- ❖ Generalized contextuality in prepare-and-measure scenarios
- ❖ **Contextual correlations**
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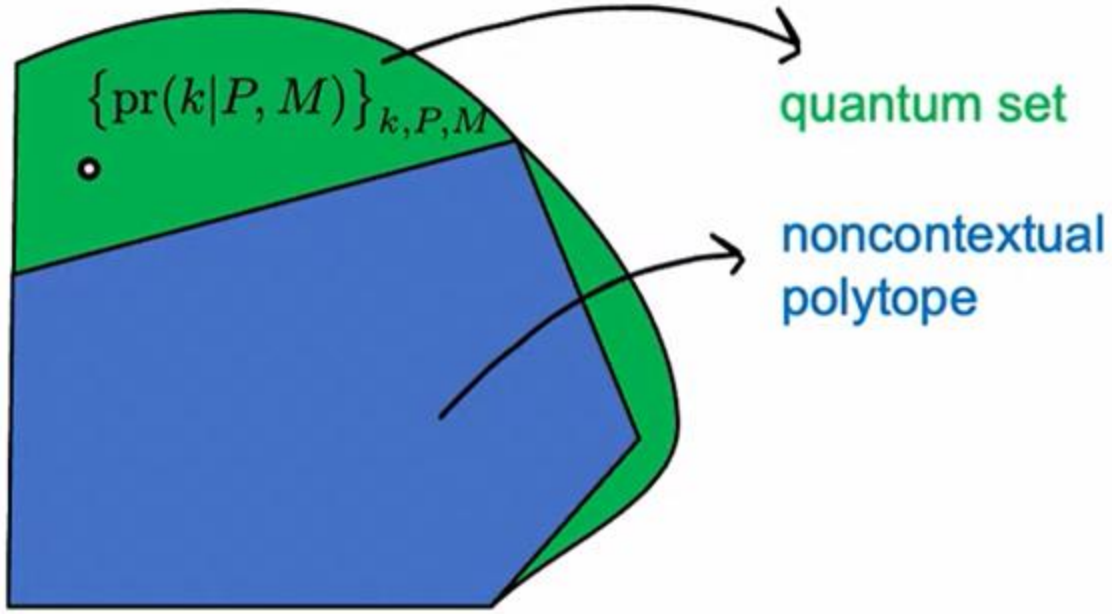
2. Contextuality certification
- ❖ Classical GPTs, simplex embeddings

3. Typicality

Noncontextuality inequalities

Violation of noncontextuality inequality

Proof of nonclassicality



Noncontextuality inequalities

$$\sum_{P,k,M} \alpha_{P,k,M}\ \mathrm{pr}(k|P,M) \leq r$$

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Why study contextuality?

Foundationally

- Bell Nonlocality
- Kochen-Specker context.
- Hypergraph context.
- Quantum Darwinism
- Anomalous weak values
- Macrorealism
- Wigner function
- Simplex embedding

Practically

- Computation
- Info. processing
- Communication
- Metrology
- Cloning
- State discrimination
- Anomalous heat flow

Phillosophically

Stems from Leibniz's principle of identity of indiscernibles, used e.g. by Einstein in Relativity Theory

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Contextuality certification

Problem: Testing whether a scenario is (generalized) noncontextual is often analytically challenging

Solution: Use the equivalence between generalized noncontextuality and simplex embeddability of generalized probabilistic theories

Result: Checking for a simplex embedding is a linear program (https://arxiv.org/abs/2204.11905)

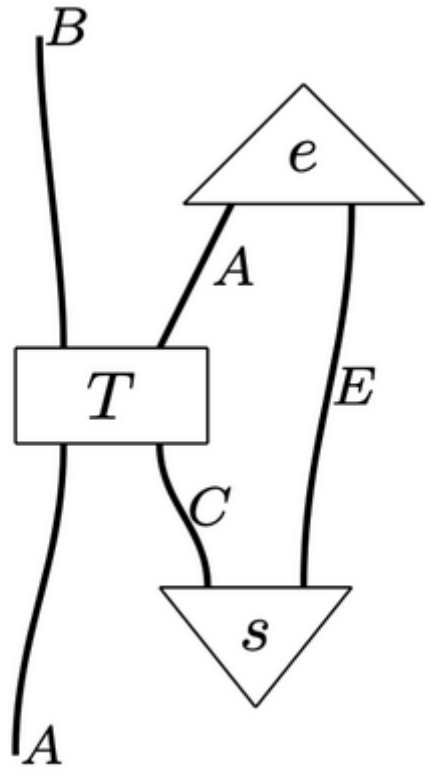
Generalized probabilistic theories (GPTs)

- Set of **states** $\mathcal{S}$

Define V as real inner product space where $\mathcal{S}$ live

- Dual of $\mathcal{S}$ is the set of vectors in V whose inner product with all state vectors in $\mathcal{S}$ lies between 0 and 1

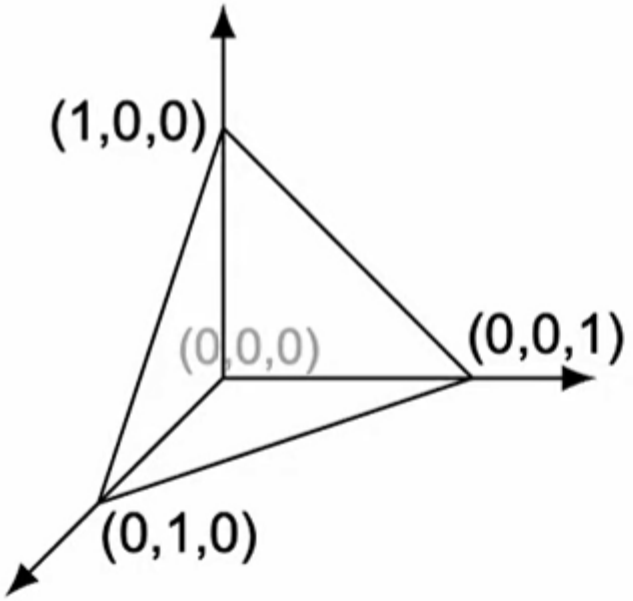
- The **effects** live in the dual space



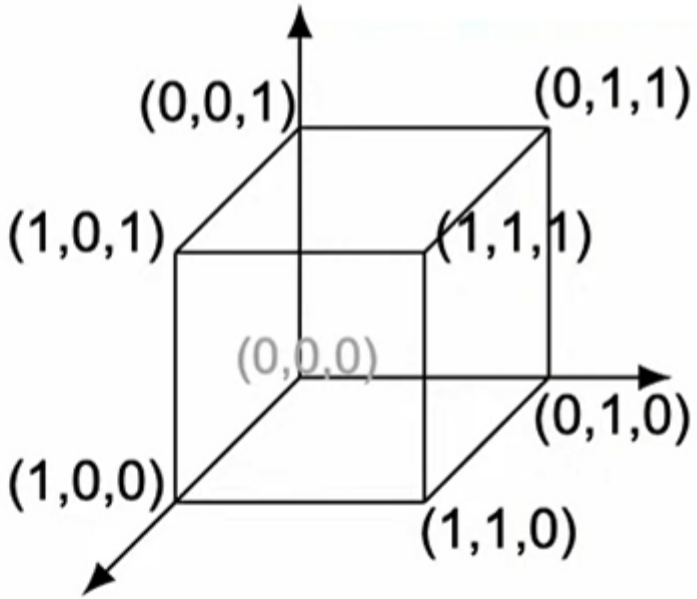
Classical GPTs are simplicial

A generalized probabilistic theory admits of a **classical explanation** if and only if there is a simplex that embeds its state space and the hypercube of effects that is dual to this simplex embeds its effect space.

states



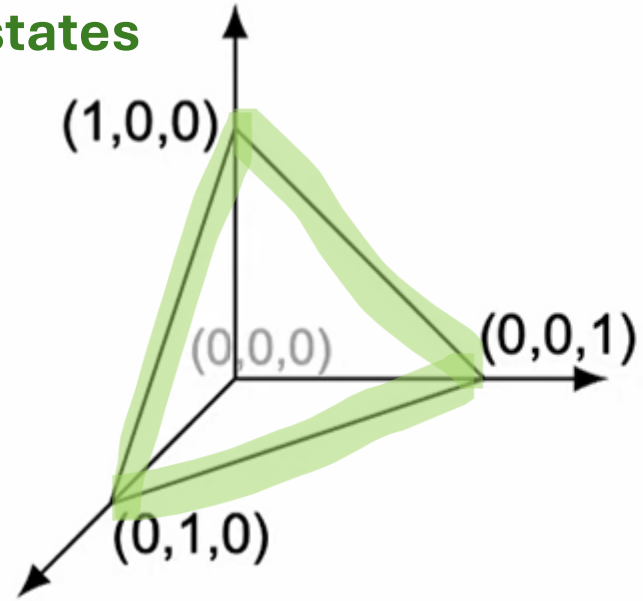
effects



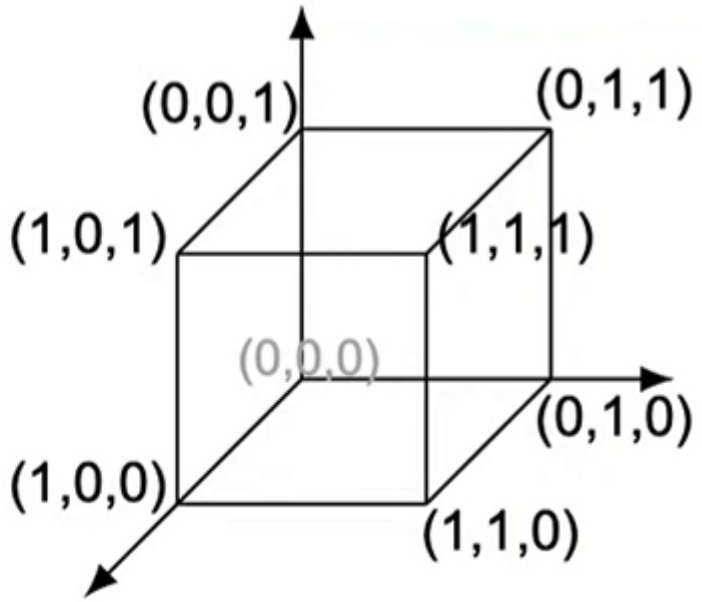
**Simplex – the simplest possible polytope in a given dimension (it is a generalization of the notion of a triangle or tetrahedron to arbitrary dimensions)*

Classical GPTs are simplicial

normalized states

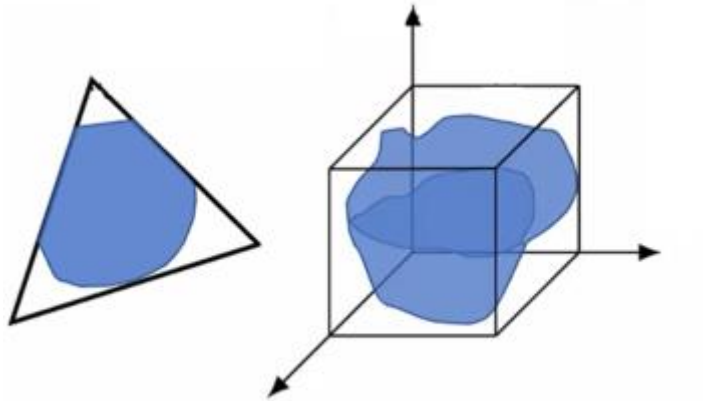


effects



- Three ontic states + probability distributions over them (with unique decomposition)
- Inner product of any point in the cube with any point in the triangle gives values between 0 and 1

How about subsets of simplicial theories?

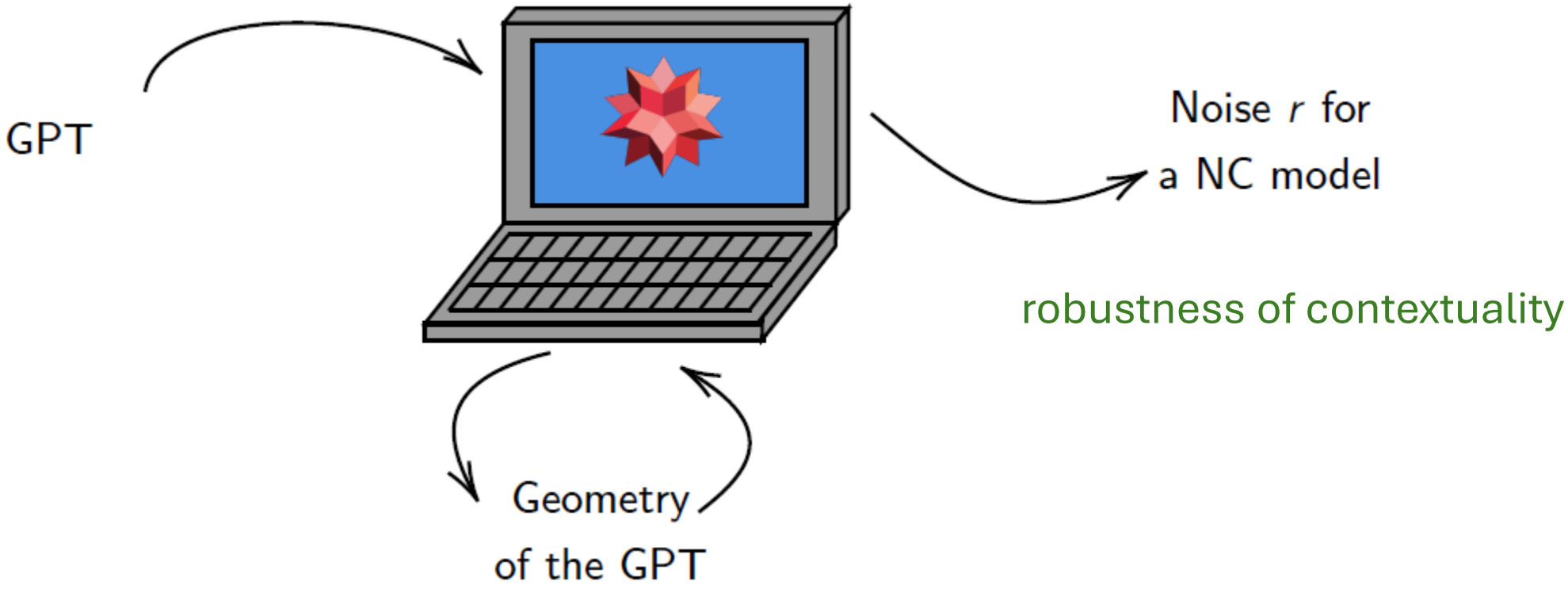


Simplex embeddable ⟷ **classically explainable**

A simplex embedding of a GPT is an ontological model for it

Linear program

How much partially depolarising noise $0 \leq r \leq 1$ must act on the states of the scenario so that a noncontextual description is possible?



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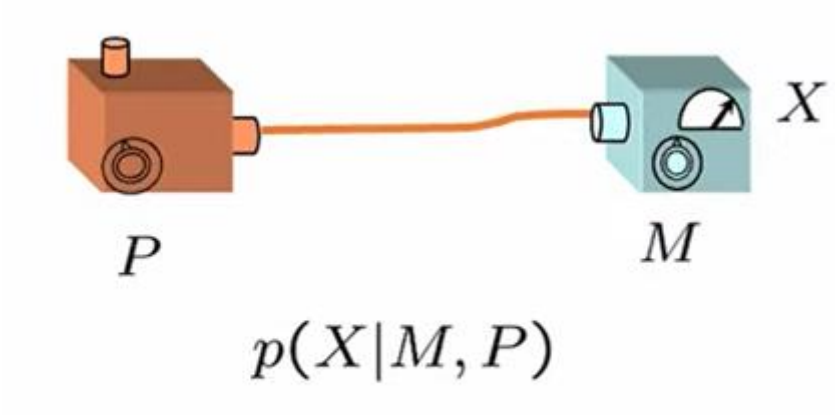
Typicality of contextuality certification

How often random quantum preparations and measurements produce nonclassical statistics?

For pure states and access to all measurements, linear dependence is both necessary and sufficient for contextuality.



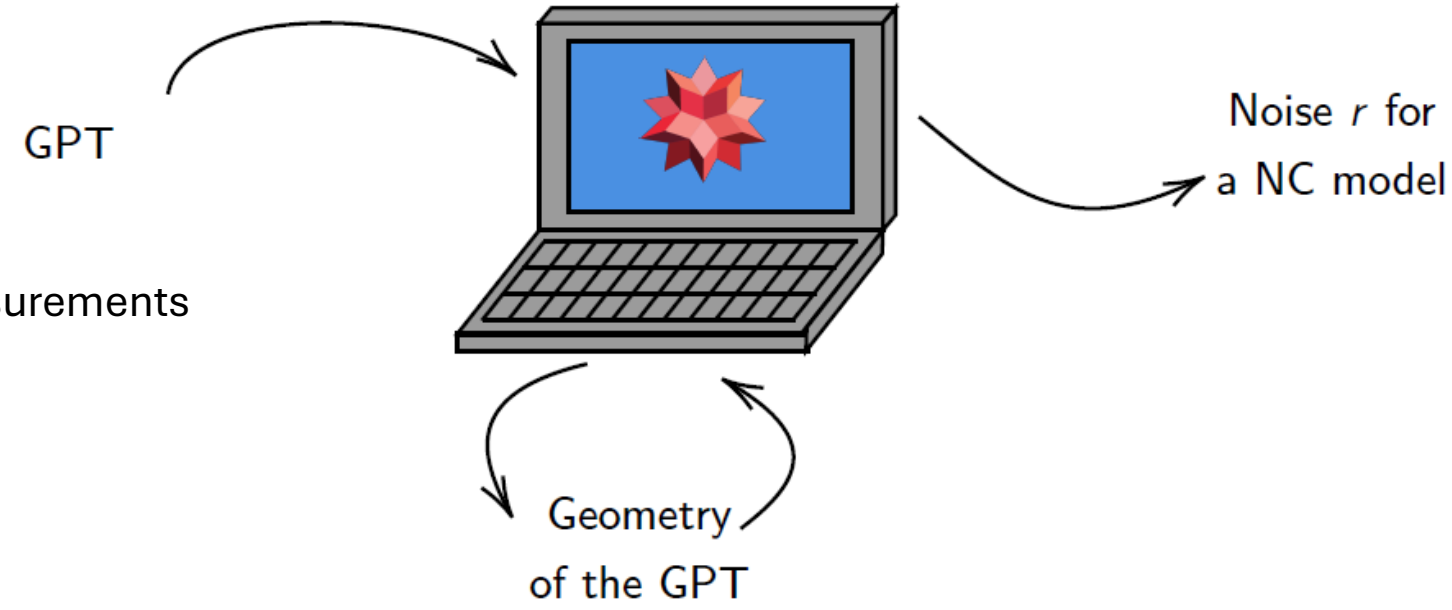
Numerical simulations

(n, m, d=2, N=10^6)

n preparations

m two-outcome measurements

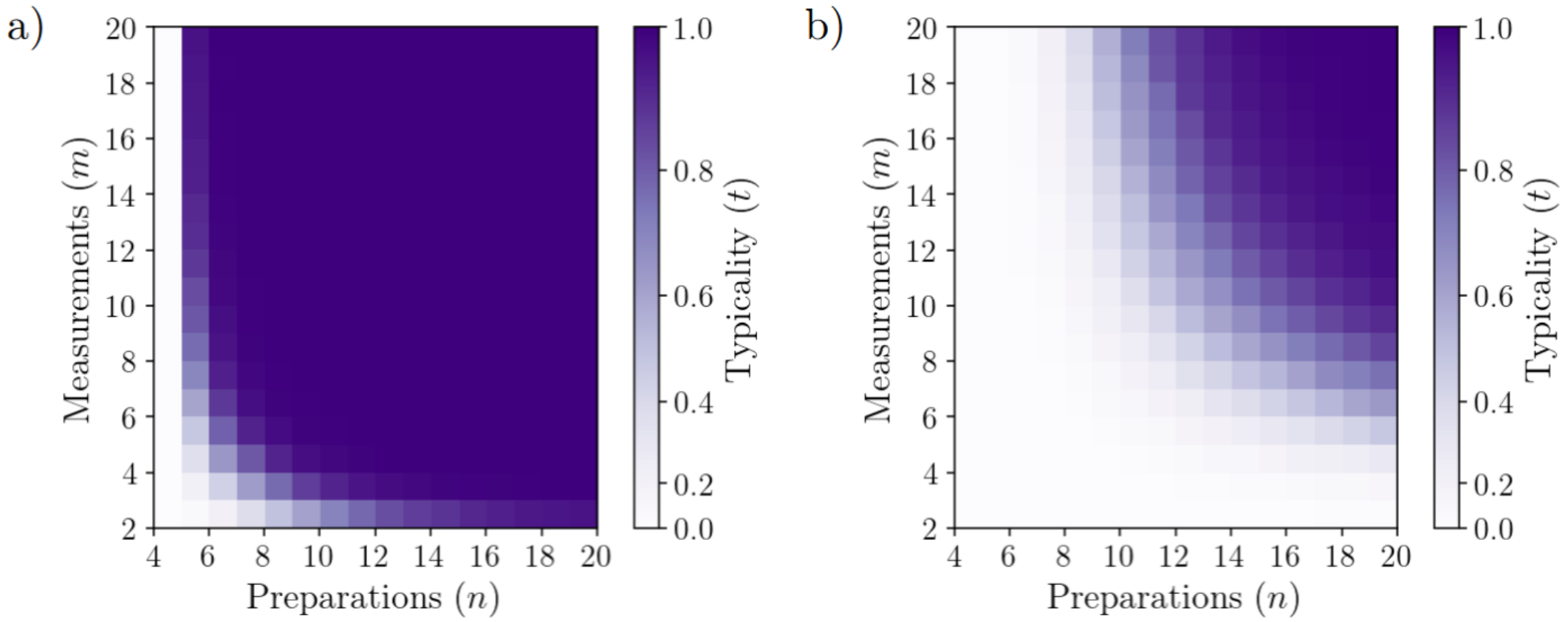
(randomly generated)



Result 1 - on number of preparations and measurements

Pure

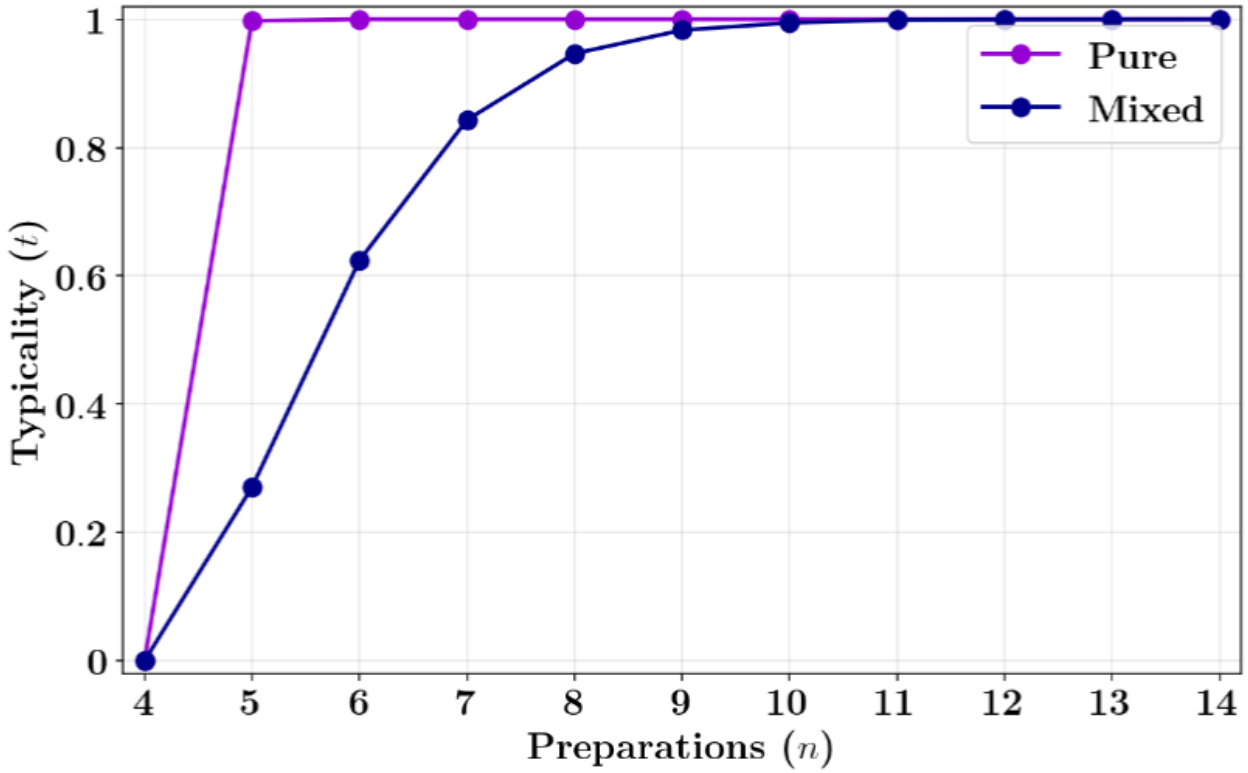
Mixed *



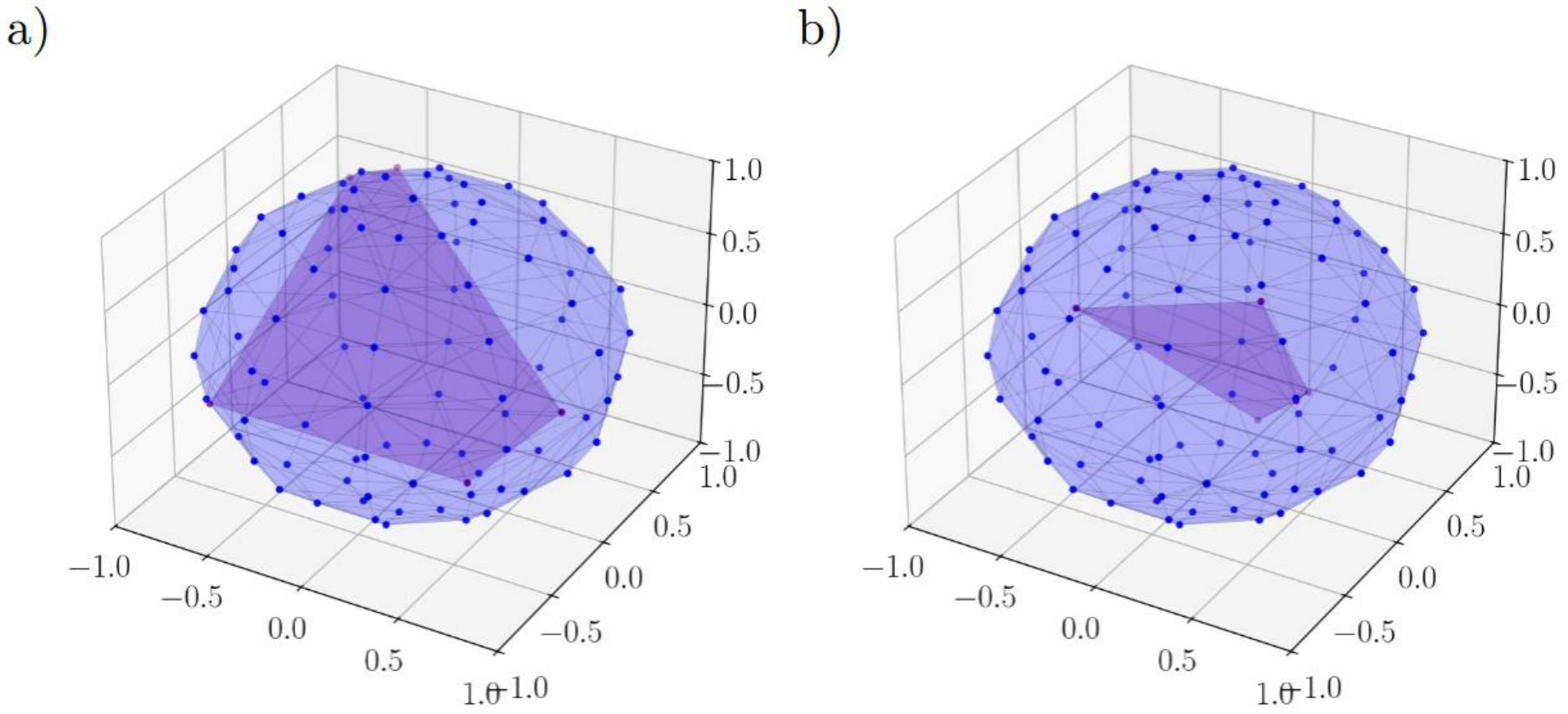
Result 2 - many fixed measurements

For pure states and access to all measurements, linear dependence is both necessary and sufficient for contextuality.

Fix m=84:

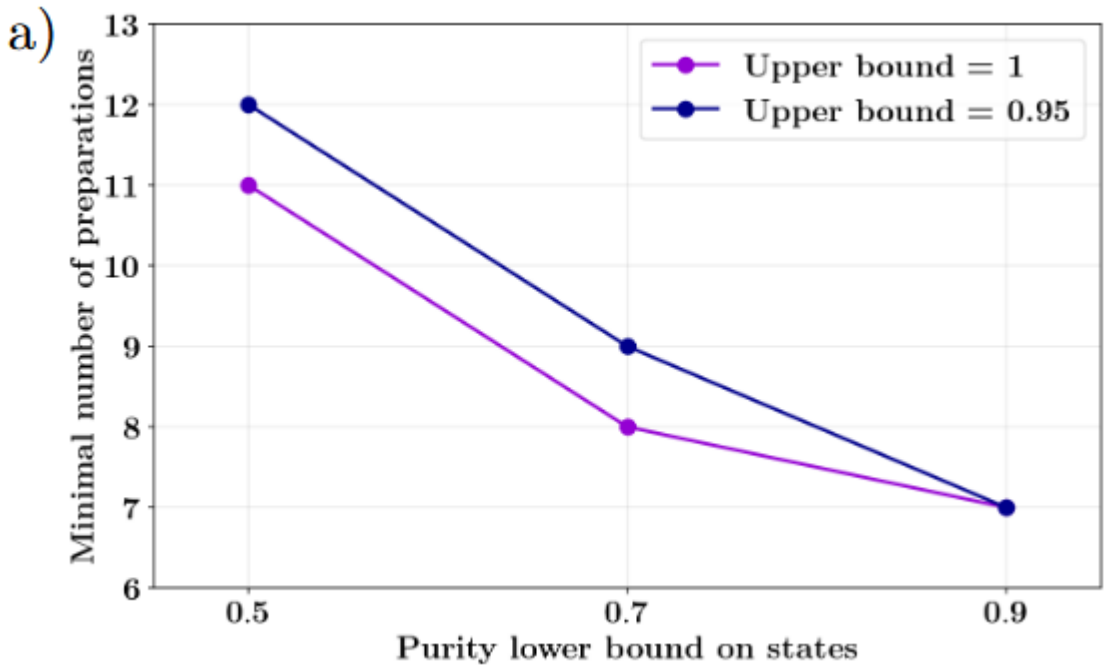


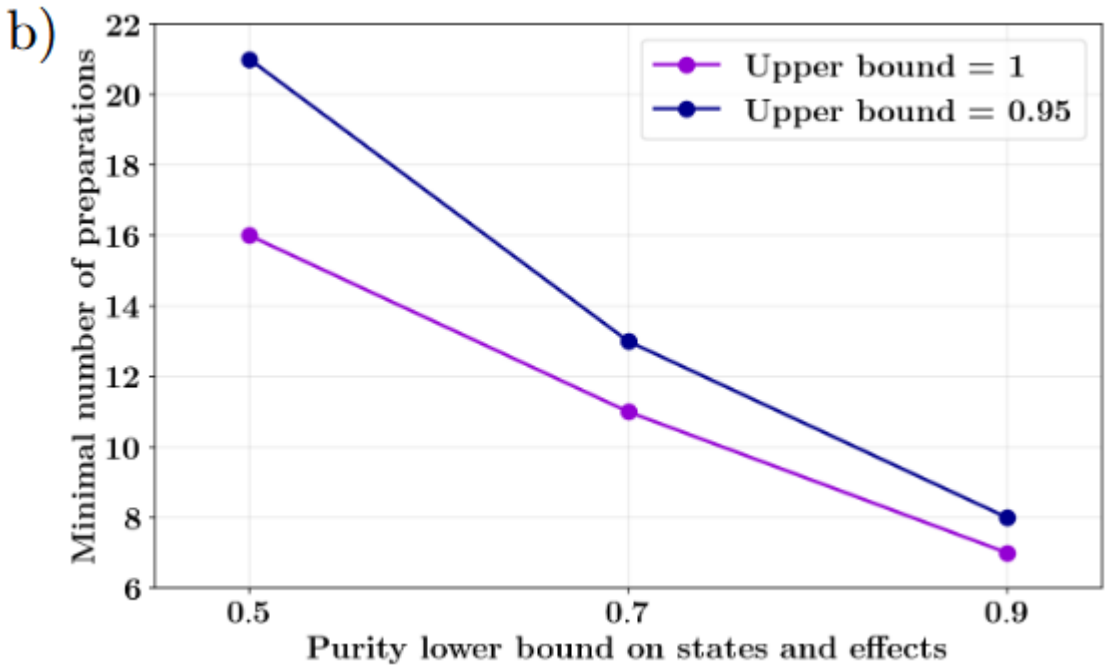
Visualization



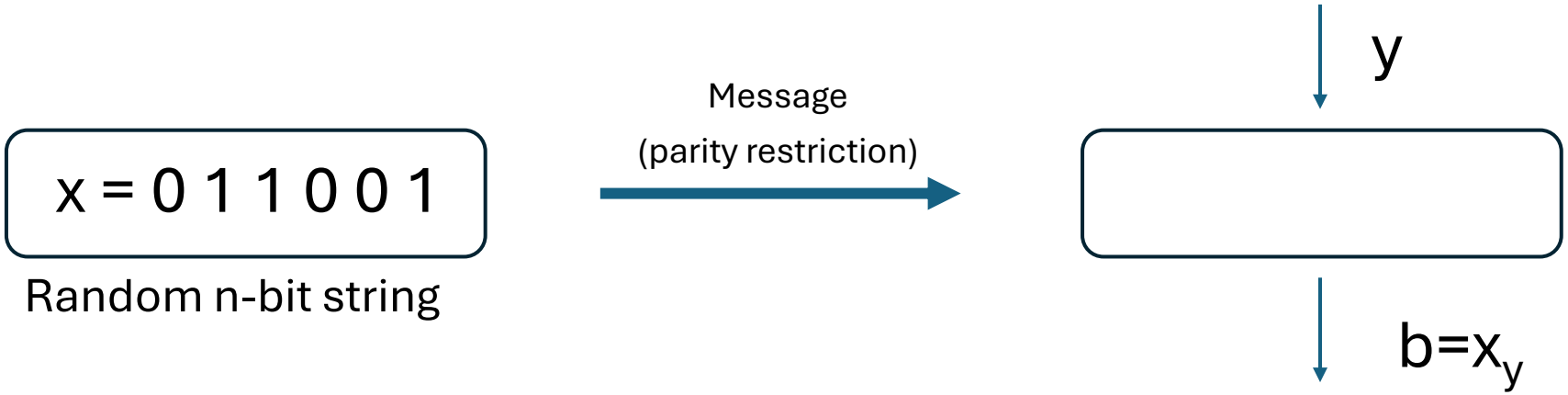
*Rescaling – shrinking of the dual

Result 3 - on purity of preparations and measurements





Result 4 – advantage in parity-oblivious multiplexing



	Typicality	Average advantage ($\bar{s}$)	Variance (σ_s)
Optimal quantum strategy	–	18.3%	–
Randomly rotated optimal meas.	100%	11%	2%
Random projective measurements	98.6%	8%	2%
Random POVMs	55.5%	2%	2%

Summary of the typicality results

- Contextuality is quite typical, even with a restricted number of preparations and measurements
- Typicality assesments don’t seem to be sensitive to small noise
- Contextuality is not equivalent to quantum advantage, but they are clearly connected
- Seems that contextuality is more typical than nonlocality

Summary of the presentation

- Contextuality is a well-motivated notion of nonclassicality
- Contextuality certification with linear programming
- Contextuality is very typical in quantum theory, but it does not always translate to quantum advantage

Thank you

Useful refs (available on youtube):

Noncontextuality by David Schmid at Solstice of Foundations 2022

Generalized Physical Theories by John Selby at Solstice of Foundations 2022

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