Out-of-Time-ordered Correlators (OTOC) and Quantum Entanglement

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Information Scrambling

- Rapid and uniform spreading of information throughout the quantum system, making the initial state impossible to recover from local measurements or partial information.
- Out-of-Time-Ordered Correlator (OTOC) - measure of a chaotic behaviour in a quantum system. Describes the correlation between two observables at different times. (Initially introduced by A. I. Larkin and Y. N. Ovchinnikov)

Classical View

- Chaotic Systems - characterized by vast sensitivity to initial conditions
- Trajectories and Lyapunov exponent λ

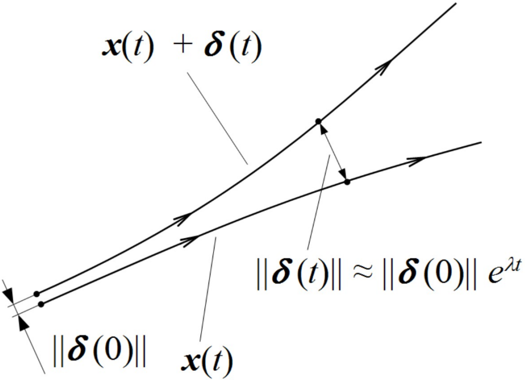

Figure: Classical trajectories in a phase space.

OTOC as a measure of Quantum Chaos

- OTOC: $C(t) = \langle [W(t), V(0)]^2 \rangle$, $W(t), V(t)$ - observables at time t.
- For example we could take the position and momentum operators.
- Naive semiclassical limit:

$$[x(t), p(0)] \to i\hbar\{x(t), p(0)\} = i\hbar \delta x(t)/\delta x(0) \sim e^{\lambda t} \tag{1}$$

thus

$$C(t) \sim \hbar^2 e^{2\lambda t} \tag{2}$$

OTOC as a measure of Quantum Chaos

1. A bound on Chaos [1] - conjecture about a bound on the rate of growth of OTOC in systems with a large number of degrees of freedom:

$$\lambda \leq 2\pi k_b T/\hbar \tag{3}$$

2. Example - kicked rotor [2]:

$$\hat{H} = \frac{\hat{p}^2}{2} + K\cos(\hat{x})\Delta(t), \quad \Delta(t) = \sum_{j=-\infty}^{\infty} \delta(t-j) \tag{4}$$

K - kicking strength, $\hbar_{eff}$ (appears in $\hat{p} = -i\hbar_{eff}\partial/\partial x$) - effective Planck constant.

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[1] Juan Maldacena, Stephen H. Shenker, and Douglas Stanford.
[2] Efim B. Rozenbaum, Sriram Ganeshan, and Victor Galitski.

Example - kicked rotor

OTOC is taken as $C(t) = \langle [\hat{p}(t), \hat{p}(0)]^2 \rangle$. Averaging is performed over Gaussian wave packets $|\Psi(0)\rangle = \sum_{n=-\infty}^{\infty} a_n^{(0)} |n\rangle$,
$a_n^{(0)} \sim exp\left(-\frac{\hbar_{eff}^2 (n-n_0)^2}{2\sigma^2}\right)$, where $\sigma = 4$ and $n_0 = 0$.

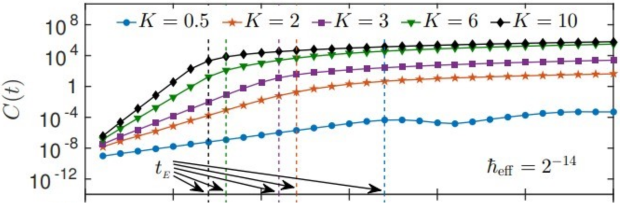

Figure: OTOC C(t) vs time (Number of Kicks)

[2] Efim B. Rozenbaum, Sriram Ganeshan, and Victor Galitski.

Example - kicked rotor

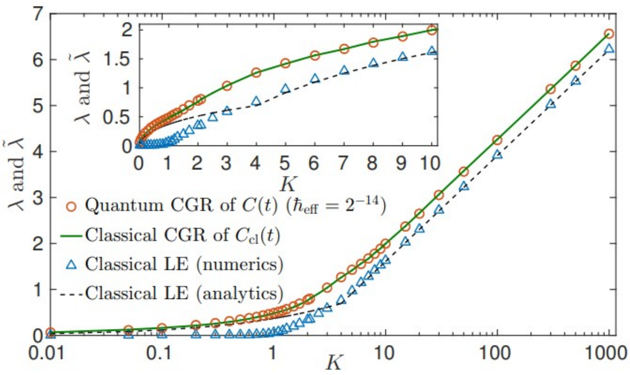

Figure: Classical Lyapunov exponent λ and growth rate of OTOC $\bar{\lambda}$ as functions of kicking strength K

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[2] Efim B. Rozenbaum, Sriram Ganeshan, and Victor Galitski.

Example - kicked rotor

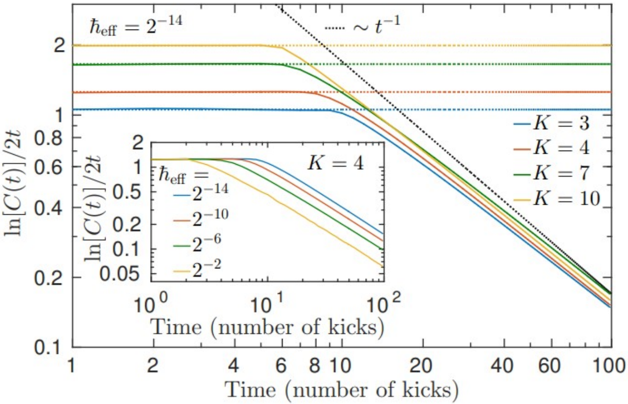
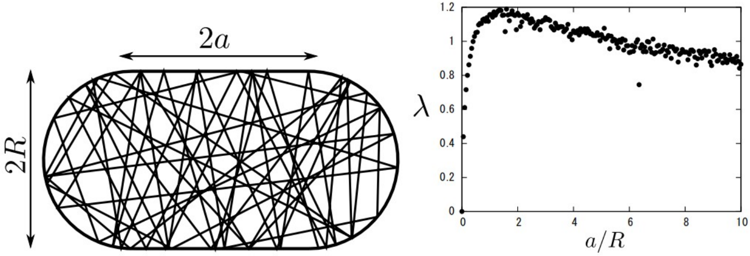


Figure: $ln[C(t)]/2t$ vs time

[2] Efim B. Rozenbaum, Sriram Ganeshan, and Victor Galitski.

Example - Stadium Billiard

Calculation of OTOC for a stadium billiard [3].



(a) A typical trajectory (b) Lyapunov exponent

Figure: Classical stadium billiard

$$H = p_1^2 + p_2^2 + V(x, y) \tag{5}$$

[3] Koji Hashimoto, Keiju Murata, and Ryosuke Yoshii.

Example - Stadium Billiard

Define OTOC as

$$C_T(t) = -\langle [x(t), p(0)]^2 \rangle = \frac{1}{Z} \sum_n e^{-\beta E_n} c_n(t) \tag{6}$$

where $c_n = -\langle n| [x(t), p(0)]^2 |n\rangle$. We can rewrite it as

$$c_n = \sum_m b_{nm} b_{nm}^*, \quad b_{nm} = b_{mn}^* = -i \langle n| [x(t), p(0)] |m\rangle \tag{7}$$

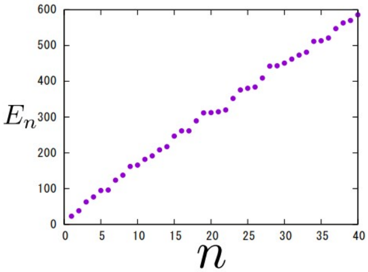
Substituting $x(t) = e^{iHt} x e^{-iHt}$ we obtain

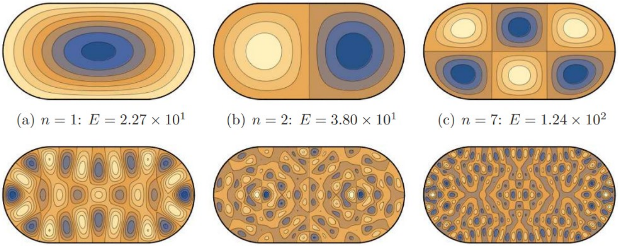
$$b_{nm} = -i \sum_k \left(e^{iE_{nk}t} x_{nk} p_{km} - e^{iE_{km}t} p_{nk} x_{km} \right), \quad E_{nk} = E_n - E_k \tag{8}$$

where $p_{km} = i/2 E_{km} x_{km}$ ($[H, x] = -2ip$).

[3] Koji Hashimoto, Keiju Murata, and Ryosuke Yoshii.

Example - Stadium Billiard





(a) $n = 1$: $E = 2.27 \times 10^1$ (b) $n = 2$: $E = 3.80 \times 10^1$ (c) $n = 7$: $E = 1.24 \times 10^2$

(d) $n = 50$: $E = 7.28 \times 10^2$ (e) $n = 200$: $E = 2.72 \times 10^3$ (f) $n = 400$: $E = 5.29 \times 10^3$

Example - Stadium Billiard

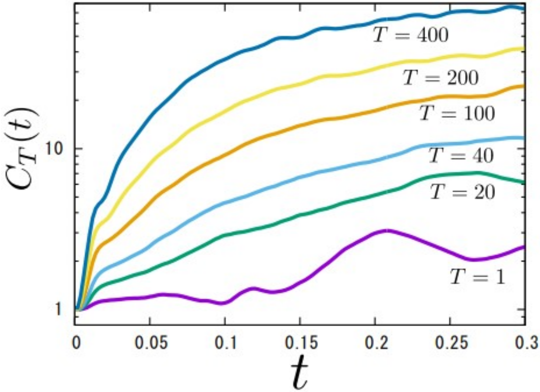
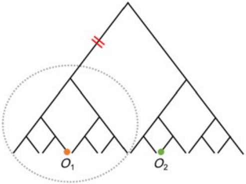


Figure: OTOC $C_T(t)$ of quantum stadium billiard at temperature T vs time t

[3] Koji Hashimoto, Keiju Murata, and Ryosuke Yoshii.

Separation of OTOC and Entanglement[4]

- Random Quantum circuit Model - Graph G with V vertices and E edges. Each vertex represents a d dimensional Hilbert space. Haar-random unitary gates are applied to endpoints of each edge randomly.
- Scaling of mixing times of OTOC and entanglement in low degree graphs:
  - t_{otoc} scales linearly with the diameter.
  - t_{ent} sclales with the number of vertices in the system.



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[4] Aram W. Harrow, Linghang Kong, Zi-Wen Liu, Saeed Mehraban, and Peter W. Shor.

References

Juan Maldacena, Stephen H. Shenker, and Douglas Stanford.
A bound on chaos.
Journal of High Energy Physics, 2016(8), aug 2016.

Efim B. Rozenbaum, Sriram Ganeshan, and Victor Galitski.
Lyapunov exponent and out-of-time-ordered correlator's growth rate in a chaotic system.
Physical Review Letters, 118(8), feb 2017.

Koji Hashimoto, Keiju Murata, and Ryosuke Yoshii.
Out-of-time-order correlators in quantum mechanics.
Journal of High Energy Physics, 2017(10), oct 2017.

Aram W. Harrow, Linghang Kong, Zi-Wen Liu, Saeed Mehraban, and Peter W. Shor.
Separation of out-of-time-ordered correlation and entanglement.
PRX Quantum, 2:020339, Jun 2021.