Hybrid Quantum-Classical Codes

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Overview

- Overview of Quantum Error Correction
- Hybrid Quantum-Classical Codes
  - Linear programming bounds for hybrid codes
  - Families of hybrid codes
  - Hybrid codes from subsystem codes
- Application to faulty syndrome measurement errors
- Future Directions

Section 1

Error Correction Background

Classical Codes

Definition

An $(n, M, d)_q$ *classical code* is a collection of M vectors from $\mathbb{F}_q^n$, that are all at least distance d apart in the *Hamming distance*: $d_H(\mathbf{a}, \mathbf{b}) = |\{i \in [n] \mid a_i \neq b_i\}|$.

- A code with a minimum distance of d can detect up to $d - 1$ or correct up to $\left\lfloor \frac{d-1}{2} \right\rfloor$ errors that occur during transmission.
- A *linear code* is a classical code with $M = q^m$ codewords that is a linear subspace of $\mathbb{F}_q^n$.
- The minimum distance of a linear code is the minimum of the weights (distance from 0^n) of all it's non-zero codewords.
- The *dual code* $C^\perp$ of a linear code is the set of codewords that are perpendicular to every codeword in C.

Weight Enumerators of Classical Codes

Definition

The *weight enumerator* $A(z)$ of a linear code is a polynomial coefficient A_i is the number of codewords of weight i in the code.

- The weight enumerator of the dual code is related to $A(z)$ using the MacWilliams identity.
- We can set up a linear programming instance that bounds the parameters $[n, m, d]_q$ of feasible codes:

$$\begin{aligned} \text{Maximize} \quad & \sum_{i=0}^{n} A_i \\ \text{Subject to} \quad & A_0 = 1 \\ & A_i \geq 0, \text{ for all } i \\ & A_i = 0, \text{ for } 0 < i < d \\ & \sum_{i=0}^{n} A_i K_j(i) \geq 0, \text{ for all } j \end{aligned}$$

Quantum Noise and Error Correction

- There are many sources of noise that affect quantum systems, which are typically far less robust than classical systems.
  - External noise from the environment and neighboring qubits.
  - Internal noise from amplitude damping.
- There are several obstacles that do not affect classical systems we need to overcome for successful quantum error correction.
- In the *depolarizing channel* the errors X, Y, Z each occur with probability $p/3$, while no error occurs with probability $1 - p$, similar to a quaternary symmetric channel.

Quantum Error-Correcting Codes

Definition

An $((n, K, d))_2$ *quantum error-correcting code* $\mathcal{C}$ is a collection of K orthogonal quantum states $\{|c_1\rangle, \dots, |c_K\rangle\}$ on n qubits, such that any error affecting less than $d - 1$ physical qubits can be detected.

- $\mathcal{C}$ defines a K-dimensional subspace of the underlying Hilbert space $\mathbb{C}^{2^n}$, and has an associated projector P.

Theorem (Knill-Laflamme [1])

An error E is detectable by a quantum code with projector P if and only if

$$PEP = \lambda_E P,$$

where $\lambda_E \in \mathbb{C}$ is a scalar depending on E.

Stabilizer Codes

Definition

A *stabilizer code* $\mathcal{C}$ with parameters $[\![n,k,d]\!]_2$ is defined by a stabilizer group $\mathcal{S}$, a group generated by $n-k$ commuting, independent Pauli elements. The 2^k shared eigenstates with eigenvalue 1 form the encoded basis states of $\mathcal{C}$.

- Stabilizer quantum codes are the quantum analog of linear and additive classical codes.
- Since encoded states are $+1$-eigenstates of elements of the stabilizer group, errors of this type have no effect on the encoded state.

Stabilizer Codes: Detectable Errors

- WLOG we can restrict our attention to Pauli errors.
- A Pauli error is detectable by a stabilizer code if and only if it anticommutes with at least one stabilizer generator.
- Anticommutation is detected by measuring the stabilizer generator observables.
- Which stabilizer generators an error commutes/anticommutes with is the syndrome of the error.
- E.g., if E commutes with S_1, S_2 and S_5 and anticommutes with S_3 and S_4 the syndrome is 00110.

Stabilizer Codes: Undetectable Errors

- The only errors we cannot detect are those that commute with all elements in $\mathcal{S}$, which is given by the centralizer of $\mathcal{S}$:

$$N(\mathcal{S}) = \{F \in \mathcal{E}_n \mid FS = SF, \forall S \in \mathcal{S}\}.$$

- Every stabilizer group can be defined by a classical additive code C over $\mathbb{F}_4$, and the centralizer is given by the dual code $C^{\perp}$. Since $\mathcal{S}$ is abelian, we have $\mathcal{S} \leq N(\mathcal{S})$, so $C \leq C^{\perp}$, that is C must be self-orthogonal.
- The minimum distance is given by

$$d = \min \mathrm{wt}\{N(\mathcal{S}) \setminus \mathcal{S}\} = \min \mathrm{wt}\left\{C^{\perp} \setminus C\right\}.$$

Shor-Laflamme Weight Enumerators

Definition

The *Shor-Laflamme weight enumerators* for quantum codes are a pair of polynomials $A(z)$ and $B(z)$, with coefficients

$$A_i = \frac{1}{K^2} \sum_{\mathrm{wt}(E)=i} \mathrm{Tr}(EP)\,\mathrm{Tr}(E^*P) \text{ and } B_i = \frac{1}{K} \sum_{\mathrm{wt}(E)=i} \mathrm{Tr}(EPE^*P)\,.$$

- For stabilizer codes, $A(z)$ and $B(z)$ count the number of elements in $\mathcal{S}$ and $N(\mathcal{S})$ respectively.
- These weight enumerators also satisfy the MacWilliams identity (each is the other's dual), so we can set up LP bounds for quantum codes.

Section 2

Hybrid Quantum-Classical Codes

Transmitting Quantum and Classical Information

- We would like to send a classical message along with a quantum message across a quantum channel.
- One approach to use a time-sharing scheme, where we share use of the channel between a quantum and a classical code.
- Devetak and Shor [2] showed that simultaneous encoding provides an advantage over time-sharing for small error rates on certain channels.

Hybrid Codes

- The first codes for simultaneous transmission were given by Kremsky, Hsieh, and Brun [3].
- While these codes provide an advantage over separate encoding, they give no advantage over encoding classical information into known quantum codes.
- Grassl, Lu, and Zeng [4] later gave sporadic codes with small parameters that provide an advantage over both.
- Our main focus will be constructions of quantum-classical hybrid codes that provide both advantages.

Hybrid Codes

Definition

A *hybrid code* has parameters $((n, K{:}M, d))_q$ if and only if it can simultaneously encode one of M different classical messages and a superposition of K orthogonal quantum states into n qubits, such that any error affecting less than d physical qubits can be detected.

- A hybrid code can be understood as a collection $\mathcal{C}$ of M orthogonal K-dimensional quantum codes $\mathcal{C}_m$ that are indexed by the classical message $m \in [M] = \{0, 1, \ldots, M-1\}$.
- To transmit a classical message $m \in [M]$ and a quantum state $|\psi\rangle$, we encode the state $|\psi\rangle$ into the quantum code $\mathcal{C}_m$.
- We call the $\mathcal{C}_m$ the *inner codes* and $\mathcal{C}$ the *outer code*.

Hybrid Codes

Theorem (Grassl-Lu-Zeng [4])

An error E is detectable by the hybrid code $\mathcal{C} = \{\mathcal{C}_m \mid m \in [M]\}$ if and only if for each index $a, b \in [M]$, we have

$$P_b E P_a = \begin{cases} \lambda_{E,a} P_a & \text{if } a = b, \\ 0 & \text{if } a \neq b \end{cases}$$

for $\lambda_{E,a} \in \mathbb{C}$, where P_a and P_b are projectors onto $\mathcal{C}_a$ and $\mathcal{C}_b$.

- We require that each detectable error is detectable by each of the inner codes (the normal Knill-Laflamme condition).
- We also require that an encoded state affected by a detectable error is perfectly distinguishable from states in other inner codes.

LP Bounds for General Hybrid Codes

- For hybrid codes, we define weight enumerators by

$$A_i^{(a,b)} = \frac{1}{K^2} \sum_{\mathrm{wt}(E)=i} \mathrm{Tr}(EP_a)\,\mathrm{Tr}(E^* P_b)\,,$$

$$B_i^{(a,b)} = \frac{1}{K} \sum_{\mathrm{wt}(E)=i} \mathrm{Tr}(EP_a E^* P_b)\,.$$

- When $a = b$ these are the weight enumerators for the inner codes, and when summed over all a, b are the (scaled) weight enumerators for the outer code.
- Using similar inequalities for the weight enumerators of quantum codes (and some others specific to hybrid codes) we can set up LP bounds on the parameters of general hybrid codes.

Hybrid Stabilizer Codes

- Hybrid stabilizers codes are defined by two nested stabilizer codes $\mathcal{C}_0 \subseteq \mathcal{C}$, with stabilizer groups $\mathcal{S} \leq \mathcal{S}_0$.
- All of the inner codes are "translations" $t_i\mathcal{C}_0 = \mathcal{C}_i$ of $\mathcal{C}_0$, $t_i \notin N(\mathcal{S}_0)$.
- Encode the quantum information as $|\varphi\rangle \in \mathcal{C}_0$, then apply the translation operator t_i corresponding to the classical message i to get $t_i|\varphi\rangle \in \mathcal{C}_i$.
- The minimum distance is given by $d = \min \mathrm{wt}\{N(\mathcal{S}) \setminus \mathcal{S}_0\}$, which may be larger than the "normal" minimum distance $d' = \min \mathrm{wt}\{N(\mathcal{S}) \setminus \mathcal{S}\}$ of $\mathcal{C}$ when viewed as a quantum code.

Hybrid Stabilizer Codes

- The generators of $\mathcal{S}_0$ are partitioned into the quantum stabilizer $\mathcal{S}_\mathcal{Q}$ and the classical stabilizer $\mathcal{S}_\mathcal{C}$.
- The outer stabilizer is $\mathcal{S} = \langle \mathcal{S}_\mathcal{Q} \rangle$ and the inner stabilizer $\mathcal{S}_0 = \langle \mathcal{S}_\mathcal{Q}, \mathcal{S}_\mathcal{C} \rangle$.

Theorem (Kremsky-Hsieh-Brun [3],Nemec-Klappenecker [5])

Let $\mathcal{C}$ be an $[\![n, k:m, d]\!]_2$ hybrid stabilizer code with stabilizer $\langle \mathcal{S}_\mathcal{Q}, \mathcal{S}_\mathcal{C} \rangle$. Then the stabilizer code $\mathcal{C}_c$ associated with classical message $c \in \mathbb{F}_2^m$ is given by the stabilizer

$$\left\langle \mathcal{S}_\mathcal{Q}, (-1)^{c_1} g_1^\mathcal{C}, \ldots, (-1)^{c_m} g_m^\mathcal{C} \right\rangle,$$

where c_i is the i-th entry of c and $g_i^\mathcal{C}$ are the generators of $\mathcal{S}_\mathcal{C}$.

Hybrid Stabilizer Codes

$$\begin{pmatrix}
X & I & I & Z & Y & Y & Z \\
Z & X & I & X & Z & I & X \\
Z & I & X & X & I & Z & X \\
Z & I & Z & Z & X & I & I \\
I & Z & I & Z & I & X & X \\
\hdashline
Z & I & I & I & I & I & X \\
\hline
I & I & I & X & Z & Z & X \\
I & I & I & Z & X & X & I \\
\hline\hline
I & I & I & I & X & Y & Y
\end{pmatrix}$$

- A $[\![7, 1\!:\!1, 3]\!]_2$ hybrid code.
- Dotted line divides quantum stabilizer generators $\mathcal{S}_{\mathcal{Q}}$ (above) from the classical stabilizer generators $\mathcal{S}_{\mathcal{C}}$ (below).
- Translation operators (classical logical operators) below double line.

Hybrid Stabilizer Codes

$$\left(\begin{array}{ccccccc}
X & I & I & Z & Y & Y & Z \\
Z & X & I & X & Z & I & X \\
Z & I & X & X & I & Z & X \\
Z & I & Z & Z & X & I & I \\
I & Z & I & Z & I & X & X \\
\hdashline
Z & I & I & I & I & I & X \\
\hline
I & I & I & X & Z & Z & X \\
I & I & I & Z & X & X & I \\
\hline\hline
I & I & I & I & X & Y & Y
\end{array}\right)$$

- A $[\![7, 1:1, 3]\!]_2$ hybrid code.
- Dotted line divides quantum stabilizer generators $\mathcal{S}_\mathcal{Q}$ (above) from the classical stabilizer generators $\mathcal{S}_\mathcal{C}$ (below).
- Translation operators (classical logical operators) below double line.
- The stabilizers of the outer code $\mathcal{C}$ are in the red box.

Hybrid Stabilizer Codes

$$\left(\begin{array}{ccccccc}
X & I & I & Z & Y & Y & Z \\
Z & X & I & X & Z & I & X \\
Z & I & X & X & I & Z & X \\
Z & I & Z & Z & X & I & I \\
I & Z & I & Z & I & X & X \\
\hdashline
Z & I & I & I & I & I & X \\
\hline\hline
I & I & I & X & Z & Z & X \\
I & I & I & Z & X & X & I \\
\hline\hline
I & I & I & I & X & Y & Y
\end{array}\right)$$

- A $[\![7, 1{:}1, 3]\!]_2$ hybrid code.
- Dotted line divides quantum stabilizer generators $\mathcal{S}_{\mathcal{Q}}$ (above) from the classical stabilizer generators $\mathcal{S}_{\mathcal{C}}$ (below).
- Translation operators (classical logical operators) below double line.
- The stabilizers of the outer code $\mathcal{C}$ are in the red box.
- The stabilizers of the inner code $\mathcal{C}_0$ are in the blue box.

"Trivial" Constructions of Hybrid Codes

Proposition (Grassl-Lu-Zeng [4])

We have the following "trivial" constructions of hybrid codes:

1. *Given an $((n, KM, d))_q$ quantum code of composite dimension KM, there exists a hybrid code with parameters $((n, K{:}M, d))_q$. (Using encoded qubits of a quantum code to transmit classical info.)*
2. *Given an $[\![n, k{:}m, d]\!]_q$ hybrid code with $k > 0$, there exists a hybrid code with parameters $[\![n, k-1{:}m+1, d]\!]_q$. (Using encoded qubits of a hybrid code to transmit classical info.)*
3. *Given an $[\![n_1, k_1, d]\!]_q$ quantum code and an $[n_2, m_2, d]_q$ classical code, there exists a hybrid code with parameters $[\![n_1+n_2, k_1{:}m_2, d]\!]_q$. (Separate encoding of quantum and classical info.)*

Genuine Hybrid Codes

- We call codes with parameters not achievable by the three "trivial" constructions *genuine hybrid codes*.

Theorem (Nemec-Klappenecker [6])

Suppose $\mathcal{C}$ is a genuine $((n, K:M, d))_q$ hybrid code. Then at least one inner code $\mathcal{C}_m$ of the hybrid code $\mathcal{C}$ is impure.

Definition

A quantum code is *impure* if there are distinct detectable errors that share an error syndrome.

- Our goal is to construct the first families of genuine hybrid codes.

Parameters of Hybrid Codes in Grassl-Lu-Zeng [4]

n/k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1	1																					
2	2	1	1																				
3	2	1	1	1																			
4	2																						
5	3																						
6	4	3	2	2	2	1	1																
7	3	3	2	2	2	1	1	1															
8	4	3	3	3	2	2	2	1	1														
9	4	3	3	3	2	2	2	1	1	1													
10	4	4	4	3	3	2	2	2	2	1	1												
11	5	5	4	3	3	3	2	2	2	1	1	1											
12	6	5	4	4	4	3	3	2	2	2	2	1	1										
13	5	5	4	4	4	3	3	3	2														
14	6	5	5	4-5	4	4	4	3	3														
15	6	5	5	5	4	4	4	3	3														
16	6	6	6	5	5	4-5	4	4	3	3	3	2	2	2	2	1	1						
17	7	7	6	5-6	5	4-5	4-5	4	4	4	3	3	2	2	2	1	1	1					
18	8	7	6	5-6	5-6	5	5	4	4	4	3	3	2	2	2	2	2	1	1				
19	7	7	6	5-6	5-6	5-6	5	4-5	4	4	3-4	3	3	2	2	2	2	1	1	1			
20	8	7	6-7	6-7	6	5-6	5-6	4-5	4-5	4	4	3-4	3	3	2	2	2	2	2	1	1		
21	8	7	6-7	6-7	6-7	6	5-6	5-6	4-5	4-5	4	4	3-4	3	3	3	2	2	2	1	1	1	
22	8	7-8	6-8	6-7	6-7	6-7	5-6	5-6	5-6	4-5	4-5	4	4	3-4	3	3	2	2	2	2	2	1	1

There is no $[\![9,4,3]\!]_2$ code, but there is a hybrid $[\![9,2:2,3]\!]_2$ code.

Where is there an $[\![n,k:m,d]\!]_2$ hybrid code but no $[\![n,k+m,d]\!]_2$ stabilizer code?

Parameters of Best Known Nonadditive Codes

n/k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1	1																					
2	2	[illegible]	[illegible]																				
3	2																						
4	2	2	2	1	1																		
5	3	3	2	1	1	1																	
6	4	3	2	2	2	1	1																
7	3	3	2	2	2	1	1	1															
8	4	3	3	3	2	2	2	1	1														
9	4	3	3	3	2	2	2	1	1	1													
10	4	4	4	3	3	2	2	2	2	1	1												
11	5	5	4	3	3	3	2	2	2	1	1	1											
12	6	5	4	4	4	3	3	2	2	2	2	1	1										
13	5	5	4	4	4	3	3	3	2	2	2	1	1	1									
14	6	5	5	4-5	4	4	4	3	3	2	2	2	2	1	1								
15	6	5	5	5	4	4	4	3	3	3	2	2	2	1	1	1							
16	6	6	6	5	5	4-5	4	4	3	3	3	2	2	2	2	1	1						
17	7	7	6	5-6	5	4-5	4-5	4	4	4	3	3	2	2	2	1	1	1					
18	8	7	6	5-6	5-6	5	5	4	4	4	3	3	2	2	2	2	2	1	1				
19	7	7	6	5-6	5-6	5-6	5	4-5	4	4	3-4	3	3	2	2	2	2	1	1	1			
20	8	7	6-7	6-7	6	5-6	5-6	4-5	4-5	4	4	3-4	3	3	2	2	2	2	2	1	1		
21	8	7	6-7	6-7	6-7	6	5-6	5-6	4-5	4-5	4	4	3-4	3	3	3	2	2	2	1	1	1	
22	8	7-8	6-8	6-7	6-7	6-7	5-6	5-6	5-6	4-5	4-5	4	4	3-4	3	3	2	2	2	2	2	1	1

There is no $[\![9,4,3]\!]_2$ code, but there is a nonadditive $((9,12,3))_2$ code.

Single-Error Detecting Quantum Codes

- There exists a family of even-length quantum codes with parameters $[\![n, n-2, 2]\!]_2$, but not for odd lengths.
- By slightly relaxing the stabilizer restrictions, many families of odd-length *nonadditive codes* have been constructed such that $K \approx 2^{n-2}$.
- By slightly relaxing the KL conditions for quantum codes, we can construct the first infinite family of hybrid codes.

Theorem (Nemec-Klappenecker [6])

For odd $n \geq 5$, there exists an $[\![n, n-3{:}1, 2]\!]_2$ genuine hybrid code.

$$\left(\begin{array}{ccccc} X & X & X & X & X \\ Z & Z & Z & Z & I \\ \hline I & I & I & I & X \end{array}\right)$$

A More General Construction

Theorem (Nemec-Klappenecker)

Let $\mathcal{C}$ be an $[\![n,k,d]\!]_2$ with normalizer $N(\mathcal{S})$. Then there is an $[\![n+m,k:m,d]\!]_2$ code if there are m elements $\{t_i\}_{i\in[m]}$ from separate, independent cosets of $G_n/N(\mathcal{S})$ such that $\langle t_iN(\mathcal{S})\rangle_{i\in[m]}$ forms a group and for each coset $tN(\mathcal{S})$ we have

$$\min \mathrm{wt}\{tN(\mathcal{S})-\mathcal{S}\} \geq d-\mathrm{wt}(\mathbf{a}_t)\,,$$

where $t=t_1^{a_1}\cdots t_m^{a_m}$ and $\mathbf{a}_t$ is the binary vector $a_1\ldots a_m$.

Corollary

If a non-perfect $[\![n,k,3]\!]_2$ stabilizer code exists, then there is an $[\![n+1,k:1,3]\!]_2$ hybrid code.

Best Known Hybrid Codes

n/k	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
1	1	1																					
2	2	1	1																				
3	2	1	1	1																			
4	2	2	2	1	1																		
5	3	3	2	1	1	1																	
6	4	3	2	2	2	1	1																
7	3	3	2	2	2	1	1	1															
8	4	3	3	3	2	2	2	1	1														
9	4	3	3	3	2	2	2	1	1	1													
10	4	4	4	3	3	2	2	2	2	1	1												
11	5	5	4	3	3	3	2	2	2	1	1	1											
12	6	5	4	4	4	3	3	2	2	2	2	1	1										
13	5	5	4	4	4	3	3	3	2	2	2	1	1	1									
14	6	5	5	4-5	4	4	4	3	3	2	2	2	2	1	1								
15	6	5	5	5	4	4	4	3	3	3	2	2	2	1	1	1							
16	6	6	6	5	5	4-5	4	4	3	3	3	2	2	2	2	1	1						
17	7	7	6	5-6	5	4-5	4-5	4	4	4	3	3	2	2	2	1	1	1					
18	8	7	6	5-6	5-6	5	5	4	4	4	3	3	2	2	2	2	2	1	1				
19	7	7	6	5-6	5-6	5-6	5	4-5	4	4	3-4	3	3	2	2	2	2	1	1	1			
20	8	7	6-7	6-7	6	5-6	5-6	4-5	4-5	4	4	3-4	3	3	2	2	2	2	2	1	1		
21	8	7	6-7	6-7	6-7	6	5-6	5-6	4-5	4-5	4	4	3-4	3	3	3	2	2	2	1	1	1	
22	8	7-8	6-8	6-7	6-7	6-7	5-6	5-6	5-6	4-5	4-5	4	4	3-4	3	3	2	2	2	2	2	1	1

Codes from Grassl et al. [4]
Improved parameters for hybrid codes
New parameters for hybrid codes

New $[\![9,3:1,3]\!]_2$ code.

Hybrid Codes from Stabilizer Pasting

$$\begin{pmatrix} X & X & X & X & X & X & X & X & I & I & I & I & I \\ Z & Z & Z & Z & Z & Z & Z & Z & I & I & I & I & I \\ X & I & X & I & Z & Y & Z & Y & X & X & Z & I & Z \\ X & I & Y & Z & X & I & Y & Z & Z & X & X & Z & I \\ X & Z & I & Y & I & Y & X & Z & I & Z & X & X & Z \\ I & I & I & I & I & I & I & I & Z & I & Z & X & X \end{pmatrix}$$

- *Stabilizer pasting* is a technique introduced by Gottesman [7] for extending pure stabilizer codes, and was used by Yu et al. [8] to construct two infinite families of nonadditive codes.
- Instead of nonadditive codes, we use small hybrid codes from the previous construction, pasting on the pure $\mathcal{S}_Q$.

Hybrid Codes from Stabilizer Pasting

Theorem (Nemec-Klappenecker [6])

Let m be a non-negative integer and n a positive integer given by

$$n = \frac{2^{2m+5} - 32}{3} + a$$

where the parameter a is a small positive integer that is specified below. Then there exists

- *an* $[\![n, n-2m-6 : 1, 3]\!]_2$ *hybrid code for* $a = 7, 9$, *and*
- *an* $[\![n, n-2m-7 : 2, 3]\!]_2$ *hybrid code for* $a = 10, 11$.

- The first two can squeeze in an extra classical bit over an optimal stabilizer code, while the second two exchange an encoded qubit for two classical bits.

Hybrid Codes from Stabilizer Pasting

Theorem (Nemec-Klappenecker)

Let m be a non-negative integer and n a positive integer given by

$$n = \frac{2^{2m+6} - 64}{3} + a$$

where the parameter a is a small positive integer that is specified below. Then there exists

- *an* $[\![n, n-2m-6:1, 3]\!]_2$ *hybrid code for* $a = 18$*,*
- *an* $[\![n, n-2m-7:2, 3]\!]_2$ *hybrid code for* $a = 19$*, and*
- *an* $[\![n, n-2m-8:3, 3]\!]_2$ *hybrid code for* $a = 20$*.*

- All of these codes (and those on the previous slide) are genuine hybrid stabilizer codes.

Subsystem Codes and Gauge Fixing

- *Subsystem codes* are a generalization of stabilizer codes, and can be thought of as stabilizer codes with purposefully unused logical qubits called *gauge qubits*.
- We now want to show that any subsystem code can give rise to a hybrid stabilizer code.
- Our construction is to encode the quantum information into the logical qubits of the subsystem code and then carefully encode the classical information in the unused gauge qubits.
- *Gauge fixing* is in essence setting the gauge qubits to a certain state, effectively "fixing" them. Since the states are fixed, no information can be encoded on those qubits.
- We do so in such a way that the quantum and classical information are protected to two different minimum distances.

Separate Minimum Distances

Theorem (Nemec-Klappenecker [5])

Let $\mathcal{C}$ be an $((n, K:M, d:c))_q$ hybrid code with a subsystem structure $A \otimes B$ on it, with $\{|\varphi_i\rangle \mid i \in [K]\}$ and $\{|v_i\rangle \mid i \in [M]\}$ as orthonormal bases for A and B respectively. Then $\mathcal{C}$ can detect up to $d-1$ errors to the quantum information and up to $c-1$ errors to the classical information if and only if

1. $P_a E P_b = \lambda_{E,a,b} P_{a,b}$, *for all E such that* $\mathrm{wt}(E) < d$, *and*
2. $P_a F P_b = 0$, *where $a \neq b$, for all F such that* $\mathrm{wt}(F) < c$.

- If a code has $d = 5$ and $c = 3$ and a single-qubit error occurs, then we will always be able to correct the error, but if a two-qubit error occurs we can still correct the quantum information, but the classical information might not be correct.

Hybrid Codes from Subsystem Codes

Proposition (Nemec-Klappenecker [5])

Let $\mathcal{C}$ be an $[\![n,k,r,d]\!]_q$ subsystem code. Then there exists an $[\![n,k:r,d:c]\!]_q$ hybrid code, where

$$c = \min \mathrm{wt}\{N(\mathcal{S}) \setminus N(\mathcal{S}_0)\} \text{ and } d = \min \mathrm{wt}\{N(\mathcal{S}) \setminus \mathcal{G}\}.$$

- Given a pair $\overline{X}$,$\overline{Z}$ of anticommuting gauge operators, one is placed in $\mathcal{S}_{\mathcal{C}}$, and the other becomes a translation operator.
- Each subsystem code gives rise to different hybrid codes by choosing different gauge operators to be in $\mathcal{S}_{\mathcal{C}}$, and this choice can have a major impact on the classical minimum distance c.
- Using the Bacon-Casaccino subsystem code construction [9], we can construct hybrid codes directly from a pair of classical linear codes.

Tower of Paulis

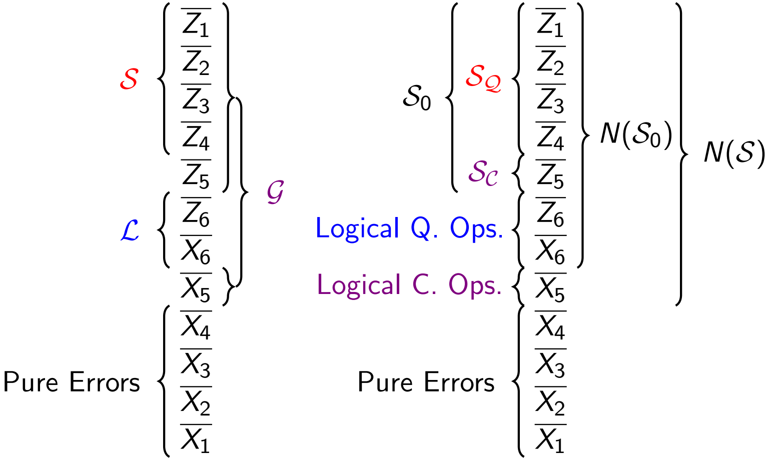

Figure 1: A new $[\![6,1:1,3:2]\!]_2$ hybrid code from a subsystem code.

Section 3

Applications to FSM Errors

Faulty Syndrome Measurement Errors

Definition

Faulty syndrome measurement errors are bit-flip errors that occur on the error syndrome, caused by faulty measurements.

- One way to protect against FSM errors is to reserve all weight-1 syndromes for them (for $d = 3$). $\rightarrow$ Allows us to correct either 1 data error **or** 1 syndrome error.
- We will try and store a parity bit for the syndrome on the encoded classical bit of a hybrid code.
- All of the syndromes associated with data errors will therefore have even weight.
- When we measure the encoded classical information we obtain an error syndrome with extra redundancy.

Using Hybrid Codes to Protect Syndrome: Idea

- Encode no classical information in the hybrid code (encode to logical $|0\rangle$).
- After the quantum state has been recieved we can do postprocessing on it to encode a syndrome parity bit on the encoded classical state.

$$\overbrace{|s_1\rangle|s_2\rangle|s_3\rangle|s_4\rangle}^{\text{Syndrome}}|0\rangle|\varphi\rangle$$

$$\downarrow$$

$$\underbrace{|s_1\rangle|s_2\rangle|s_3\rangle|s_4\rangle}_{\text{Syndrome}}\underbrace{|s_1 \oplus s_2 \oplus s_3 \oplus s_4\rangle}_{\text{Parity Bit}}|\varphi\rangle$$

Figure 2: What we want to do.

Using Hybrid Codes to Protect Syndrome: Actual

- In reality, the error from the quantum channel might anticommute with the classical stabilizer, meaning t may be either 0 or 1.
- This is fatal for some (but not all) hybrid codes, so pick so that no weight-1 data error has a weight-1 syndrome.

$$\overbrace{|s_1\rangle|s_2\rangle|s_3\rangle|s_4\rangle}^{\text{Syndrome}}|t\rangle|\varphi\rangle$$

$$\downarrow$$

$$\underbrace{|s_1\rangle|s_2\rangle|s_3\rangle|s_4\rangle}_{\text{Syndrome}}\underbrace{|s_1 \oplus s_2 \oplus s_3 \oplus s_4 \oplus t\rangle}_{\text{Parity Bit}}|\varphi\rangle$$

Figure 3: What actually happens.

Quantum Data-Syndrome Codes

- *Quantum data-syndrome codes* [10] protect data against the effects of faulty syndrome measurement by measuring an overcomplete set of stabilizer generators.
- An $[\![n, k, d : r]\!]_q$ QDS code can correct any combination of $\lfloor (d-1)/2 \rfloor$ errors to the quantum information **or** the syndrome using $n - k + r$ measurements.
- Fujiwara [11] showed that any $[\![n, k, 3]\!]_2$ stabilizer code can be transformed into an $[\![n, k, 3 : 1]\!]_2$ QDS code by measuring the product of the stabilizer generators, which provides a parity bit for the syndrome.
- Is it possible to have QDS codes with no extra measurements $(r = 0)$?

Using Hybrid Codes Construct QDS Codes

$$\begin{pmatrix} X & I & I & Z & Y & Y & Z \\ Z & X & I & X & Z & I & X \\ Z & I & X & X & I & Z & X \\ Z & I & Z & Z & X & I & I \\ I & Z & I & Z & I & X & X \\ \hdashline Z & I & I & I & I & I & X \end{pmatrix} \qquad \begin{pmatrix} X & I & I & Z & Y & Y & Z \\ Z & X & I & X & Z & I & X \\ Z & I & X & X & I & Z & X \\ Z & I & Z & Z & X & I & I \\ I & Z & I & Z & I & X & X \\ X & Y & Y & Z & I & I & Z \end{pmatrix}$$

- We replace the stabilizer generator from $\mathcal{S}_{\mathcal{C}}$ (left) and replace it with the product of all 6 generators (right).
- This is equivalent to measuring a modified stabilizer for the classical information (without having to do the encoded CNOTs), which makes the code a QDS code.
- This gives us new $[\![6,1,3:0]\!]_2$, $[\![7,1,3:0]\!]_2$, and $[\![9,3,3:0]\!]_2$ QDS codes.

Thoughts on QDS Codes

- Fujiwara [11] showed that the 7-qubit Steane code (with nonstandard choice of generators) is also a $[\![7,1,3:0]\!]_2$ QDS code, but one (seemingly) that cannot be based off of a hybrid code.
- Our new QDS codes from hybrid codes seem to imply that while impurity is not necessary for QDS codes with $r = 0$, it seems to help.
- In effect, we are taking advantage of subsystem codes which need fewer measurements to correct errors to the quantum information.

Section 4

Conclusion and Future Directions

Conclusion

- Simultaneously encoding quantum and classical information allows us to sometimes squeeze in a extra classical bit when the quantum information is maxed out.
- We construct the first families of genuine hybrid codes.
- We give a general construction of hybrid codes from subsystem codes.
- We show how hybrid codes can lead to constructions of QDS codes, and shine some light on an open question about them.

Future Directions

- Unification of EA, subsystem, and hybrid codes.
- Generalize "more general construction" to other "gadgets".
- Better bounds for hybrid codes with separate minimum distances (especially when $c < d$).
- Hybrid codes for different channels (e.g., amplitude damping, etc.).
- Beyond constructing QDS codes, are there other applications for hybrid codes?
- Are there any connections between hybrid codes and nonadditive codes beyond "parameter location"?

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Thank You!